Muffler Acoustics - Application to Automotive Exhaust Noise Control

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Lecture - 05
Solution of 1-D Helmholtz Equation: Propagation in 1-D Ducts/Pipes

Welcome to the Muffler Acoustics course. So, in the last lecture we were discussing about the Helmholtz equation; the one dimensional Helmholtz equation which is given by the following. Typically, this equation is used to model planar wave propagation in ducts, one dimensional ducts or planar wave propagation; where symbols carry the usual meaning what I explained in my last lecture.

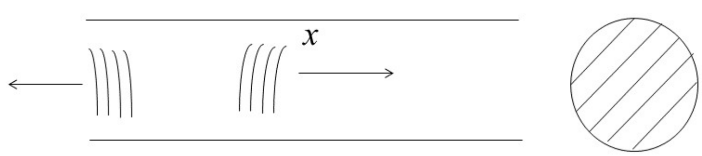
$$\frac{d^2p}{dx^2} + k_0^2\tilde{p} = 0,$$

Here,

$$k_0 = \frac{\omega}{C_0}$$

$$\tilde{p} = Acoustic\ pressure$$

So, maybe we can consider duct like this, I have some.



So the wave propagation direction is indicated of course, other wave propagates in this direction, but the idea is that if we consider the duct section to be circle, it could also be rectangular or square.

The idea is over this entire section the pressure value acoustic pressure value is constant, so is the acoustic velocity; particle velocity along the direction. So it is basically the phase or the wave is constant over this plane, so its also called planer wave fronts. So that is in this along the; if you cut any slice, it has the same properties all throughout that slice.

So basically this equation is then used to model your acoustic wave propagation where

$$\tilde{p} = \tilde{p}(x)e^{j\omega t}$$

which is assumed to very harmonically we insist harmonicity.

So basically, in the last lecture we figured out that this complete solution consisting of wave that propagates along the positive x direction is given by

$$p(x,t) = Ae^{-jk_0 x+j\omega t} + Be^{+jk_0 x+j\omega t}$$

So you know this essentially this term like the phase. So we can probably write it in a more clean and more probably slightly better form like this.

$$= Ae^{j(\omega t- \ _0 x} + Be^{j(\omega t-k_0 x} \ \ldots\ldots \qquad (2)$$

Clearly, this is forward propagating wave, the one that propagates in the positive x direction, this is in the negative x direction alright. So what do we do after this? Obviously, if you put you know let us say, we let us call this equation (2). Let us call this equation (1).

If we substitute, you know say, this part in equation 1, it will satisfy and so will the other part. So they form the complete sort of, they form the complete solution or complete basis because it is a second order differential equation, we need two solutions and A and B are the arbitrary constants also known as the amplitude A and B are the amplitudes or the wave or the progressive wave.

So this is perhaps; the first time that I am introducing this term progressive. Why progressive? let us talk about it.

As we probably see from this figure this the waves they are going in certain direction without getting attenuated or without getting reflected. So basically this solution this one talks about the wave that goes along a certain direction without suffering any reflections. And, similarly this shows the ways in the opposite direction, so they are progressive because they constantly travel along a certain direction without suffering reflections.

Of course, when reflections are there, then something other happens, what we will get is called standing wave and that, again, will depend on the boundary conditions. We will introduce all those things sequentially, but now that we have got the pressure solution, which I will probably write it down.

So here we have the full solution of acoustic pressure which is

$$\tilde{p}(x,t) = Ae^{j(\omega t - k_0 x)} + Be^{j(\omega t + k_0 x)}$$

Now, if we use our Euler equation for momentum.

$$\rho_0 \frac{\partial U}{\partial t} = \frac{\partial \tilde{p}}{\partial x}$$

What do we get? Right. Now, this probably you can see from the lecture number 3 or 4 that is the last two lectures, when you linearize this. I must mention one point here that we are still considering a stationary medium.

So U what you are saying is not like its a constant translation. U is something like oscillations about mean positions. If you have a particle right here, so it does what does is like this. It does not do like this, it does like this. So U is basically a kind of vibration or the oscillation of the air molecules; so, it pertains to the oscillatory velocity of the particles. So, of course you can have situations in where you have flow in ducts where a flow not only is particles are not only translated in time like this, but they also do this.

So basically in; so, if you have a duct with flow and you have waves propagating through that acoustic waves. So, you see something like this; there is a gradual there is a net displacement of a particle it is convicted along a certain direction, but the waves the particles are also oscillating about their mean position.

In this case, of course, there is no net displacement; there is no net transport they are just oscillating. So, it was necessary to first distinguish that case we are still in a stationary medium.

Now, since p is harmonic, that is $\tilde{p}(x,t) = \tilde{p}(x)e^{j\omega t}$,

$$U(x,t) = U(x)e^{j\omega t}$$

$$\rho_0 \, j\omega U = -\frac{d\widetilde{p}}{dx}$$

$$j\frac{\omega}{C_0}\rho_0 C_0 U = -\frac{d\widetilde{p}}{dx}$$

$$jk_0\rho_0 C_0 \, U = -\frac{d\widetilde{p}}{dx} \qquad (3)$$

so in that case your U will also be something like that. So here we have ordinary derivative. So, what we do is that we write this thing in terms of a wave number and the characteristic impedance what I discussed in my last lecture. So here you have this thing. So, let me call this equation number (3).

So now we can plug the solution that we saw in equation (2) or probably this one in on the right-hand side of the equation (3) to get what? We will get

$$jk_0\rho_0 C_0 \, U = -\left\{Ae^{j(\omega t - k_0 x)}(-jk_0) + Be^{j(\omega t + k_0 x)}(jk_0)\right\}$$

$$= jk_0\rho_0 C_0 \, U = jk_0\left\{Ae^{j(\omega t - k_0 x)} - Be^{j(\omega t + k_0 x)}\right\}$$

$$U(x) = \frac{1}{\rho_0 C_0}\left\{Ae^{j(\omega t - k_0 x)} - Be^{j(\omega t + k_0 x)}\right\} \qquad (4)$$

So, this gets cancelled and the solution U x becomes; so then this is the solution, for the acoustic particle velocity, alright. So, let us call this as equation (4).

So we have U, we have p and what we essentially see here there are quite a few important things that come out of the solutions given by (3) and (4) is a proper formal solution of Helmholtz equation 1D Helmerich vision and U is the particle velocity that is found from the momentum equation which is combined with the continuity equation. So, once you get Up you can also automatically get U.

It is just a matter of putting the solution for p in the momentum equation and simplifying so we get both the solutions. So, the thing to be noted here is the quantity $\rho_0 C_0$ in the denominator the one that I am underlining. So, this I was just mentioning a while back, and also the last lecture is a characteristic impedance.

$$\beta C_0 = 12\frac{kg}{m^3} \times 343\frac{m}{s} \simeq 414\, kg\, m^{-2}s^{-1}$$

so as a characteristic impedance is basically it tells you the impedance of the opposition of the wave propagation offered by the media.

So, let us go back to this equation 4 and also simultaneously recall pressure equation, so rho naught C naught is the thing that comes in denominator. Now if you divide by U x, what do we get? We

$$\frac{\tilde{p}(x)}{U(x)} = \rho_0 C_0 \frac{\{Ae^{-jk_0x} + Be^{jk_0x}\}}{\{Ae^{-jk_0x} - Be^{jk_0x}\}} = Z(x) \qquad (5)$$

Now suppose this a generalized expression for impedance for the lack of space. I am putting this here. We will use this equation. Let us call this equation number 5. Now, suppose if you have a duct if we have only unidirectional propagation. That is the wave goes only along the positive *x*- direction like this.

So, there is no wave that is going in the backward directions to B is 0. If that is the case, the numerator cancels out and what we are essentially,

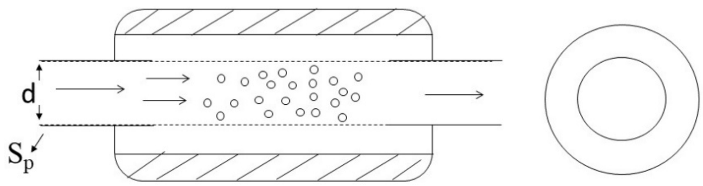
$$Z_0 = \frac{p(x)}{U(x)} = \rho_0 C_0 (6)$$

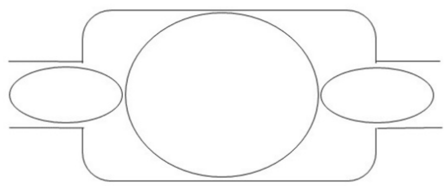
So, this is the ratio of the acoustic pressure, to the acoustic velocity of the forward propagating wave.

That is a progressive wave alright, and that is given by $\rho_0 C_0$and the value is known. Similarly, if you have a wave that propagates the negative x direction, it is very easy to see that ratio then is minus $\rho_0 C_0$. So, it depends on the direction in which the wave propagates.

We can define the convention, but important point to be noted is that $\rho_0 C_0$is constant, so it does not depend on the cross-section area of the duct. It just depends on the physical properties like the density and sound speed, which also depends on the temperature. So it means independent of the geometry of the physical dimensions of the duct.

So, it is called characteristic because it is very inherent to that medium inherent to air, which is the medium in which the acoustic wave propagating. We need to know this physical constant, so that is why it is called the characteristic impedance, $\rho_0 C_0$of the medium. However, they are other things also which will be particularly useful in the context of muffler acoustics. That is basically when considering system of ducts.





Muffler nothing but basically, let me draw some typical muffler configuration. So, you see this inlet port outlet port here inside; it would be a complicated thing. But we are I am just trying to give you a glimpse, a sneak peek into what you can expect in the later part of the course.

You can have a simple expansion chamber or something like perforates. Something like this; they could be flow here, uniform plug flow and all that. They could neck extensions forming quarter wave resonators at the inlet outlet. They could be a lining here; they could be lining.

This shape could be more complicated rather in a circle and you could have multiple such things. You can have more resonator cavities and you can make the structure as complicated as you want. We will of course, begin with very simple systems and you can have temperature gradients.

The point is it is all the study of how to manipulate wave propagation within a ducted system to reflect the acoustic power back; or to absorb it. Those form the those constitute the principles of muffler acoustics and we are going to discuss all that in due course, but for now, coming back to the fundamental equation, let me call this equation number (6). So, it is important to define a couple of other impedance parameters.

So, we saw Ux, which is the particle velocity, but now if we consider this thing, this has certain area. Let us say this diameter and Sp for the area of the port inlet outlet port area is the diameter is same; so Sp is the diameter. So, what is the volume velocity? So, if you multiply U(x) with a constant cross section area. What do you get? We get things like volume velocity, so

$$U(x)\ \cdot S_p$$

So, if you basically divide this Z_0if you multiply the denominator of equation number (6) by

We will get something

$$Z_v\ =\ \frac{\tilde{p}(x)}{S_p U\ (x)} = \ \ \frac{\rho_0 C_0}{S_p}\ =\ \rho_0 Y_0$$

So, this is also the characteristic impedance, but this takes into account the cross-section area of the inlet port or the or of a duct in general.

That is the dimension of the duct. Typically, we will

$$where, \qquad Y_0 = \frac{C_0}{S_p}$$

we will name it, name this particular thing in due course, but this is called the characteristic impedance in terms of the volume velocity. So, let me call it Zv volume velocity for the waves that propagates among a certain direction and we have yet another quantity called the mass velocity.

So mass velocity is nothing but

$$V(x) =\ \rho_0 S_p\ U(x)\frac{m}{M^3}\frac{m^3}{s}\ \ = kg\ s^{-1}$$

So, the reason that I am introducing this quantity and as is conventionally used in muffler acoustic; duct acoustics is that typically the mass; mass flow remains constant when you multiply by density.

Density might be at the upstream part, it might be the air is hotter, so it is a lighter thing. So, density is relatively lower compared to the density at the downstream, but what essentially remain constant. So, the volume velocity might fluctuate, but mass velocity is always constant; so people do work in terms of mass velocity.

As a result, what happens is that if you just consider this equation and divide,

$$Z_M = \frac{\tilde{p}(x)}{S_p U\,(x)\rho_0} = \frac{\tilde{p}(x)}{U\,(x)\rho_0} = Y_0 = \frac{C_0}{S_p}$$

So, we are eventually left with Y naught. What we define in the last slide, so Y naught is just like bringing this thing, this particular thing here at in the denominator. So basically, Z mass velocity is given by Y_0, which is nothing but sound speed divide by the cross-section area of the duct.

So, this takes into account the dimensions of the duct of course and it is regardless of the density and considers only sound speed. So obviously, for duct with smaller dimension small diameter, characteristic impedance is more and vice versa for a larger duct. So, a smaller that has more impedance towards or it offers more opposition to the propagation of acoustic waves.

And we will see that the moment we use things like this sudden expansion that is happening here partly or maybe things like this the impedance of this port and this port while they are same, the characteristic impedance of this part is vastly different from this one.

So, this will be one of the reasons why lot of cool power is reflected back into the system and why that innovation is being produced. But it is important to see this part C_0/Sp which is the characteristic impedance and with this, I think we will conclude our discussion for this week's lecture.

In the next week, we will probably discuss about resonance is in the one-dimensional system for a closed end pipe for an open-end pipe; Some simple solutions that we can see when you have piston excitation and one end and open or closed end at the other things. Before we move on to something like including mean flow effects in a duct; So, these two or three topics perhaps are important before we go into other things.

Thanks a lot.