Muffler Acoustics - Application to Automotive Exhaust Noise Control

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Week-12
Lecture - 56 and 57
Analytical Mode-Matching for Extended-Inlet and Outlet Muffler: Setting-up of the Equations

Welcome to the final week of this NPTEL course on Muffler Acoustics.

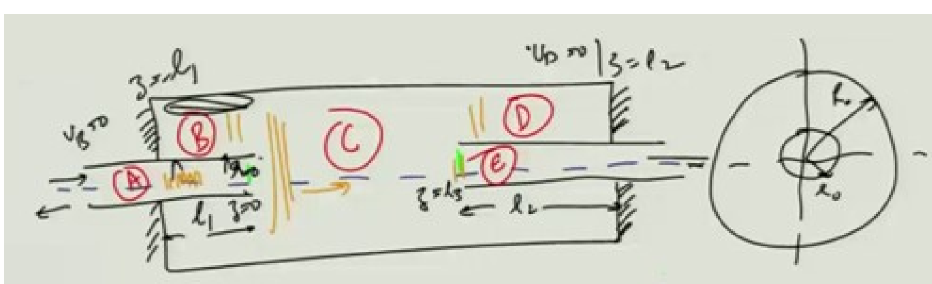
So, we are into week 12 of this course. So, just to keep all of you guys updated what we will be doing in this final week on this course is basically do a couple of things, do couple of important things. First is your, analyze your extended inlet and outlet system using an analytical formulation.

$$l_1 = l_2 + l_3 = L$$

You know all this in the last week, we just had a glimpse of 3D analysis, where we presented your piston driven model for rectangular and circular chamber mufflers, where we did you know side inlet and side outlet; sorry end inlet and end outlet systems and talked about end inlet, side outlet, side inlet, side outlet systems presented some MATLAB simulations.

So, well, what we intend to do really in this week is basically talk about the extended inlet and outlet system which is basically something like this you know things like this.

Extended-Inlet and Extended-Outlet



So, we will consider only a circular chamber muffler and what we do let us consider this region A, B, C, D and E and we will follow all the conventions that were that are presented in Selamet's paper which I am going to present to you shortly along with this derivation.

So, this is an extended inlet and outlet system as we have been you know reading a lot about this thing, since week I guess 4 onwards. So, there is no perforated pipe. So, this kind of a thing is not there ok; this kind of a thing is not there, it is just extended inlet and outlet system.

So, all this while you know in the last ever since we presented the system in week 4, we have considering only planar wave propagation and you know to accommodate for the high order mode effects, you know we use to incorporate some end correction things. Basically, you know having the notions that the geometric length, let us say the length is l_1 here ok, the physical length, the actual length and this is your *l* 2, the physical length ok. These end plates are rigid. So, this annular plate is rigid ok, that is important.

So, the point is that we were considering the physical length *l* 1, *l* 2 and to physical length these physical length, we were adding something called a end correction. So, the actual length for the acoustic length or the effective length that was slightly larger than the physical or geometric lengths by certain factor *l*.

Let us,

$$l_{1a} = l_1 + \delta$$

$$l_{2a} = l_2 + \delta$$

and fortunately, the end correction that is delta is the same even at the outlet. So, this was again delta.

$$L - l_1 - l_2 \; - (2\delta)$$

The idea is that this small you know to account for the higher order modes effect, this delta term is added which basically increases the length or the increase the length and hence, it is called the acoustic length.

And as a result, the peaks that occur in the transmission loss spectrum, they would occur slightly before then just by considering the length physical length *l* 1. So, you know in a very clever in a very subtle manner, we used to consider the higher order mode effect that is generated at the extended inlet and outlet systems. So, all these things are happening.

Now, however, if you want to now this the question is that how do you determine these deltas at the inlet and outlet? So, one can do a proper finite element full three-dimensional finite element simulation, something that we have not covered in this course so far; possibly in the next advanced course on muffler acoustics that I might offer.

All these topics numerical treatment of these mufflers might be considered, will be considered. Along with some experimental techniques, we will talk about that in the very last lecture of this course, lecture 5 of week 12, final lecture, we will talk more about that. But for now, you know what I just tell you these deltas were computed the once the expressions presented in the paper by Chaitanya and Munjal appearing in Applied Acoustics, that was done using three dimensional finite element analysis.

But there are other techniques also like analytical techniques or you know things like mode matching techniques, numerical mode matching techniques which is also I have kind of reserved that topic exclusively in the next course. But basically, your analytical mode matching technique is something that will be used to determine the overall transmission loss spectrum for such a system.

And in the process, we will find out we can evaluate these deltas in a from the analytical transmission loss spectrum by matching the peaks. And then, one can obtain you know the exact the corrected spectrum. What we will do is that we will keep the scope of the next 2 or 3 lectures of this week very sort of limited to analyze the simple system and not go too much into detail.

So, in this particular week, sorry in this particular lecture, lecture 1 here, what we will do is that we will set up the system of equations required to do the mode matching and this might spill over this and this all these things might spill over to the lecture the lecture 2 as well.

So, let us see how we proceed. So, these regions are symbolically denoted by A, B alphabets; you know A, B, C, D, E and all that they are circular in shape; the ports are circular; the chamber is circular and everything is considered rigid. So, there is no structural compliance as usual. Now, how about we start writing down the equations, the modal summation solution in each of these regions A, B, C, D and E in terms of the unknown modal coefficients. So, let us begin with the port A; you know let us begin with the port A.

So, what we get essentially is you know as usual we have Helmholtz equation in cylindrical polar coordinate system.

$$(\nabla^2 + \mathrm{k}_0^2)\,\tilde{p} = 0 \quad p \;=\; \sum_{m=0,1}^{\infty}\;\sum_{n=0,2}^{\infty}\; I\left(\frac{\alpha_{mnr}}{k_0}\right)$$

We will not employ rigid wall boundary conditions here or here. I mean in the sense that we will express this in terms of forward and backward travelling waves.

when you will have your and note another thing that I want to tell you here which I sort of forgot. Sorry for that is that you know this such a for such a configuration, only the axisymmetric mode will propagate.

So, what it means that only you know m = 0 mode that is Bessel functions or for in this annular region Neumann functions, so only with such a thing will propagate ok.

$$J_0\left(\frac{\propto_{0n}}{k_0}r\right) \quad n = 0,1,2,3,4,5,6$$

So, basically cos m theta sin m theta azimuthal mode or circumferential modes will not be there because such modes are not sort of excited ok. So, we will get this sort of a thing ok and then now, what we can do is that write down the solution.

$$(\nabla^2 + \mathrm{k}_0^2)\,\tilde{p} = 0$$

So, let me just write down the modal solution for region A which is

$$\tilde{p}_A\,(r,\;\; z) = \sum_{n=0}^{\infty}\;\left(\vec{A}_n^{+}\,e^{-jk_nz} + A_n^{-}e^{-jk_nz}\right)\psi_n(r)$$

So, this will become n is equal to 0 to infinity and A n plus; so, A n plus means the waves that goes in the along the positive propagates along the positive direction and A minus, you will see will be the wave that propagates in the negative z direction and we will follow that convention for all the region.

So, k n, I will tell you what k n is and A n again, this is the negative propagating wave coefficient and this is your ψ_n (r) ok. So, as usual $j = \sqrt{-1}$. So, keep that aside and A_n^+ and A_n^- are the modal amplitudes ok and for the region A, ψ n is just the Bessel function.

So, let us write it down. Let me just make some small correction here. Let me call this psi n A; n is the nth modal this thing and psi A means for the Ath region A here the port A here; and this is

$$\psi_n^A = J_0\left(\frac{\alpha_{mn}}{R_0} r\right)$$

So, I will just drop m. So, initially it was m n.

So, mn by let us say the radius of this thing is let us have equal radiuses let us say A and A of these ducts as is usually the case and the diameter or the radius of this is you know that the radius of this can be you can think of this as R_0 and actually this can be thought of as small r_0, small r_0 ok; a small change. Now, the thing is this is R_0, r_0 and well.

Now, the thing is that we will put directly m is equal to 0 and n. So, we will just drop m, we just call it alpha mn in with the understanding that alpha mn for alpha 0 is this thing is 0n. So, n can be 1, we can have 01, actually 00 that is the planar wave which is 0, alpha 01, 02, 03.

$\alpha_{m,n}$	***m = 0***	***m = 1***	***m = 2***	***m = 3***	***m = 4***	***m = 5***	***m = 6***	***m = 7***	***m = 8***
n = 0	0	1.8412	3.0542	4.2012	5.3176	6.4156	7.5013	8.5778	9.6474
n = 1	3.8317	5.3314	6.7061	8.0152	9.2824	10.5199	11.7349	12.9324	14.1155
n = 2	7.0156	8.5363	9.9695	11.3459	12.6819	13.9872	15.2682	16.5294	17.7740
n = 3	10.1735	11.7060	13.1704	14.5858	15.9641	17.3128	18.6374	19.9419	21.2291
n = 4	13.3237	14.8636	16.3475	17.7887	19.1960	20.5755	21.9317	23.2681	24.5872
n = 5	16.4706	18.0155	19.5129	20.9725	22.4010	23.8036	25.1839	26.5450	27.8893
n =6	19.6159	21.1644	22.6716	24.1449	25.5898	27.0103	28.4098	29.7907	31.1553
n =7	22.7601	24.3113	25.8260	27.3101	28.7678	30.2028	31.6179	33.0152	34.3966
n =8	25.9037	27.4571	28.9777	30.4703	31.9385	33.3854	34.8134	36.2244	37.6201

Table 26. The Roots of the derivative of Bessel function of the first kind: non-dimensional cut-on frequency in a circular cylindrical waveguide $D_2 = D_1 = D_0$ ($e = 0$, $\xi \to \infty$)

So, based on the based on the monograph that I have shown, let me just show to you my the springer monograph the table. So, you know in this thing as we see from this particular the highlighted; I am not able to highlight, but basically where I am pointing my mouse at, the values shown in red these are all axisymmetric modes.

Other one, these are all the ones shown here these are the resonance frequency or the circumferential mode and from here onwards all these modes, you know for m not equal to 0, m = 1 onwards and n = 1 onwards, you have your cross modes.

For m = 1, 2, 3, 4 and n being 0, these are circumferential modes; but we will be considering only m = 0, n = 0 that is the planar wave mode for which j is 0 j 0 of 0 that is argument in the argument of the Bessel function of order 0 is 0, then we just have unity.

So, basically this is the plane wave mode and it comes out nicely mathematically. This is the first radial mode, second radial mode, third radial mode and so on. So, these are the radial modes and that is what we will be dealing with. So, I straight away write this as alpha n. So, instead of all these, we will just replace this by

$$\alpha_{mn} \quad for\, \alpha_0 n \quad \left[\begin{array}{l} \alpha_0 n \quad \alpha_{00} = 0 \\ \quad \alpha_{01} \\ \quad \alpha_{02} \\ \quad \alpha_{03} \quad \alpha_n \\ n = 0,1,2,3,4 \;\; \infty \end{array} \right.$$

And till ∞, we will consider only the first few modes.

So, $\frac{\alpha_n}{R_0} r$, this is your modal summation ok. This is what we get. Now, $J_0\left(\frac{\alpha_{mn}}{R_0} r\right)$ ok and now, what we need to do is that we have figured out or expanded the solution for this region and kn is something that we need to worry about. So, k n, there is no mean flow.

Mean flow, of course, is absent here; I just forgot to mention here, **mean flow is absent or we just ignore stationary medium is considered**; stationary medium and now you have your kn. So, what is

$$k_n = \sqrt{k_0^2 - \left(\frac{\alpha_n}{l_0}\right)^2}$$

So, clearly when now this is something important, when k under what conditions do higher order modes propagate? Now, when n = 0, like I was mentioning you know alpha 0 is 0. So, kn always propagates and this is the argument is 1 here.

$$\psi_0^A = J_0(0) = 1$$

So, it is a planar wave mode and $k_n = k_0$. It is very simple to see because $\alpha_0 = 0$, that that we have seen from the monograph.

Now, when alpha when n is not 0; n is 1, 2, 3 whatever it is, then the real business starts and we are into your proper three-dimensional considerations. At least the first few modes should be considered because it is not possible to consider you know infinite set of the such modes, we will consider only the finite sets that set of mode say 5 modes in the inlet pipe and maybe 10 modes in the chamber and so on or in the annular cavity.

Let us have some representation n here and when

$$k_0 > \frac{\alpha_n}{r_0}$$

$$k_0 r_0 > \alpha_n$$

$$k_0 r_0 < \alpha_n$$

that is excitation frequency is greater than this or k naught r once you know once you know f naught, we can know k naught. So, if the excitation non dimensional excitation frequency is greater than the non-dimensional resonance frequency of that particular mode say the first radial mode, then, the first radial mode will propagate.

If not, then the if you have such a thing you know, then the mode will not propagate. If this is less than n, then it is evanescent; if it is greater than equal to this thing, then these modes will propagate, it is just about to propagate when this is the equality occurs. So, basically what happens is that if you have greater than k naught is greater than this thing, then there is no problem.

It is a this is a complex exponential what we have is a real number ok. There is no problem in that. But now, when k naught is such that this is less than alpha n, then the trick happens that you know root over k naught r square, this becomes a purely imaginary number.

$$k_n = \sqrt{-1}\ \sqrt{\left(\frac{\alpha_n}{k_1}\right)^2 - k_0^2}\ = j\sqrt{(-)}$$

So, this is now this is greater ok. So, here you will have j times the entire argument which is of some positive number. So, why am I telling you this? Because j times, so here you will have a purely real part of your purely real part which is actually exponentially decaying or exponentially growing.

So, depending upon the convention that we you know follow, if we consider this as you know root over minus 1 as you know this is j; you know if we consider this as j square root, we can also consider - j depending upon what we want. So, however, if we consider this as j, then j times j.

So, you know the. So, basically A n, coefficient associated with A n will become

$$A_n^+ \, e^{-j} j \sqrt{\left(\frac{\alpha_n}{k_1}\right)^2 - k_0^2} \; z = A_n^+ \, e^{(-)z}$$

So, something positive you know. So, this is something positive and this of course, is multiplied by z right.

So, this will grow; this will grow in if over space and hence, this will this can cause numerical problems. On the other hand, if you have things like this is plus this is minus as I was mentioning. If you have things like well,

$$A_n^+ \, e^{-j} j\sqrt{(-)} \; z = A_n^- \, e^{(-)z}$$

So, this is decaying exponentially alright; A_n^- this thing.

So, there is no problem. There should not be any numerical problems with this term, there should be, there would be a problem. So, a simple a clever take which is trick which is not quite given in books, but I am trying to present that in a very simple manner is that let us have a change of variable.

We will when we set up the equations, we will have that. So, such a thing you know wherever such a thing would occur, we will and not at the inlet pipe when you have z = l. At z = l, in the other section because you know you see we are fixing the coordinate system z = 0 here for this C part and z is equal to you know let us let me call this z = l_3.

Now, l_1 physical length; $l_1 + l_2 + l_3 = L$. This thing will not change even when you put this. So, the effective length will become $L - l_1 - l_2 - (2\delta)$; the clear length or the length between from tip to tip. So, the effective length effective clear length will be slightly different; but that is a later part of the story.

Main thing is that we are fixing z = 0 for the region here and z = l_3 for this region and then other things will follow. But what I am saying is that at some stage, we will have to make a change of variable. When z = 0, then there is of course, no problem this is unity and things will be relatively simple; but for the chamber C, we might have to do something different. So, we will come to it.

So, this is this was the story for the duct A ok and for duct and annular region, now similar actually similar thing would be there for duct E, the solution for you know the port E, let me deal with that thing in a I mean since I am dealing with the inlet and outlet ports; how about I just write down the solution for that as well.

Now, $\tilde{p}_E\ (r, z) = \sum_{n=0}^{\infty}\left(E^{+} e^{-jk_n z} + E^{-} e^{-jk_n z}\right) \psi_n^E\ (r)$

A are pretty much the same. So, we get this sort of a thing and we have this guy ok. Now, the same things would

$$\psi_n^E\ (l) = J_n\left(\frac{\alpha_n}{r_0} r\right)$$

And actually I guess, I made a small mistake here. This is sorry; this is r_0 here. I am sorry and yeah, everything else seems to be alright.

$$k_n = \sqrt{k_0^2\ \left(\frac{\alpha_n}{r_0}\right)^2}$$

And this is r_0. We get this sort of a thing and k n is the same; k naught square minus alpha n by r naught square, we get the same thing. Now, we have this. Now, how about the chamber region? Let us worry about the chamber region now that is basically C region and here also, you know like I was mentioning in the last lecture of week 11 that you know annular region is included for A, E as well as C. So, the Neumann function will not be there; there will only be Bessel functions.

So, under such a situation, the analytical solution for the chamber region that is C is given by

$$\tilde{p}_C\,(r,z) = \sum_{n=0,1,2}^{\infty} \left(C_n^+ e^{-jk_n^2 z} + C_n^- e^{-jk^2{}_n z}\right) \psi_n^E\,(r)$$

Now,

$$\psi_n^C\,(r) = J_0\left(\frac{\alpha_n}{R_0} r\right)$$

$$k_n^C = \sqrt{k_0^2 - \left(\frac{\alpha_n}{l_0}\right)^2}$$

So, well, instead notice the difference. Here actually this can you can think of this as k_n E and k_n A; k_n E and k_n A are pretty much the same. They are the same because the radius of the inlet and outlet pipe are same; but in this thing, this is R_0. So, R_0 is different ok

So, typically, we consider in for automotive muffler expansion chambers of expansion, this ratio can be something like or the radius ratio expansion ratio can be really varies from 3 to something like 6 typically and so, the area ratio can be something like 9 to 36 or something like that.

$$\left(\frac{k_0}{r_0}\right) = \begin{matrix} 3\ to\ 6 \\ \\ 9\ to\ 36 \end{matrix}$$

But the main trick now is what do you do for this thing. Now, there is one thing that I want to make it very clear that this analytical business of analytically finding or analyzing these things will work only for such a configuration, when you have a concentric pipe. You know you see you have a you have a concentric pipe. So, because it is very simple to apply this boundary condition, at r is equal to for the annular region B, at r is equal to r_0, we are assuming mind you the thickness of the wall is 0 that is a it is a infinitesimal wall, there is no wall thickness for this one ok.

But in actuality, in actual application, you will have certain thickness. So, those effects can be considered sort of you know in advanced courses or you know in as more advanced topics. So right now, just for mathematical simplicity, we are ignoring tw although some authors like Chaitanya and Munjal, they have considered; they have employed finite element formulation taking into account the finite thickness of the wall to get the end correction ok. So, that they have done.

So, what we will do is that we will have to worry about the analytical solution for the region B and D. Analytical cavities, they will be the same; the modal summation at least based on the length l 1, l 2; there the net pressure field will be different. But again, like I said this analytical thing is only available only sort of suitable or amenable to analytical solution, if the ports are concentric.

Because at r is equal to r_0, the you have the condition $\frac{\partial p}{\partial r} = 0$ and this is also true for r is equal to capital R_0; then, the radial velocity is also 0. So, the idea is that we can find out a set of orthogonal function for such a thing you know this will. So, this particular form will remain same. So, for region B and D;

For region say B, we,

$$\tilde{p}_B\,(r,z) = \sum_{n=0,1,2}^{\infty} \left(B_n^{+} e^{-jk_n^B z} + B_n^{-} e^{-jk^B{}_n z}\right) \psi_n^B\,(r)$$

$$k_n^B \sqrt{k_0^2 - \left(\frac{B_n}{k_0}\right)^2}, \qquad where$$

We have only axisymmetric mode in the annular cavity as well and your psi n B function, let me write down this guy first. So, this is nothing but your Bessel function, this thing where,

$$\psi_n^\beta = J_0\left(\frac{\beta_n r}{k_0}\right) - \frac{J_1\,(\beta_n)}{N_1\,(\beta_0)}\, N_0\left(\frac{\beta_n l}{k_0}\right)$$

and here, you will have

$$J_1\left(\beta_0 \frac{l_0}{k_0}\right) - \frac{J_1\,(\beta_n)}{N_1\,(\beta_n)}\, N_1\left(\frac{\beta_n l_0}{k_0}\right) = 0$$

Now, these are all obtained by some identities for Bessel function, and here you will have the Neumann function for the first time because we have the origin is not sort of included here. So, this is this and Neumann function β_n / R_0 ok. We will get this sort of a thing. And what about the other things? You know you have this. So, beta you know is obtained by solving this set of equation and in my previous papers, I have listed down for typical values you know this is β_n r_0 and this is R_0 here and here, we have J_1 β_n / N_1 and beta n into N_1. So, this equation has to be solved numerically only ok.

For a circular cylindrical chamber with a rigid concentric pass tube,

$$Z_{EkEi} = \frac{jk_0C_0}{S_{Ek}S_{Ei}} \left[\begin{array}{c} \sum_{l=0,1,2\cdots}^{\infty} \sum_{m=0,1,2\cdots}^{\infty} \sum_{n=0,1,2\cdots}^{\infty} \\ \left(\iint_{S_{Ek}} \cos(m\theta) \left(J_m \left(\alpha_{mn} \frac{r}{R_{outer}} \right) \right. \right. \\ \left. \left. -\gamma_{mn\beta}^{1} N_m \left(\alpha_{mn} \frac{r}{R_{outer}} \right) \right) r dr d\theta \right) \\ \times \left(\iint_{S_{Ei}} \cos(m\theta) \left(J_m \left(\alpha_{mn} \frac{r}{R_{outer}} \right) \right. \right. \\ \left. \left. -\gamma_{mn\beta}^{1} N_m \left(\alpha_{mn} \frac{r}{R_{outer}} \right) \right) r dr d\theta \right) \\ \times \overline{\left\{ \left(\frac{l\pi}{L} \right)^2 + \left(\frac{\alpha_{mn}}{R_{outer}} \right)^2 - k_0^2 \right\} N_{l,m,n}} \end{array} \right] \qquad (19)$$

so this one this is this was how the Green's function look like, but we are not really considering the Green's function. The form is the same you know J, N_m; is again here I am calling it for n even for nonzero I mean nonzero values N m that is 1 2 3 4, but for the x concentric extended inlet and outlet m is only 0.

$$-\gamma_{mn\beta}^{1} = \frac{J_{m-1}(\alpha_{mn\beta}) - J_{m+1}(\alpha_{mn\beta})}{N_{m-1}(\alpha_{mn\beta}) - N_{m+1}(\alpha_{mn\beta})} \qquad (20)$$

And γ^1 is basically this thing. These are obtained you know these values, these β values were obtained from solving the Eigen equation or the equation subject to rigid wall condition.

And for different values of beta, I have sort of tabulated some of the higher order mode. So, let me just go quickly to the table and show you know these things to you so well.

Table 3. Non-dimensional cut-on frequencies (rigid wall modes) of a circular cylindrical chamber without pass tube ($\beta = 0$) and that with a concentric rigid pass tube with $\beta = \{02.05\}$

$\alpha_{m,n}$		*m = 0*	*m = 1*	*m = 2*	*m = 3*	*m = 4*
n = 0	$\beta = 0$	0	1.8412	3.0542	4.2012	5.3176
	$\beta = 0.2$	0	1.7051	3.0347	4.1991	5.3173
	$\beta = 0.5$	0	1.3547	2.6812	3.9578	5.1752
n = 1	$\beta = 0$	3.8317	5.3314	6.7061	8.0152	9.2824
	$\beta = 0.2$	4.2358	4.9609	6.4950	7.9638	9.2734
	$\beta = 0.5$	6.3932	6.5649	7.0626	7.8401	8.8364
n = 2	$\beta = 0$	7.0156	8.5363	9.9695	11.3459	12.6819
	$\beta = 0.2$	8.0551	8.4331	9.5495	11.1060	12.6102
	$\beta = 0.5$	12.6247	12.7064	13.1704	13.3476	13.8923
n = 3	$\beta = 0$	10.1735	11.7060	13.1704	14.5858	15.9641
	$\beta = 0.2$	11.9266	12.1651	12.8997	14.1493	15.7127
	$\beta = 0.5$	18.8889	18.9427	19.1032	19.3684	19.7354
n = 4	$\beta = 0$	13.3237	14.8636	16.3475	17.7887	19.1960
	$\beta = 0.2$	15.8210	15.9932	16.5193	17.4324	18.7548
	$\beta = 0.5$	25.1624	25.2035	25.3224	25.5214	25.7978

Non-dimensional cut on frequencies for a circular chamber without a pass tube and that with a pass tube. So, this is essentially a you can instead of a pass tube, it is a basically concentric annular region and $\beta = 0.2$ plane wave mode always exist. But let us focus only on the column here, these things are not quite relevant right now.

What it means basically is that for the higher order modes, the first radial mode for a $\beta = 0.2$, 0.5. They get cut on at a much higher frequency compared to the circular cylindrical chamber. I mean for 0.2 is this is small difference.

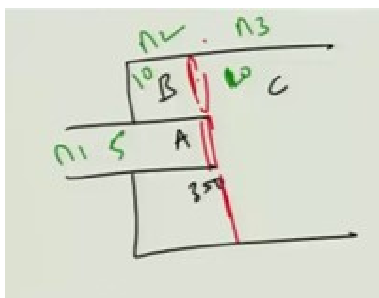
As we keep increasing beta there, the ratio of the inner the that is basically r naught to small r_0 to R_0 value, if this ratio keeps increasing, then we have in the radial modes, the excitation frequency keeps getting shifted in the higher frequency range as its seen by these values ok.

So, such values have been prepared also for me and we will what we will do is that right now, we will just consider we will just consider this guy and then, we will use that and you know and the modal solution for the region D will also look exactly like region B. So, instead of B is you will have

$$\left\{D_n^+ e^{-jk_n^D z} + D_n^- e^{-jk^D{}_n z}\right\} \psi_n^D\,(r)$$

This will remain the same; you know then, you can consider find set of modes. So, once we got the modal solution, what to do now? How do we proceed ahead? ok. So, for proceeding ahead, we need to have the matching condition. So, right now for the first time after all this background, I am presenting to you some matching condition.

So, let me draw this guy again. You know let us focus only on the extended inlet. This is the region A, B and C.



$$\tilde{p}_A|_{z=0} = \tilde{p}_C|_{z=0}$$

$$\tilde{p}_{\boldsymbol{B}}|_{z=0} = \tilde{p}_C|_{z=0}$$

and because the field continuity must always be satisfied. Over this region here, the pressure is the same and for this one at each and every point, the pressure in the annular port just at z is equal to 0; the same as z is equal to C ok.

So, we get that and what about the velocity condition? Velocity,

$$\tilde{U}_A\big|_{z=0} = \tilde{U}_C\big|_{z=0}$$

$$\tilde{U}_B\big|_{z=0} = \tilde{U}_C\big|_{z=0}$$

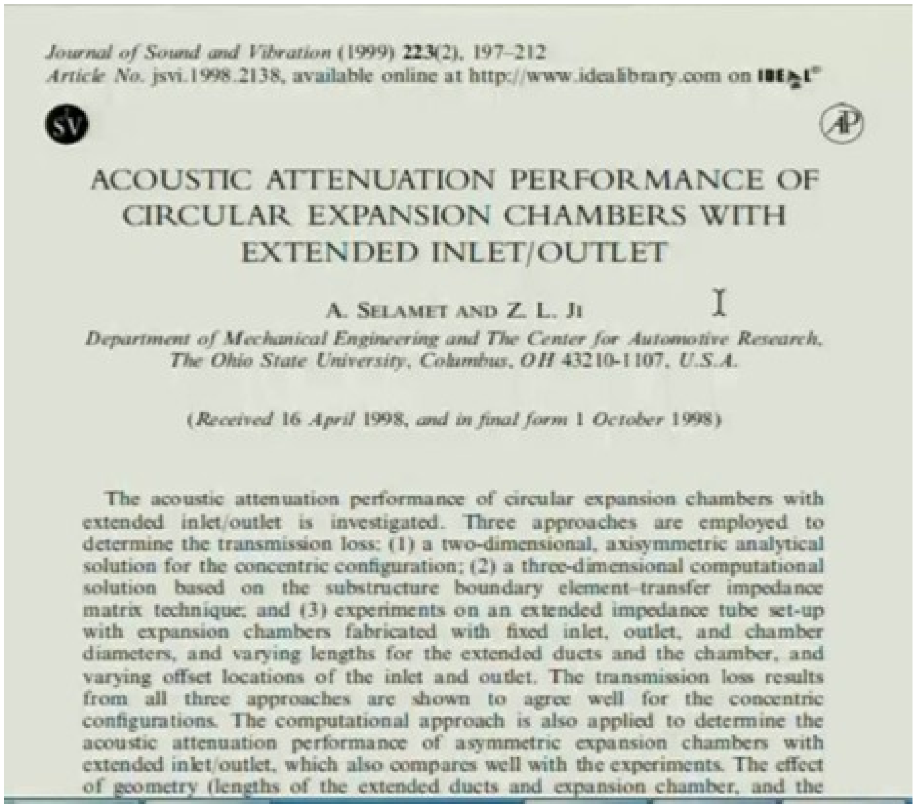
So, we basically get this sort of a thing. So, annular velocity here it is the same as the one in the chamber and in the annular cavity that is also the same.

Now, basically what you know this is the paper that I am following is the one by Selamet and his co-worker, long time back published about almost 20 years back or so, I guess it was published showing a paper later in 1999. So, more than 20 years, but a Abomb and others, what they have done is that they have used separate equations.

Selamet and his co-worker, Selamet and Ji, they have basically combined these two equation in a certain manner which we will soon see. So, what we need to do that just using the modal summation solution that we have derived or explained so far, we will you know multiply, we will basically do some sort of a mode matching.

Basically, use again exploit again and again you know three-dimensional analytical things or you know whenever you have a mode matching thing or a Piston driven model or a Green's function model, wherever basically one has to incorporate the three-dimensional effects, in such a thing without resorting to full 3D analysis, you know without completely discretizing the 3D continuum into finite elements rather just discretize the to work on the two-dimensional problems.

You know we all repeatedly have to make use of orthogonality principle. Basically, rely on modal summation multiply both sides by modal appropriate modal functions, integrate over a certain domain and if they are orthogonal, it is fine. If they are not orthogonal, we still have to well deal with them. And then, what we can do is that, once we have this system, now let us go to the solution here. So, what we do? We need to set up the system of equations ok.



Journal of Sound and Vibration (1999) 223(2), 197–212
Article No. jsvi.1998.2138, available online at http://www.idealibrary.com on IDEAL

ACOUSTIC ATTENUATION PERFORMANCE OF CIRCULAR EXPANSION CHAMBERS WITH EXTENDED INLET/OUTLET

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(Received 16 April 1998, and in final form 1 October 1998)

The acoustic attenuation performance of circular expansion chambers with extended inlet/outlet is investigated. Three approaches are employed to determine the transmission loss: (1) a two-dimensional, axisymmetric analytical solution for the concentric configuration; (2) a three-dimensional computational solution based on the substructure boundary element–transfer impedance matrix technique; and (3) experiments on an extended impedance tube set-up with expansion chambers fabricated with fixed inlet, outlet, and chamber diameters, and varying lengths for the extended ducts and the chamber, and varying offset locations of the inlet and outlet. The transmission loss results from all three approaches are shown to agree well for the concentric configurations. The computational approach is also applied to determine the acoustic attenuation performance of asymmetric expansion chambers with extended inlet/outlet, which also compares well with the experiments. The effect of geometry (lengths of the extended ducts and expansion chamber, and the

So, this is the paper that I was talking about Acoustic Attenuation Performance of Circular Chamber Mufflers with Extended Inlet and Outlet by Selamet and Ji; quite widely cited paper published in JSV, Journal of Sound and Vibration more than 20 years back.

While the main emphasis and contribution of the work is on the multidimensional wave propagation and attenuation, the limiting case of the planner wave behavior is also simply superimposed to illustrate its application bunds as applied to the present configuration

So, there is a formulation that I have sort of presented so far is basically following the work by Selamet only.

So, these are the functions that we just saw just now. Now, other thing that is very important of course, is the velocity condition. You know if you go back to the presentation to this thing, you know you see that the matching condition really means there is a matching condition. Yeah. So, we have velocity also that is to be matched.

Velocity fields have to be continuous; the pressure field also has to be continuous. So, but so far in the modal solution, we have derived only the pressure modal summation expansion of the pressure field. But we need to do, we need to get that thing for the modal summation for the acoustic particle velocity as well.

So, you know what do we do? We use the Euler equation you know $j\rho\omega U = -\frac{\partial P}{\partial z}$. Where, U is the velocity along the z-direction is $= -\frac{\partial P}{\partial z}$. Now, we once you substitute you know and this time harmonicity, you get $j\omega$ and U is just U, I mean you just solve for the spatial $j\rho\omega U = -\frac{\partial P}{\partial z}$.

So, once you know P because the modal summation solution you already know. The trick here is that you need to carefully be careful with the signs. So, moment you do that you know $j\rho\omega$can come down now and you know j once you take you know - jk you know on the modal summation you have

$$U(r,z) -= \frac{1}{\rho\omega}\sum_{n=0}^{\infty} k_n \left(A_n^+ \, e^{-jk_n z} - A_n^- \, e^{jk_n z}\right) \psi_n(r) \qquad (6)$$

So,

$$P(r,z) -= \sum_{n=0}^{\infty} \left(A_n^+ \, e^{-jk_n z} + A_n^- \, e^{jk_n z}\right) \psi_n(r)$$

take common and here that is why you will get a minus sign and minus minus will cancel here and j j will also be cancelling. So, you know you will get $\frac{1}{\rho\omega}$ and k_n. So, k_n if it was just a plane wave mode, it would have been very simple nice clean analytical solution k was ω by c or C_0.

So, ω would have cancelled and you would just get $\frac{1}{\rho_0 C_0}$ that is your characteristic impedance. By now, the thing is these things should be something like a bread and butter for you. So, but however, in a higher order modes, you have you know you will something like this. So, k_n is not obviously, it is not k_n is not k naught in the general case. It is you know root over k_0 square minus your α_n by r square root over the entire thing. So, k_n. So, basically k_n you cannot take out of the summation sign, it must remain like this. So, let me just zoom this thing for you so that you guys can have a look at it a bit more clearly.

So, this is what it is you know. So, this is the one that I am talking about ok. So, these you know this will be valid for any region; only psi function will change and A_n^+ and A_n^- that will also change and the rigid wall boundary condition, obviously, this is one thing that I sort of forgot to mention apologies for that. So, basically you know if you go to your if you go to your region, the well the muffler region, at z if you fix the condition on here at z is equal to at this end plate.

So, at z = - l_1 velocity U_B = 0 because the rigid plates because the end plates are considered to be rigid, they are not they are not yielding or they basically they do not deform. So, rigid wall conditions will be applied and there will be. So, we have a standing wave pattern set here and similarly, at U_D = 0 at z = l_2. So, you know let us get back to our paper and see what these guys have done.

$$U_B|_{z_1=-l_1} = 0 \quad (on\ S_B) \qquad (7)$$

we have this. So, when you do that, what we do? You know you essentially, we have to look sorry for going back and forth in this thing, but that is the nature of the things. So, we are expressing

$$B_n^+ = B_n^- \ e^{-2jk_{B,n}l_1} \qquad (8)$$

So, you know B_n^- is the wave that goes in the negative x z direction; so, we and similarly, D_n^+.

$$D_n^- = D_n^+ \ e^{-2jk_{D,n}l_2}$$

So, what about the other one? You know D_n^- the wave that goes in this direction is propagated is written in terms of D_n^+. So, these are some of the clever tricks that you need to do.

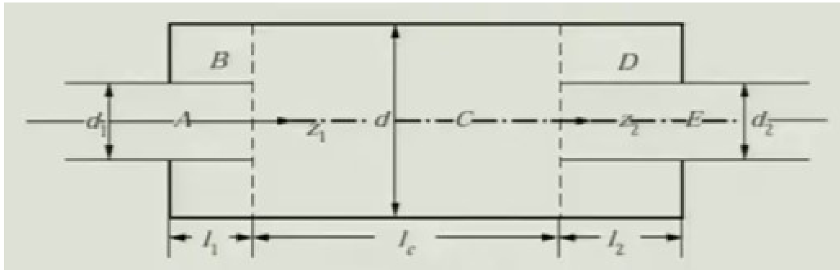

Figure 1. Circular expansion chamber with extended inlet and outlet ducts.

Now, basically let us get back to our matching conditions. This is obviously, another this is one boundary condition; at the other end, we just talked about in the annular cavity that is at z is equal to *l* 2 here that this plate is also rigid. So, we get this condition. Now, comes the matching condition which I have just written down in the presentation.

So, for you know we put the modal summation expansion for, how do we how do?

$$P_A|_{z_1=0} = P_C|_{z_1=0} \qquad U_A|_{z_1=0} = U_C|_{z_1=0} \qquad (on\ S_A) \qquad (9,10)$$

$$P_{\boldsymbol{B}}|_{z_1=0} = P_C|_{z_1=0} \quad U_B|_{z_1=0} = U_C|_{z_1=0} \qquad (on\ S_B) \qquad (11,12)$$

We go about data? What do you do further to process the equations? Well, we expand P_A and P_C in the modal summation solutions that we just got and multiply as Selamet and Ji mentioned for the pressure continuity condition, multiply both the sides of the equation (9).

That is, equation (9) is basically nothing but the first one and equation (10) is this (11, 12). So, equation (9) is multiplied by the modal function psi A, s and that is for the duct and this is multiplied you know again you know mode orthogonality will apply. So, you multiply this by a generic function, if this is n, well if this is let me just get to this one.

If this is n, you multiply this by n; here you multiply by e n N_1 you know you multiply this by N_1 and then, you integrate it. So, only that mode will survive when n = N_1 ok. So, that is why you just writing this as only one term

$$(A_S^+ + A_S^-)\langle \Psi_{A,S}\ \Psi_{A,S}\rangle_A = \sum_{n=0}^{\infty}(C_n^+ + C_n^-)\left(\Psi_{C,n}\ \Psi_{A,S}\right)_A$$

$$B_S^-\left(e^{-2jk_{B,S}l_1}\right)\langle\Psi_{\mathrm{B,S}}\ \Psi_{\mathrm{B,S}}\rangle_{\ b}=\ \sum_{n=0}^{\infty}(C_n^+ +\ C_n^-)\left(\Psi_{\mathrm{C,n}}\ \Psi_{\mathrm{B,S}}\right)_{\ B}$$

So, or n is equal to N_1. So, you can call that s. So, that s mode will survive and this you know so and we you. So, for multiplying over that. So, this we need to integrate over the port area that is this thing.

So, you multiply with r. So, dS is nothing but r dr d theta and this is a $\Psi_{\mathrm{A,S}}$ function and using mode orthogonal mod orthogonality only this function will survive and you have only this term and then, the mode function $\Psi_{\mathrm{A,S}}$ and this is integrated this one with the mode function in this one that is the Bessel function with the in the larger radius in the numerator; I am sorry in the denominator that is basically your this guy R_0.

So, psi is this thing. So, essentially what it does is basically you know you when you talk about the $(\psi)[\psi]$ thing you what we do is basically integrate say well let us

$$(\psi)[\psi]\quad \iint_{ds} J_0\left(\frac{\alpha_n}{R_0}r\right)J_0\left(\frac{\alpha_s}{r_0}r\right)\ r\ dl\ d\theta$$

Basically, we multiply this by this thing and so well and then, basically only this mode will survive here when like I was mentioning and here, we have summation over entire thing.

Similarly, when you multiply the other pressure continuity equality condition with the function $\Psi_{\mathrm{B,S}}$. So, again you know here you have your only one modal coefficient as we have explained just a while back and only that particular mode will survive in the annular region, this one because of orthogonality.

Because you know we are basically considering orthogonality of such function over the domain of the port or the annular cavity. When you multiply, when you integrate, when you multiply with the corresponding function what I mean is that $\Psi_{\mathrm{A,}}$ when you multiply by Ψ_{A} and integrate over the port area.

So, here things will be all right; here you will have a lot of nonzero modes also, that is why you have the summation thing and similarly, here this is orthogonal over the annular region. So, you have your $\Psi_{\mathbf{A},\mathrm{S}}$ you know this kind of a thing and again you have a summation.

So, there is no problem here I believe until this part because again you know k B s this can be again like I said this can be written as j times root over whatever we are getting; j times whatever we are getting and with the convention that is followed. What is the convention?

So, j is your they have followed the convention that j is root over minus 1 imaginary unit. So, when j square j square is minus 1 minus 1 is plus. So, they probably would have done something to suppress the numerical instability. So, we will work with our own codes here. So, this is what they are getting and these things will be there.

Now, and the other condition is the velocity condition; other set of conditions. So, this one and this one. So, what now here the other trick is to multiply let us say this equation with the modal function modal function for the chamber C and integrated over the port area, similarly multiply the other velocity condition that is this condition with the modal summation solution modal function for this one and integrate over the annular area and then, add them. Now, only when you add them, only this particular term survives and in all the other things, the other set of equations other terms will be there ok.

$$\sum_{n=0}^{\infty} k_{A,n}\,(A_n^+ - A_n^-)\,\langle \Psi_{\mathrm{A,n}}\ \Psi_{\mathrm{C,S}}\rangle_A + \sum_{n=0}^{\infty} k_{B,n}\,B_n^-\left(e^{-2jk_{B,n}l_1} - 1\right)\langle \Psi_{\mathrm{B,n}}\ \Psi_{\mathrm{C,S}}\rangle_B$$

$$k_{C,S}\,(C_S^+ - C_S^-)\,\langle \Psi_{\mathrm{C,S}}\ \Psi_{\mathrm{C,S}}\rangle_C$$

So, this will be there. Now, integrals designated into this thing are deferred to the appendix of this paper.

Similar conditions will apply will be set for the outlet that is where is the outlet? Outlet is this part you know so pressure here in the chamber is equal to the pressure here and pressure here is equal to the pressure right in the chamber D. So, that is how you know they

$$P_C|_{z_1=l_c} = P_E|_{z_2=0}, \qquad U_C|_{z_1=l_c} = U_E|_{z_2=0} \qquad (on\ S_E) \quad (19,20)$$

$$P_{\boldsymbol{C}}|_{z_1=l_c} = P_D|_{z_2=0}, \qquad U_C|_{z_1=l_c} = U_D|_{z_2=0} \qquad (on\ S_D) \quad (21,22)$$

So, they have translated the coordinate system. So, for the port, this port, the coordinate system here is defined in the new coordinate system z_2 is equal to 0 and z_2 is equal to 0 here for the annular cavity as well ok and z_2 is equal to l_2 at the outlet port.

Now, once we have this sort of a thing, we do the same business for the pressure condition multiply this with the modal summation for the P_E port or P_D port and then, integrate it. Only this term, only a certain term is survived in the port and the annular cavity and the velocity, you multiply by $\Psi_{C,n}$.

In comparison with Abom, the present approach adds the two integral equations for the velocity continuity conditions (equations (10), (12), and (19), (21)) to get one analytical expression, at the expansion and contraction, respectively, thereby reducing the number of equation

And then, go about adding these things and only particular mode will survive in the chamber. Now, there is one thing now you know basically what is happening here is that your we have we want to evaluate the sorry the transmission loss this thing. So, transmission loss performance. So, E plus is a wave that propagates in this direction, E minus is a wave that propagates here in the negative direction where I am where I am trying to point ok.

So, E minus has to be set to 0, all the modal coefficients E n minus, they have to be set to 0 you know that is what an anechoic termination is exposed at the is imposed at the exit of the chamber by setting the reflected wave coefficients E_n to 0. So, this like this the number of unknowns reduces.

$$\sum_{n=0}^{\infty} \left(C_n^+ \, e^{-jk_{C,n}l_c} + C_n^- \, e^{jk_{C,n}l_c}\right) \langle \Psi_{C,n} \, \Psi_{E,S} \rangle_E = \left(E_S^+ + E_S^-\right) \langle \Psi_{E,S} \, \Psi_{E,S} \rangle_E$$

$$\sum_{n=0}^{\infty} \left(C_n^+ \, e^{-jk_{C,n}l_c} + C_n^- \, e^{jk_{C,n}l_c}\right) \langle \Psi_{C,n} \, \Psi_{D,S} \rangle_D = D_S^+ \left(1 + e^{-2jk_{D,S}l_2}\right) \langle \Psi_{D,S} \, \Psi_{D,S} \rangle_D$$

$$k_{C,S} \left(C_S^+ \, e^{-jk_{C,S}l_c} + C_n^- \, e^{jk_{C,n}l_c}\right) \langle \Psi_{C,S} \, \Psi_{C,S} \rangle_C$$

$$\sum_{n=0}^{\infty} k_{E,n}\,(E_n^+ - E_n^-)\,\langle \Psi_{\mathrm{E,n}}\ \Psi_{\mathrm{C,S}}\rangle_{\,E} + \sum_{n=0}^{\infty}\ k_{D,n}\,D_n^+\big(1 - e^{-2jk_{D,n}l_2}\big)\langle \Psi_{\mathrm{D,n}}\ \Psi_{\mathrm{C,S}}\rangle_{\,D}$$

So, this guy gets is set directly to 0, you know this guy is directly set to 0 and once, you do that and you get all these things and d plus is there. Now, what about we might have some trouble as I was mentioning earlier regarding.

So, this becomes this is the growing term. So, what we same thing same logic applies here. So, what the clever trick then is that the clever trick then is that you set C_n let me go to the presentation.

$$C_n^+ e^{-jk_n^C}\, l_3 \quad \rightarrow \tilde{C}_n^+$$

So, we trick is that let us set E; e to the power you know you have this. Even in this term and even in the velocity term, you set this guy - jk_C n l_3 like I was mentioning. This is l_C, they have written; I am calling it l_3 I am calling ok. This entire thing you know this k_C is such that its root over alpha n by

$$-jj\sqrt{\left(\frac{\alpha_n}{l_1}\right)^2 - k_0^2} \quad \tilde{C}_n^+\ C(-)$$

So, we will call this entire thing as $\tilde{C}_n^+$. We will solve for this guy rather than this thing because this might lead to numerical issues. They most likely it will. So, $\tilde{C}_n^+$ this will we are setting this here. Coming back here, so well, we have resolved this problem but then, other thing has to be done.

Now, let us check if we have plus j j j sign. So, j square j square is minus in case of k_C is imaginary. So, this will be exponentially dying term. It should not create a problem. So C, so, basically C_n^- you know something this will be left as it is, but then we have to be careful with the other set of equations. What do we do?

Now, at the other boundary condition, there is something that I want you to understand properly. So, here also we have you know these three equations. So, if we are setting this guy as like this. We cannot increase our number of coefficients, you see how many set of coefficients we have?

We have you know A_S^-s reflected waves in the inlet pipe travelling wave, forward traveling wave, negative traveling wave in the chamber and B minus. So, three sets of, four sets of equation 1 2 3 4 and similarly, we have you know this is set to 0 5 here and 6 and these two are known.

Now, moment we set C_n^+ e to the power -jk C_n l_3 to other coefficient 6 will be another coefficient, but then we are increasing unknown variable to 7. Now, we in order to, but we will have only 6 conditions; 3 here and 3 above; three sets of equations. So, to resolve this issue, we have to be careful with the other variable. Let us worry about what we are going to do with the other thing.

So, C_n^+ is something that we have done. So, we need to do something with the this variable. What we did now was that we just set this guy C_n e to the power -jk_n Cl_3 is equal to C n the conditions at the outlet. Now, we because, but when we should not be increasing a number of unknown variables because we have only six set of equations ok.

So, what we do is that we do a clever trick. This is the trick1 that we did at the outlet. At the trick 2 at the inlet is that you know just put just multiply this analytically be e with e jk n C l_3. So, this will be C_n^+ is equal to $\tilde{C}_n^+$ is equal to this thing.

So, there won't be any numerical problems because when this is j. So, j j square is minus and - k C_n l_3. So, this will become exponentially decaying term. So, C_n^+, is an exponentially decaying thing and so, basically what we need to do really is that in here right in this term C_n^+, we will substitute this with $\tilde{C}_n^+$ into e to the power jk_n l_{C3}. This we will do for the pressure equation here, this condition here as well as, as well as the velocity condition in this equation here.

So, with this, you know we are good to go we can have different number of constants; we can have different number of constants in the chamber, in the port and number of different number of modal summation value n in the modal sorry in the inlet port as well as the as well as in the cavity here on the chamber.

So, basically, let us consider what I mean is that if you go to the presentation, you know in this thing we can consider say first 5 modes here, 10 modes here, 20 modes here or actually 10 modes here; same this would not be too different. But you know or you can have 10 modes here; 10, 10, 10 you know basically you know n_1 modes here, n_2 mode here and n_3 here.

Generally, we consider n_2 as equal to n_3 that is the number of first few modes in the cavity annular cavity and the expansion chamber are considered the same. Inlet pipe, because you know those pipe diameters are typically quite small compared to the chambers about 40-50 mm for automotive mufflers. So, we consider you know limited number of modes given the operating frequency and so on. So, basically, you know with this, what we will do is that we will we have set up the system of equations.

And they eventually, you know what other thing that I sort of must point out here is that once these things are set up, we will we have we have set up the system to determine the transmission loss then, what we need to do is that the dimensions of the inlet duct are assumed such that the incoming wave A plus its planar and its magnitude is chosen to be unity and anechoic termination is imposed like I was mentioning.

So, we has infinitely theoretically infinite set of equations because n is going to infinity, but we have to truncate it to the first q mode. So, we have typically you know 6 q plus 1 number of equations q being the first q mode, you know first you know the 1 means the planar wave mode and then, on the top of that whatever number of modes you want to consider.

So, 6 q or 6 q plus 1 whatever you want to say so that many set of equations and that many set of unknown coefficients we have. So, all the things we are solving in terms of A plus. We are basically one thing that I must kind of clarify here that is very important that we are considering the excitation by only the planar wave ok. So, A_S^+ is there or you know I would say I would say if you go on the top A_n^+.

So, n = 0 will give you the planar wave mode and there will be no in the excitation part, in the inlet port here and the excitation domain here, I mean in this chamber in the for the A region, there will be no we are what we are doing actually is that we are considering let me mention this point a bit more clearly. So,

$$
\begin{aligned}
A_0^+ &= 1 \quad A_n^- \neq 0 \\
A_1^+ &= 0 \quad A_1^- \neq 0 \\
A_2^+ &= 0 \quad A_2^- \neq 0 \\
&\vdots \\
&\vdots \\
A_n^+ &= 0 \quad A_n^- \neq 0
\end{aligned}
$$

$$E_n^+ = 0 \quad \text{anechoic condition}$$

we do not know this cannot be 0. Similarly, A_2 these are reflected wave coefficients, this we need to solve. So, all the variables you know what are those variables A and also of course, minus this cannot be 0.

$$
\begin{array}{lll}
A_n^-\Big|_{n=0,\ n_1} & n = n_3 = q & \\
B_n^-\Big|_{n=0,\ n_2} & q & \\
\tilde{C}_n^+\Big|_{n=0,\ n_3} & q & \\
 & & 6q \\
\tilde{C}_n^-\Big|_{n=0,\ n_3} & q & \\
D_n^+\Big|_{n=0,\ n_2} & q & \\
E_n^+\Big|_{n=0,\ n_1} & q &
\end{array}
$$

So, we have you know that many variables or we assume all the variables, all the number if you assume n_1 is equal to n_2 is equal to n_3, then we have is equal to q. So, q unknowns, q unknowns, q unknowns, q, q; so, 6 q. But if you consider plane wave separately, you can consider q 6 q plus 1 like that. But you know all these will be expressed eventually as a big matrix, where you have this block of equations.

I am sorry unknown coefficients,

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{Bmatrix} A_n^+ \\ B_n^- \\ \tilde{C}_n^+ \\ \tilde{C}_n^- \\ D_n^+ \\ E_n^+ \end{Bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{Bmatrix} A_0^+ \\ \\ \\ \\ \end{Bmatrix}$$

and then we need to and then we need to invert that matrix and express all these variables in terms of A_0; sorry A_0 and find out the transmission loss as you know as given by this expression including the higher order mode.

But usually, higher order modes will not survive in the outlet pipe, there are small diameter. So, this will reduce through the expression given in; expression that we have discussed so far.

For planar wave, this will reduce to this form. So, in the next class, we will do the numerical analysis of such things. Till that time, I will hang up. I will just stop here and I will see you in the next class. I will do some numerical simulations in MATLAB.

Thanks.