Muffler Acoustics - Application to Automotive Exhaust Noise Control
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Week-11
Lecture - 52 and 53
Rectangular Chamber Mufflers: Characterization and TL Analysis using 3-D Piston-driven model

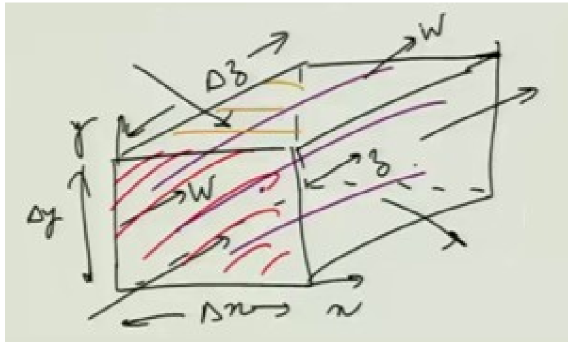
Welcome back to this NPTEL course on Muffler Acoustics. So, we are in week 11 and for the first time, we will be doing the three-dimensional analysis of Rectangular Chamber Mufflers as well as Circular Cylindrical Mufflers in this lecture.

So, this is here week 11 and what we have today is lectures 2 and 3 combined of week 11. So, what we are going to do exactly in this lecture is that we will present a proper theoretical formulation of how to characterize different single inlet, single outlet rectangular chamber mufflers as well as circular cylindrical mufflers with arbitrary location of ports.

Although, this formulation like I mentioned in the last class can be used to analyse system with multiple ports, I mean more than two ports I mean to say like single inlet and double outlet or single inlet and triple outlet or two inlets and one outlet and different combinations.

But we will restrict our discussion to single inlet single outlet system or SISO system. So, we essentially have a two port system; two port rectangular chamber or a circular cylindrical chamber. So, you know let us start with your basic equations that govern wave propagation, you know for simplicity what we will do is that we will consider you know a cartesian coordinate system something like this you know and ok.

So, now we have this; we have this. So, what exactly are we going to do? Our first job is to rewrite the **continuity equation** which is now **inhomogeneous and momentum equation,** which is homogeneous and combine them to get one single inhomogeneous wave equation or rather the Helmholtz equation I am sorry.



So, you know the final form that we are looking at

$$(\Delta^2 + k_0^2)\tilde{p}$$

$$\tilde{p} = \tilde{p}\ (x, y, z)e^{j\omega t}$$

$$= -j\omega\rho_0 Q_0 \delta\ (-)$$

So, this is for a point source. So, how do we get to this form for this **Cartesian system cartesian and circular cylindrical system**. This is what we are going to; we going to study in this class well. So, let us move ahead and you know we will define a coordinate system something like this. So, here you will have your say the z direction, this is your x, this is your y and what is the flux that what is the flux that enters in this phase?

So, here, we will have your rho u. And this is the say let us say this is this direction, this length is $\Delta x,\ \Delta y, \Delta z$ and the flux that enters here is your velocity W and one that leaves here is also W. But we will we will look at the mass flux that leaves and go goes inside and leaves from the other phase and so on.

So, you know let us let us just focus on your to begin with the this particular phase, where I am which I am shading, this rectangular phase net mass flux that enters this phase is rho W at z and the one that leaves and of course,

$$\{\Delta x\ \Delta y\ (\rho w)_z - \ (\rho w)_{z+\Delta z}\ \Delta x \Delta y\}$$

Similarly, if you consider the phase here, mass flux that goes here and leaves here, what is at that?

$$+\ \{\Delta y\ \Delta z\ (\rho U)_x - \Delta y \Delta z\ (\rho U)_{x+\Delta x}\ \}$$

So, this will have ρU term because this is the x direction. So, plus this thing and the one that leaves on the y side. I mean this suffix means,

$$+\{\Delta z\Delta x\ (\rho V)_y - \Delta z\Delta x\ (\rho V)_{y+\Delta y}\} = \frac{\partial(\rho\Delta x\Delta y\Delta z)}{\partial t}$$

So, this is the net mass flux that is created within the differential volume. Now, the mass flux that comes inside leaves comes inside and leaves and so on and what this is equal to what? Plus all this while we were equating this thing to the net mass flux, the net a temporal change of the mass flux of the mass inside this thing that is your density times the volume.

So, we have this particular guy sitting here. So, this is a small elemental volume in obviously, in Cartesian coordinate system $\Delta x\Delta y\Delta z$ and what we are going to do now is that all this while equating here. So, now of course, if we want to derive all this thing in circular cylindrical systems, things would be a bit more; things would be a bit more complicated in the sense that we need to transform from x, y, z to θ z.

And basically, it essentially involves you know doing something to the Laplacian to get your del square by del x square terms and all those things or else, we can you know do these things from you know from the continuity equation, momentum equation side of things as well.

But for now, we stick to Cartesian coordinate system and using that will present one sort of a formulation. Now, what is happening here is that this is what we are getting all this while when there was no source inside this volume ok. However, if you consider general source term inside this volume, let us say you know this entire thing is has some sort some sort of sources, which inject mass or take out mass and the distribution of such source is such that it is you know it is defined as I am using a different colour just to you know highlight this fact.

So, you know let us say this is m mass, m as a function of t which is distributed over the volume.

$$+\dot{m}(t)\dot{\Delta} x\Delta y\Delta z$$

So, why did we multiply here? This plus term, of course, is there because now we are assuming some sort of a term and this m Δt is really the mass which is a function of time that is being injected or taken out of the system or this elemental cube and the mΔ or $\dot{m}(t)\Delta$ basically tells you that this is per unit volume.

So, you just have to multiply this to the entire you know differential volume the or the small elemental volume to get the total mass that is being injected or taken out over the small elemental volume.

Now, one thing we can we can straight away do here is that mass is nothing but density times say the volume velocity. So, basically,

$$+\rho\dot{Q}(t)\ \Delta x\ \Delta y\Delta z$$

So, the units of this is kg by

$$\frac{4}{M^3}\ \frac{\frac{M^3}{S}}{M^3}$$

$$\frac{4}{\cancel{M^3}}\ \frac{1}{S}\quad \cancel{M^3}$$

$$\frac{kg}{S}$$

You this you can consider this as a flow rate. So, here you will have your kg meter cube; but you know this is again we are defining this as floor flow distribution per unit volume.

So, this will have a unit of per second only and then, you have this meter cube term. So, these things will get cancelled and eventually, what we will have is kg by second. So, that is the mass flow rate and it is dimensionally consistent with this thing.

Because now, you can clearly see you know here is kg by meter cube and meter cube. So, you will get kg, just you will just get kg and in the denominator you have del t dou t. So, you get kg by second and same thing one can verify for all the terms here. The point I am trying to make here is that this is a dimensionally, consistent $\rho\dot{Q}(t)\ \Delta x\ \Delta y\Delta z.$ You know for special cases like you know if you have a thing like point source.

Then you would have terms like rho into

$$\rho Q_0 e^{j\omega t} \cdot \delta(x - M_0) \,\&\, (Y - Y_0)\, \delta(z - z_0)\, \Delta x \Delta y \Delta z$$

But it is localized at a point.

$$\rho U_0 f(x, y)\, \delta\, (z - z_0)\, \Delta x\, \Delta y\, \Delta z \quad f(x, y) = 1$$

So, we get this ok. So, this term would be something like this. So, for now, you know this is just I am trying to get you a feel of this. So, what these direct delta function means? So, this is the first time, I guess we are talking about these functions in this course. So, small digression, I must make.

So, you know a direct delta function delta x is defined say that it is 0 for all values. Or you know you can consider this as delta direct delta function in Cartesian system. And this is defined as,

$$\delta(x - x_1) = 0 \qquad for\ all\ \ x \neq x_0$$

$$\int_{-\infty}^{\infty} \delta(x - x_0)\, dx = 1$$

$$\delta(x - x_0) = \frac{1}{2\epsilon} \quad at\ x = x_0$$

$$\overset{4}{\in \rightarrow 0}$$

So, that means, that this is equal to a very small quantity exactly at x is equal to x naught, where this guy is limit is tending to epsilon is tending to 0. So, this will kind of blow up.

So, this is direct delta function is used in lot in engineering you know by solving problems for engineering mechanics, structural mechanics and basically, presenting some physical systems, where you have point excitation or a point you know you do something like you know you just have impulsive sound something like if you do this.

So, this is just like this is happening just at that instant of time; if you burst a balloon. So, this is you, then in such a case this direct delta function would be

$$F_0 e^{-j\omega t}\delta(t = t_0)$$

So, whatever force or whatever something whatever disturbance is happening that is localized or that is acting at right at t is equal to t naught and for all other values it is 0.

So, mathematically, you know it is defined like that. Although, it is tending to infinity at that instant in space that incident time, I am sorry or that particular point in space you know in the overall sense, it comes out to be finite; nice finite value. So, if you can have a beam something like this and you can have a force point load acting in this direction.

So, this is modelled really by a direct delta function, force acting like this or a point excitation is also there. So, in which case you will have this additional ejωt term; meaning that there is an excitation force, harmonic excitation force; but localized very very much localized in space at a single point.

So, coming back to our discussion, we will keep this guy aside and move with the generality and what we will do? We will we will go around doing our regular business you know divide throughout by $\Delta x\, \Delta y\, \Delta z$ or probably, take this guy common $\Delta x\, \Delta y\, \Delta z$ common on the our right hand side left hand side and then, take limit of $\Delta x \rightarrow 0$, $\Delta y \rightarrow 0$, $\Delta z \rightarrow 0$ and when we do that, we will its it is not a very great thing to see.

I mean it is rather simple match to see that in the limit of

$$\Delta x \rightarrow 0, \quad \Delta y \rightarrow 0, \; \Delta z \rightarrow 0$$

$$-\left\{\frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} + \frac{\partial \rho W}{\partial z}\right\} \Delta x\, \Delta y\, \Delta z$$

what will happen is that you will have this is equal to this thing

$$+\rho \dot{Q}(t) \Delta x\, \Delta y\, \Delta z$$

$$= \Delta x \, \Delta y \, \Delta z \, \frac{\partial \rho}{\partial t}$$

$$=> \frac{\partial p}{\partial t} + \nabla \cdot \rho U$$

I would do it something like this is equal to rho naught is take this is again taken common and so, this what we can do is that move this guy on the right hand side, actually write it something like this, where this bold term is means a vector or I would for now just for convenience sake.

$$=> \left\{ \frac{\partial p}{\partial t} + \left(\frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} + \frac{\partial \rho W}{\partial z} \right) - \rho \dot{Q}(t) \right\} \times$$

What I am going to do is that write down the full blown form. And this is equal to $\rho \dot{Q}(t)$and I will take this guy also on the right hand side and

$$\Delta x \Delta y \Delta z = 0$$

$$\rho V_0 \, e^{j\omega t} f(x, y) \delta(z - z_0)$$

This is of course, multiplied here. So, you know for this equation to hold for any generic volume, the term inside this thing must go to 0.

Because this you see this is a generic volume; $\Delta x \Delta y \Delta z$. In a big fluid, you can consider a small volume since we are considering this volume to be arbitrary, this should hold for any $\Delta x \Delta y \Delta z$ that small dv volume and for that, this entire thing the term inside must go to 0. So, what do we eventually get?

We get,

$$\frac{\partial p}{\partial t} + \frac{\partial \rho U}{\partial x} + \frac{\partial \rho V}{\partial y} + \frac{\partial \rho W}{\partial z} = \rho \dot{Q}(t)$$

we get this guy continuity equation with the inhomogeneous volumes source term sitting on the right hand side. Clearly, if Qθ $\dot{Q}(t)$ is 0, we get back our familiar homogeneous equation. So, that is the beauty of this formulation. It is a kind of reduces back to the homogeneous form, in case there are no artificial or though sources within that volume.

Now, clearly, if you know if you like I was discussing a while back, you know if you have this term only if you have a source term which is governed by given by this thing you know if this particular form of $\dot{Q}(t)$ is like this. Then, that means, for a big volume no matter how big it is, everywhere this $\dot{Q}(t)$ 0; at only at a certain point in space x naught y naught z naught, this is nonzero and in the integral since it comes out to be finite ok.

So, we get this sort of a thing. Now, what about the momentum equation? Momentum equation, there are no there are no inhomogeneous term of the right hand side. So, I would just sort of write it down like this and actually, before I even go to momentum equation, it is a good idea to actually you know kind of linearize the things because we are we will be working in a linear formulation.

So, rho is just your you know rho is nothing but rho naught plus rho tilde and rho tilde is equal to $\frac{\tilde{p}}{C_0^2}$, you know from our isentropic equation of state and let us consider a stationary medium that is no mean flow.

$$\rho = \rho_0 + \tilde{\rho};\ \tilde{\rho} = \frac{\tilde{p}}{C_0^2}, \qquad \begin{aligned} V &= \tilde{U} + U_0 \\ V &= \tilde{V} \\ W &= \tilde{W} \end{aligned}$$

So, under such conditions, this equation gets simplified to I am directly writing down the steps without going into the details again.

$$\frac{1}{C_0^2}\frac{\partial \tilde{p}}{\partial t} + \rho_0 \left\{\frac{\partial \tilde{U}}{\partial x} + \frac{\partial \tilde{V}}{\partial y} + \frac{\partial \tilde{W}}{\partial z}\right\} = \rho_0 Q e^{j\omega t}$$

Because it is something that we have sort of done before. So, we get this this is equal to and here, we will get your ρ_0. And let us say we have a point source representation, we will get this e j omega t sitting here.

So, I would just say this is Q (t) and it is localized in space.

$$\rho \dot{Q}(t)\ \delta(x-x_0)\ \delta(x+Y_0)\ \delta(z-z_0) \qquad (1)$$

So, this is where the source is ok. What we could probably do here is that we can sort of insist on time harmonicity. So, we can put p ejωt and those sort of terms. But before, in fact, we even do that. Let us just box up this equation; meaning that this is an important result, let us say 1; we call it 1.

And momentum equations like I said will remain the same, they are homogeneous.

$$\rho_0 \frac{\partial \widetilde{U}}{\partial t} = -\frac{\partial \tilde{p}}{\partial x} \quad => \frac{\partial^2 \widetilde{U}}{\partial x \partial t} = -\frac{\partial^2 \tilde{p}}{\partial x^2}$$

$$\rho_0 \frac{\partial \widetilde{V}}{\partial t} = -\frac{\partial \tilde{p}}{\partial y} \quad => \rho_0 \frac{\partial^2 \widetilde{U}}{\partial y \partial t} = -\frac{\partial^2 \tilde{p}}{\partial y^2} \qquad (2)$$

So, we get this. Similarly, and finally, rho naught into this thing is your. So, what we get is again taking a derivative with respect to the space. So, here,

$$\& \quad \rho_0 \frac{\partial \widetilde{W}}{\partial t} = -\frac{\partial \tilde{p}}{\partial z} => \rho_0 \frac{\partial^2 \widetilde{W}}{\partial z \partial t} = -\frac{\partial^2 \tilde{p}}{\partial z^2}$$

So, idea is to eliminate the particle velocity components along each direction to get this. And you know what we do is that we will we will keep all this result aside and name these as equations (2).

And now, take a time derivative of this equation and that will set up the matters. So, once you do that you will get. So, this term will become

$$\frac{1}{C_0^2} \frac{\partial^2 \tilde{p}}{\partial t^2} + \rho_0 \left\{ \frac{\partial \widetilde{U}}{\partial t \partial x} + \frac{\partial \widetilde{V}}{\partial y \partial t} + \frac{\partial \widetilde{W}}{\partial z \partial t} \right\} = \rho_0 j\omega Q_0 e^{j\omega t}$$

and here,

$$\delta(x-x_0)\ \delta(x+Y_0)\ \delta(z-z_0)$$

$$j\omega \rho_0 U_0 e^{j\omega t}\ j(x,x_0)\delta(z-z_0)$$

We have been insisting time harmonicity. So, I would put it this as $e^{j\omega t}$. So, here we get,

$$\rho_0 j\omega Q_0 e^{j\omega t}$$

So, you know we get this and now, if we substitute equations to a - c back in here ok.

We will get you know we will get your Laplacian. Essentially, we are going to get the Laplacian ah. So, we will get this ρ_0 t is also there here. So, let me just figure out. We will put this there are number of minus signs here.

$$\frac{1}{C_0^2}\ \frac{\partial^2 \tilde{p}}{\partial t} - \Delta^2 \tilde{p}\ j\omega\rho_0 Q_0 \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)e^{j\omega t}$$

$$\Delta^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

So, we get this Laplacian operating upon this instead of each of these term, this one, this one, this one; we put we take rho naught of course inside and you will just get minus Laplacian times whatever your whatever you are getting and this is equal to rho naught j. I would write this as, we will get

$$\Delta^2 \tilde{p} - \frac{1}{C_0^2}\frac{\partial^2 \tilde{p}}{\partial t^2} = -j\omega\rho_0 Q_0 \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)e^{j\omega t} \qquad (3)$$

$$\tilde{p} = \tilde{p}\ (x,y,z)e^{j\omega t}$$

Now, here of course, Δ^2 means in Cartesian system and we will soon see how do we get to the r theta system in a while. But before we do that, now let us sort of simplify this guys further. We will multiply, we will put this in the standard form; you multiply throughout by minus.

So, we get this guy. So, this is the inhomogeneous wave equation and now, if we; so, this is your inhomogeneous in homo equation (3) you can say ok. So, this is what we are going to get. Now, if we assume that this is time harmonic ok. So you will get

$$\left(\Delta^2 \tilde{p} + \frac{\omega^2}{C_0^2}\tilde{p}\right) = -j\omega\rho_0 Q_0 \delta(x-x_0)\delta(y-y_0)\delta(z-z_0) \qquad (4)$$

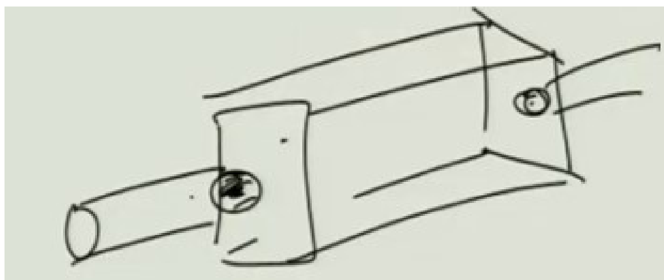
We get this sort of a thing and omega by

$$\frac{\omega}{C_0} = k_0 = \text{ excitation wave number.}$$

The dimensions of that is per meter or k naught square is 1 by meter square. So, you eventually, will get

$$(\Delta^2 + k_0^2)\ \tilde{p} = -j\omega\rho_0 Q_0 \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \qquad (4a, b)$$

So, this is a very important result. Now, this is your in **homogeneous in Cartesian coordinate system,**



Inhomogeneous Helmholtz equation.

So, there are lots of things, we can do from here. You know the first thing that I would like to tell you in a small digestion here. So, you have to pay attention here, especially you know to this term that is sitting here.

So, this again, if you set Q_0 to 0 you get back our old friend homogeneous Helmholtz equation in which case your you get homogeneous; but we have inhomogeneous. So, we get this. What does it mean? It means that like I said the point, the source is localized on the right at a particular point; but then, you know what we are essentially trying to do?

Let us consider you know a rectangular chamber muffler something like the port inlet port is like this and it is just a rough figure. You know this is something there is a you have a chamber something like this and so, you are you know trying to squeeze out the entire finite points from into finite area into one point you know this thing.

So, as if the you know the entire thing is the inlet and out the inlet or the outlet ports are you know condensed down to a single point, the point of singularity and that is what is represented by the direct delta function. So, they are basically the ports, the finite size ports are represented by a point mathematical point.

- Finite Sized or Cross Section Area is Represented by Mathematical Points
- Green's Function Definition Response to a Point Source (Space) or Impulse Time
- Uniform Piston Driven Model Planar Wave Fronts in The Port, Right at the Chamber Interface
- Mode coupling is not considers
- Mode matching Approach.

Finite sized or cross section area is represented by mathematical points in space, basically on the surface of the chamber. So, the you know the you can also you know later on to solve this thing you can invoke the Kershaw Helmholtz theorem.

But we will probably not go into that detail now, we will just stick to the, we will just stick to this particular formulation and see how to use the homo solution of homogeneous equation to solve this. But before that, you know whatever solution we are we are going to get for this is called the Green's function solution.

So, Green's function is by definition response to a point, point source in space or impulse; impulse happens in time like I said a clapping or bursting of a balloon. So, this is the Green's function representation of when you represent the piston by a point source.

Now, of course, like I said a much better representation at least of this thing is the at least when you actually consider the complete span of the ports. Now, the different theories that are used to you know characterize mufflers, the simplest one of course is you know the point source representation which like assumes a finite size ports into points like I said, but then you have a theorems like you know methods like I am sorry a uniform piston model. So, basically the finite port is still represented by finite thing only; but then, over this entire cross section area, the velocity on the pressure fund fronts are assumed to be uniform.

That means, to say in other words, the other the next mathematically the more accurate model which we are going to use for doing some results and compare it with 3D simulations which were computed previously is uniform piston driven model.

Uniform piston driven model in which basically you tend to assume planar wave fronts in the port, planar wave fronts right at the chamber right in chamber port interface, we assume it right from the start ok.

So, this is one of the, this is one of the salient features of piston driven model. But it is much better than you know than the Green's function. Because now, at least you are representing the ports by this thing and we see we will soon see that you know you integrating the variance functions over the port area can do a lot of good things for you.

Here of course, there is modal coupling is not considered between the port and the chamber that is one of the drawbacks of the Piston driven model, but is still is much better than the Green's function of point source model because of the fact that we are representing the ports, we are actually considering the finite size and considering some sort of an average representation of the variation of the ports.

So, we are you know kind of taking into account the pressure variations across the chamber over the over the surface area, where the port is located and using that to get the pressure response function of the Green or the impedance matrix parameters and we will compare this with you know the finite element results, wherever they are available and the next thing is basically your mode matching, which is perhaps the most accurate mode matching approach.

Mode matching; so, basically it considers coupling in in this chamber and this chamber and once you do that, then basically you tend to set up a set of equations and solve for the unknown coefficients and eventually, it can lead to either calculation of the four pole parameters that is characterization of the system or the computation of the transmission loss expressions transmission loss thing.

So, for, but most of the automotive mufflers, you know typically which are less than D less than you know 50 mm or something like that, this is this kind of thing is pretty much all right. Because for the most of the frequencies of interest, the first higher of frequency in the circular ports will be at a at a much higher frequency.

So, this would this would suffice for most of the automotive mufflers for larger mufflers like industrial mufflers, we might have to consider mode matching. In fact, mode matching can be important for some quite a few popular muffler configurations like external inlet and outlet systems.

We will hopefully see that in the next week. Let us see how we go about it. But this is what it is. Now, let us go back to our previous slides, where we have this kind of a thing. Now, we have Q_0t like this and then you know we have Q_0, but suppose you know let us have another representation which is your you know Piston driven model.

So, Piston driven model what are the changes that one needs to make? So, in such a system, you know you still have your $\rho_0 Q_0$; I am not sorry Q_0 is basically velocity that is there. So, remember that units of Q_0, where Q dot t were 1 by second and this was kg by meter cube and here, it was meter cube. So, things were working out.

So, well, so basically if you assume that the source is distributed uniformly over the port area and velocity,

$$\rho U_0^{e^{j\omega t}} \ f\,(x, y)\delta(z - z_0)\Delta x \Delta y \Delta z$$

and we define a function such that over the face x,y, let us say the port is located on the face x,y at any generic point. So, $f(x, y) = 1$ over this port and 0 elsewhere and in the in set dimension this is still this guy.

So, and then of course, this entire the other terms pertaining to this thing will be delta y delta z thing. So, what about the, what about the dimensionality of such things? Dimensionality wise, it should not be a, be a problem ok. So, kg / m^3 and if we assume here by meter square meter by second and this is of course, per meter you know you have your direct delta function representation and here, you have your meter cube. So, under such a, such a situation you know this *f x y*.

So, this gets cancelled. I am sorry this these two gets cancelled and this meter gets cancelled. So, f x y must be 1 which is basically your non dimensional function. This is $f(x, y)$ is 1 over this area and 0 over the area shaded in green colour.

So, you know basically all we need to do for modelling the uniform piston driven model, which I have been telling you to be more accurate is get you know use this.

And of course, U naught is varying as e j omega t. And you know when we have such a kind of model with. So, in such case, we can work out the algebra instead of if we have this ρQ_0 ejωt and all that, so what will happen? So, let us go to your this thing ρQ_0.

So, instead of let me write down, let me use another colour.

$$\rho U_0^{e^{j\omega t}} \; f\,(x, y)\delta(z - z_0)$$

and the other term. So, instead of this term, you will have this term in the in the bracket. So, as usual, you know you will have these things will be there. So, instead of this will have rho naught into u naught and you I would say like this f x y something like this and you will have this kind of a term. And so, now, once, we have this particular thing what we will do is basically we e j omega t is of course there. So, then let us do the, you know other things. So, here

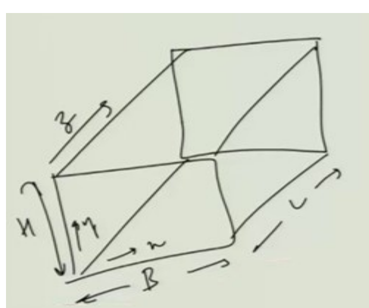
$$\rho U_0^{e^{j\omega t}} \; f\,(x, y)\delta(z - z_0)$$

Instead of this thing will have this. So, finally, what do we get? You know let us get to the equation. So, instead of this thing, we

$$-j\omega\rho_0\, U_0 f\,(x, y)\delta(z - z_0)$$

So, this is your Piston driven model representation. Now, coming back, after this small digression, what we will do?

You know for a rectangular chamber, let us say of something like this. You have things like this, you know this is let us say the, this is B, H, L. This is the x direction, y direction and z direction. You know for subject to the rigid wall boundary condition,



So, we have

$$at\ x = x, B$$

$$\frac{\partial \tilde{p}}{\partial x} = 0, \qquad U = 0$$

Similarly,

$$\frac{\partial \tilde{p}}{\partial y} = 0, \qquad V = \ \ y = 0, 4$$

So, the corresponding,

$$\frac{\partial \tilde{p}}{\partial z} = 0, \qquad W = \ \ z = 0, L$$

Now, once you get that you know I leave some of the details for you to work.

This is the thing and of course,

$$\tilde{p}(x, y, z) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{lmn} \cos\left(\frac{l\pi z}{L}\right) \cos\left(\frac{m\pi y}{H} z\right) \cos\left(\frac{n\pi x}{B} z\right) e^{j\omega t}$$

So, time harmonically, it is behaving like this and one thing that is there is nothing but

$$k_{lmn}^2 = \left(\frac{l\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{B}\right)^2$$

So, you know what we will do? that we will we will substitute let us say equation (4 a, b); where a corresponds to the point source representation and b corresponds to the piston driven representation ok. We will have this sort of a thing.

Now, once we do that, we will substitute this guy back in equation 4 let us say a ok. We will we will substitute this guy in here and see what we get? So, obviously, you know this you know del square, if you operate say del square by del x square, if you operate upon say this term, you are going to get again get back cos, but to the minus sign. So, you will get minus m pi H whole square ok. So, you are going to get m pi sorry this is not I am sorry $\frac{n\pi}{B}$. So, you know eventually, what you will get is basically,

$$\sum_{lmn} \left\{ \left\{ -\mathrm{k}_0^2 \left(\frac{l\pi}{H} \right) + \left(\frac{n\pi}{B} \right)^2 + \left(\frac{n\pi}{L} \right)^2 \right\} \right\} C_{lmn} \cos\ (-) \qquad (5)$$

So, you know we get this and then, you get your k^2 C_{lmn} and all this cos functions ok. This summation is there of course.

This is a triple summation over l m n. So, I am not writing down this thing explicitly. But

$= -j\omega\rho_0 Q_0 \delta\ (x - x_0)\ \delta\ (y - y_0)\ \delta\ (z - z_0)$ functions and you just get rid of the time term, let us understood its time harmonic. So, here also this $ej\omega t$ and here also I am just so it just cancels out; you just do not worry about it.

Here, we are stuck up with one important thing. What do we do? We have only well one equation and infinite constants how many constants do we have C_{lmn}; l varies from say 0, 1, 2, 3, 4 till infinity and m also varies from 01 to infinity; n also does so. So, technically, we have infinite constants; but only one equation.

So, how do we resolve this issue? How do we get these modal coefficient C_{lmn}? This is what we are after, these are modal coefficients and unfortunately, they are infinite of them and we have unfortunately only one equation. So, how do we get you know calculate these values appropriately? So, had this been in person class, I would have you know really expected one of you with you know lot of you would have decent background on engineering, maths and you know related things.

So, one of you would have actually come down to the board and worked it out; but given the online nature, I would probably give you the solution now and so, we need to do something or invoke something called **Mode Orthogonality**.

So, what we do? We should realize by now that that cos of m pi let us say yeah m and this is well H and

$$C_{lmn}\ \frac{1}{2} \int 2\, cos^2 \frac{m\pi y}{H}\ cos \frac{m, \pi y}{H}\ day$$

$$= 0\ \ , j\ m \neq m_1$$

$$m = m_1$$

Then things are different ok. If m is equal to m 1, then this is not 0; then, it becomes cos square. So, you divide and multiply by 2. So, it becomes 2 cos square and then, you invoke your cos 2 - 2 θ rule.

And you will see that this will be half of H by 2. So, this sufficient enough hint, this is not a high school algebra or I mean calculus or engineering math stuff. You know things will become more complicated, when you have Bessel functions. We are soon going to arrive at that. So, we will multiply you know this equation. Let us say equation (5) with.

$$\cos \frac{m_1 \pi}{H} y \, cos \frac{n, \pi}{B} x \cos \frac{l\pi}{L} z$$

We will multiply both sides by this; multiply both sides and then, when we integrate, we multiply both sides and integrate. So, you know only we do integral, I mean multiply I integrate over the volume.

We can pair this term similarly for x term, similarly for z term $dxdydz$. So, only when,

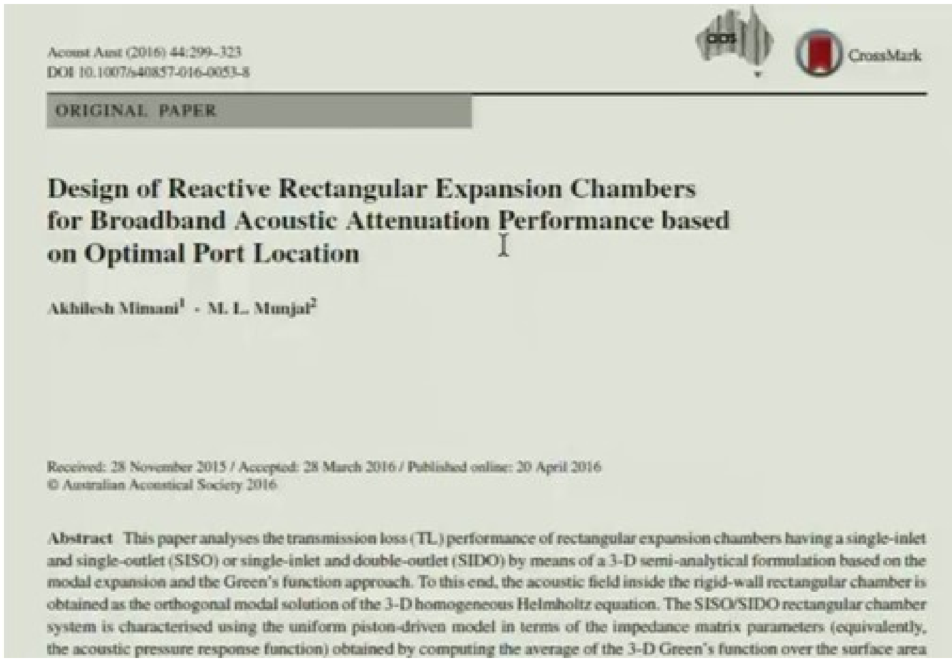
$$N_{lmn} \int_L \int_H \int_B \left(cos \frac{m\pi x}{H} \; cos \frac{m\pi y}{H} \right) (-) \, (-3) \; dxdydz$$

$$\left. \begin{matrix} m = m_1 \\ n = n_1 \\ l = l_1 \end{matrix} \right] \; C_{l_1 m_1 n_1}$$

only when you have such a thing it will survive ok.

So, basically, what we will get? We will get you know the modal coefficients. So, once we get, we once we all the other all the other model coefficients will not survive only $C_{l_1 m_1 n_1}$ will survive and we will have these integral values which is we denote by N_{lmn} and once, we get this thing, we will substitute back in the in the solution that we get here to get the Green's function response.

So, what I am going to do instead of writing it up all together, I will present to you a paper that I published a few years back and this is somewhere here.



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ORIGINAL PAPER

Design of Reactive Rectangular Expansion Chambers for Broadband Acoustic Attenuation Performance based on Optimal Port Location

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Abstract This paper analyses the transmission loss (TL) performance of rectangular expansion chambers having a single-inlet and single-outlet (SISO) or single-inlet and double-outlet (SIDO) by means of a 3-D semi-analytical formulation based on the modal expansion and the Green's function approach. To this end, the acoustic field inside the rigid-wall rectangular chamber is obtained as the orthogonal modal solution of the 3-D homogeneous Helmholtz equation. The SISO/SIDO rectangular chamber system is characterised using the uniform piston-driven model in terms of the impedance matrix parameters (equivalently, the acoustic pressure response function) obtained by computing the average of the 3-D Green's function over the surface area

So, this was a paper on Design of Rectangular Chambers published by another co-author, I mean also there was there was another co-author in this paper. This article appeared in acoustics Australia in 2016, it is been some time now.

$$\tilde{p}(x,y,z) = \sum_{l=0}^{l=\infty} \sum_{m=0}^{m=\infty} \sum_{n=0}^{n=\infty} A_{mnl} \cos\left(\frac{\pi z}{L} x\right) \cos\left(\frac{m\pi}{H} y\right) \cos\left(\frac{l\pi}{B} z\right).$$

$$f_{mnl} = \frac{C_0}{2} \sqrt{\left(\frac{l}{L}\right)^2 + \left(\frac{m}{H}\right)^2 + \left(\frac{n}{B}\right)^2}$$

$$N_{n,m,l} = \left\{ \int_{x=0}^{x=B} \left(cos\left(\frac{n\pi x}{B}\right)^2 \right) dx \right\} \left\{ \int_{y=0}^{y=H} \left(cos\left(\frac{m\pi y}{H}\right)^2 \right) dy \right\} \left\{ \int_{z=0}^{z=L} \left(cos\left(\frac{l\pi z}{L}\right)^2 \right) dz \right\}$$

$$\begin{aligned}
\varepsilon_{nml} &= 1 \quad for \quad (m = n = 1 = 0) \\
&= 0.5 \quad for \quad (m = n = 0. l \neq 0) \\
&\qquad (m = 0, n \neq 0. l = 0), (m \neq 0, n = 0. l = 0) \\
&= 0.25 \quad for \quad (m = 0, n \neq 0. l \neq 0) \\
&\qquad (m \neq 0, n \neq 0. l = 0), (m \neq 0, n = 0. l \neq 0) \\
&= 0.125 \quad for \quad (m \neq 0, n \neq 0. l \neq 0)
\end{aligned}$$

So, you know we have this regular modal expansion business and natural frequencies of the rectangular box and these are our integral values. So, this is one when m is equal to n is equal to l and 0.5 for you know you can have a look at the different permutations and this is the inhomogeneous point source equation.

And we substitute as it said, orthogonal model solution of equation (2 a), for this one that is basically this one is substituted in equation (5) and using mode and evaluated using modal coefficients are valid using mode orthogonality of circular function, back substituted to obtain the Green's function representation for arbitrary location of mode.

So, instead of naught or x naught y naught z naught, we have x S, y S, z S were basically the same thing; they are; they represent the source coordinate and R suffix represents the coordinates for the centre the receiver port, that is where do you want to measure the function for a given excitation point source at x naught, y naught, z naught or x_S, y_S, z_S.

$$(\nabla^2 + k_0^2)\, p = j\omega\rho_0 Q_0 \delta(x - x_S)\delta(y - y_S)\delta(z - z_S) \qquad (5)$$

$$\frac{p(x_R, y_R, z_R | x_S, y_S, z_S)}{\rho_0 Q_0} = \frac{G(x_R, y_R, z_R | x_S, y_S, z_S)}{\rho_0 Q_0}$$

$$= jk_0 c_0$$

$$\times \left\{ \sum_{l=0}^{l=\infty} \sum_{m=0}^{m=\infty} \sum_{n=0}^{n=\infty} \frac{\cos\left(\frac{n\pi x_R}{B}\right) \cos\left(\frac{m\pi y_R}{H}\right) \cos\left(\frac{l\pi z_R}{L}\right) \cos\left(\frac{n\pi x_S}{B}\right) \cos\left(\frac{m\pi y_S}{H}\right) \cos\left(\frac{l\pi z_S}{L}\right)}{N_{n,m,l}\left(\left(\frac{n\pi}{B}\right)^2 + \left(\frac{m\pi}{H}\right)^2 + \left(\frac{l\pi}{B}\right)^2 - k_0^2\right)} \right\}$$

$$e^{j\omega t} \qquad (6)$$

So, this is the Green's function representation looks nasty; but it is not that nasty. You know it is a little bit of algebra and would get you there. So, this is the inhomogeneous solution of inhomogeneous point source equation ok. Clearly, when x R is equal to x S and all this R is equal to S, you have the source, source and receiver to be the same and point source representation in such a thing is known to cause singularity problems ok. When you have.

So, basically this is nothing but you know another thing that I want to tell you is that p look at the units; p is the acoustic pressure and $\rho_0 Q_0$ is the mass velocity kg by meter cube and meter cube by second ok. So, you get you know you get kg by second. So, mass velocity, acoustic mass velocity; pressure by acoustic mass velocity that is your impedance.

So, this is nothing but the impedance matrix parameter. So, you know once you get this, so for a two port system, if you have excitation at say one of the ports and measure the you know get the transfer function or get the outer the response function; then, just by substituting R is equal to S, you can get z11 parameter.

Similarly, by choosing another if you choose the receiver port to be the same as the outlet port, then you get the z21 parameters and z12 is equal to z21. So, the moment you get z21, you know z12. Similarly, if you put R and S same for the other port which can be located anywhere, you get the source port, we get another response function ok. So, like this, we can characterize in principle any number of system, I mean this this these things can be sort of anywhere. The number of ports as well as for we will consider concentrated only on a two port system. So, p is p is equal to z1 z matrix times v, where z is z11, z21, z12, z22; the 2 cross 2 matrix.

$$f_1(x,y) = 1, \forall S_S$$

$$= 0, \quad \forall S_{x-Y} - S_S,$$

$$p(x_R, y_R, z_R | x_S, y_S, z_S) = \frac{1}{S_S} \iint\limits_{S_S} \{G(x_R, y_R, z_R | x_S, y_S, z_S)\} dx_S dy_S \qquad (11)$$

Now, like I have been mentioning you know this point source representation not only is least accurate in terms of you know the underlying assumptions, the other issues of course, the singularity issue. So, people have you know like I mentioned in this paper about the singularity issue published by Zhao and Kim. Now, you consider a piston driven model, so which is the same thing what you derived a while back this is 1 over the port area and 0 over the annular area.

Now, what we do in order to get the solution of that we notice that this function G, you know G is equal $\rho_0 Q_0$ times $jk\rho_0 c_0$ multiplied by this big expression is a solution of this thing ok. So, what we essentially do is that we substitute equation (6) that is Green's function solution in equation number 5, by knowing that this is a solution and then, as I mentioned, we have to go a little fast.

So, what we do to get the solution is that we substitute this guy here and do the following mathematical operations. Once, we do that, we will realize that Green's function, if you just integrate the greens function of the port area and divide by this thing, the we will get the solution.

$$L = (\Delta^2 + k_0^2)$$

$$(\Delta^2 + k_0^2)G(-) = -j\omega\rho_0 Q_0 \delta(-)$$

$$L\{\delta(-)\} = -j\omega\rho_0 Q_0 \delta(-)\ \delta(-)\delta(-)$$

$$\iint L\{\delta(-)\}f\ (x, y)dxdy = j\omega\rho_0 Q_0 \int \delta(-)dx \int \delta(-)dy\, \delta(-)$$

$$L\left\{\iint_\rho \frac{6(-)dxdy}{S}\right\} = -j\omega\rho_0 U_0\, j(x_1^2\ y^2)\delta\ (x - x_0)$$

Because you see this particular thing this will be very comparable. This particular you know Q_0, we are insisting that it is U_0 into the port area. So, we are dividing this by we are taking this s inside the integral sign and U_0 into f of *xryr*. So, what is going to happen is that this particular thing will be highly comparable or not highly will be exactly the same as this equation. So, you know if we. So, this entire thing what I am sort of writing here, this is the solution this particular thing, if we compare this entire equation here, where I am pointing you know this entire thing. So, this is the same as the equation shown here.

So, if we were to just integrate the Green's function of the port area and divide it, we are going to get the solution by visual comparison, we can just sort of get it. So, Green's function, I mean it is not all useless work. It is actually quite useful; it is a good intermediate step to get the Green's function.

$$z_{RS} = \frac{p(x_R, y_R, z_R | x_S, y_S, z_S)}{\rho_0 Q_0}$$

$$= \frac{1}{S_S}\iint\limits_{S_S} \; = \frac{1}{S_S}\iint\limits_{S_S} \left\{\frac{G(x_R, y_R, z_R|x_S, y_S, z_S)}{\rho_0 Q_0} dx_S dy_S\right\}$$

For port R located in X-Y face.

$$= \frac{1}{S_S}\iint\limits_{S_S} \; = \frac{1}{S_S}\iint\limits_{S_S} \left\{\frac{G(x_R, y_R, z_R|x_S, y_S, z_S)}{\rho_0 Q_0} dx_S dy_S\right\} \; dx_R dy_R$$

For port R located in Y-Z face.

$$= \frac{1}{S_S}\iint\limits_{S_S} \; = \frac{1}{S_S}\iint\limits_{S_S} \left\{\frac{G(x_R, y_R, z_R|x_S, y_S, z_S)}{\rho_0 Q_0} dx_S dy_S\right\} dz_R dx_R$$

For port R located in Z-X face. (12a-c)

So, and then we would also try to get the average response over the receiver port. So, that is why these formulations are there. I suggest you should probably try to download this paper on the internet or else, I can give it to you and try to do the derivations by yourself and to learn further. So, this is how we get the general z_S r_S parameter for different location of the receiver port on the X-Y face and the Y-Z at face and on the Z-X face. So, once we do that, we get the impedance matrix parameters.

$$\int_{x=(x_S+r_S)}^{x=(x_S+r_S)} \left(cos\left(\frac{n\pi x}{B}\right)^2\right) \left\{\int_{y=\left(y_S+\sqrt{r_3^2(x-x_S)^2}\right)}^{y=\left(x_S+\sqrt{r_3^2(x-x_S)^2}\right)} cos\left(\frac{m\pi y}{H}\right) dy\right\} dx$$

$$= 2\int_{x=(x_S+r_S)}^{x=(x_S+r_S)} \sqrt{r_3^2(x-x_S)^2} \;\; cos\left(\frac{n\pi y}{B}\right) dx, \;\; m = 0 \qquad (14)$$

$$= \frac{2H}{m\pi} cos\left(\frac{m\pi y_S}{B}\right) \int_{x=(x_S+r_S)}^{x=(x_S+r_S)} \sin\left(\frac{m\pi\sqrt{r_3^2-(x-x_S)^2}}{H}\right)$$

$$\times \; cos\left(\frac{n\pi y}{B}\right) dx, \;\; m \neq 0$$

So, now, we probably can develop some particular you know particular expressions for the integral, when the port is located you know for to get the definite integral.

So, basically what it does is that in this paper exact you know I mean the integrals were evaluated numerically using some Newton quadrature rules more precisely Simpson's three-eighth rule, but the limits of integration you know where the ports is located x r minus r S; r S is the radius of the port and all these limits were carefully evaluated.

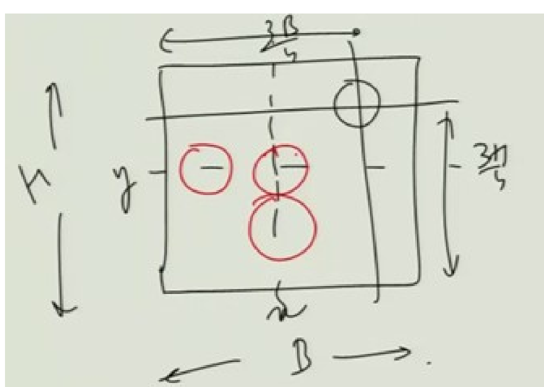
And or you know these things are there. So, basically one can characterize a rectangular chamber muffler. With this sort of a approach, one can characterize any number of port muffler and we can sort of get the evaluate the transmission loss parameters, once we get the impedance matrix parameter.

So, this they after that is the regular stuff, what we discussed in the first lecture of this week. And you know other thing that I want to quickly tell you is that you know let us consider; let us consider the function, the you know influence of location of the port. So, this something that you have to understand for particular cross section. So, you know cos say let us,

$$\cos\frac{m\pi x}{B} \;\; cos\frac{n\pi y}{H}$$

$$m = o \;\; \cos\;(-)$$

So, when m is equal to 0, you know this is cos of 0. So, this is just 1 that is a planar wave mode or you know that is something that should we come out very nicely, you know what we discussed while back. Now, when you have y is equal to 0, you are going to get this and all that. So, now, you get the planar wave mode.



Now, when,

$$cos\frac{\pi n}{B}$$

When x is equal to B by 2, this is 0. So, you know you get a modal line here and when y is equal to H by 2, you get a modal line.

So, this is something that we discussed a long time back now, you know second week, the last couple of lectures or so. The point I am trying to make is that if the port is located at any point here, then m = 1 mode can be suppressed you know the first higher order mode along the x direction.

Similarly, if it is located here the, this mode can be suppressed; but if it is located somewhere here (0, 1), (1, 0), and (1, 1) modes can be suppressed. Similarly, if the port is located at this point, it is easy to say this is 3 / 4B; this is 3H / 4.

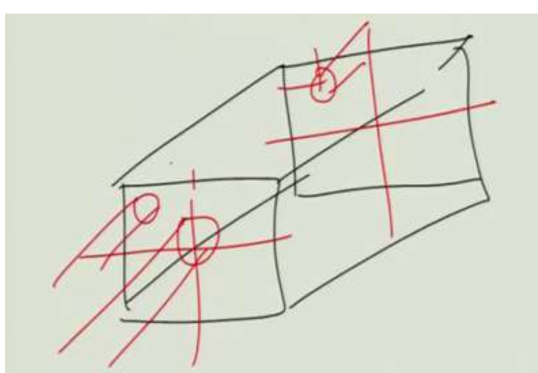
$$cos\frac{2\pi x}{H} \qquad cos\frac{2\pi y}{B}$$

for such a function, when you know when you have,

$$\frac{2\pi}{11}\frac{3H}{4} \quad 2$$

$$cos\left(\frac{3\pi}{2}\right) = 0$$

Similarly, you know the mode, this particular mode can also be suppressed. So, basically in such a situation you have (0, 2), (2, 0), (2, 2) modes can be suppressed.



So, one of the optimal configurations for a, you know for a, expansion chamber muffler, you know can be something like this. Hopefully, we should be able to do some cases. One is at the centre, you know the port is somewhere here; other can be a staggered one somewhere here at you know B / 4 and that sort of a thing or if it is or it can be also be located somewhere here for a reversal chamber kind of a configuration.

And you know the other thing is your configuration something like you know let me just open up the paper for you. So, other configuration that we did analyse the lots of configurations that we did analyse in this paper.

So, there I mean how are, we going to get a broadband transmission loss. So, I will probably show you some results; maybe run a couple of codes.

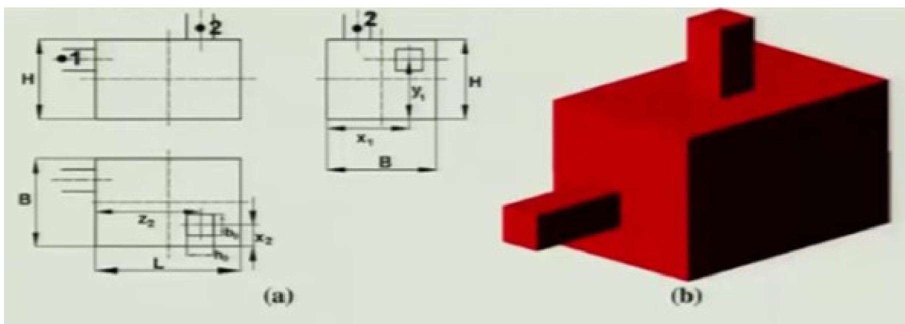


Figure 7: A 2-port cross-flow configuration of rectangular expansion chamber having an end-inlet port (marked as 1) located on the B-H end face and a side-outlet port (markedas 2) located on the B-L face a orthogonal projections and b 3-D view.

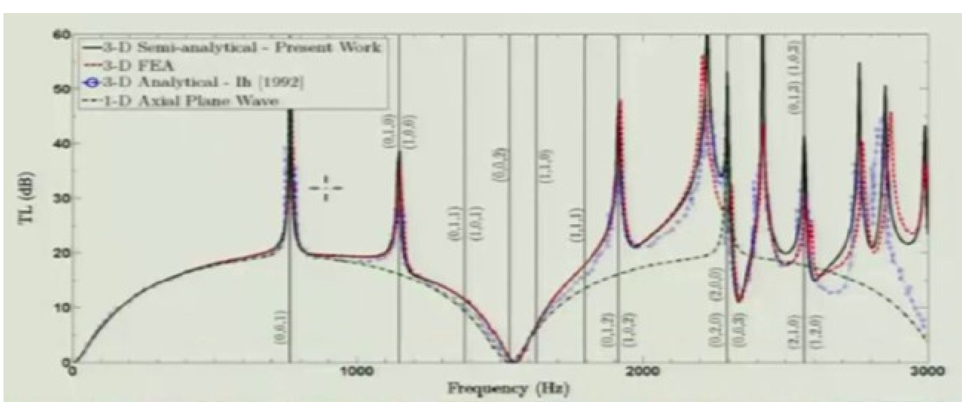


Figure 8: TL performance of the cross-flow configuration shown in Fig.7 having the following chamber and port dimension: B = H = 150mm, L = 225mm,b_0 = h_0=50mm and port location given by x_1= 0.5B, y_1= 0.5H,z_1 = 0, x_2= 0.5B, y_2= 0.5H,z_2 = 0.5L-

comparison of the 3-D semi-analytical approach used in this work, the 3-D analytical approach used by Ih [2], 3-D FEA and the 1-D axial wave theory.

And so, you know for us end in the side outlet thing, we you know if the side outlet port is located l / 2 along the z axis, we can nullify the first axial trough. And you know the location of the centre of the port; location of the port at the centre somewhere here

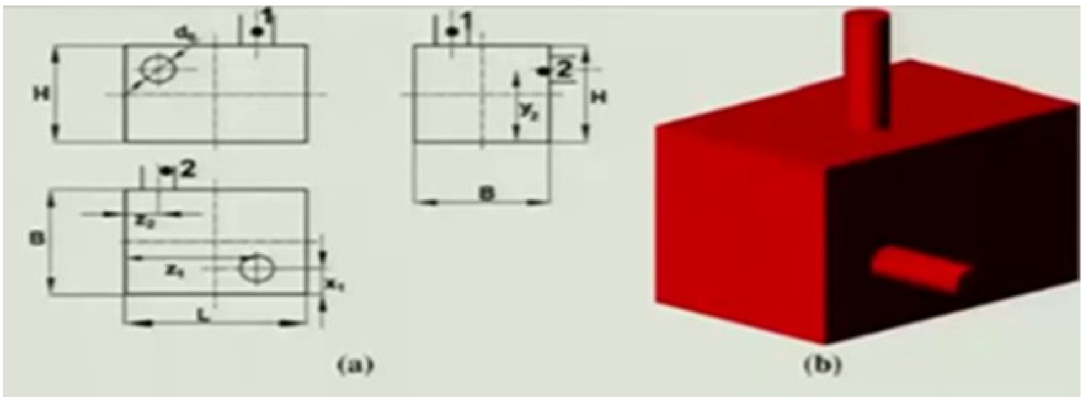


Figure 9: A 2-port cross-flow configuration of rectangular expansion chamber having an end-inlet port (marked as 1) located on the B-L face and a side-outlet port (markedas 2) located on the h-L face a orthogonal projections and b 3-D view.

So, ah, but then this is going to fail at the second actual resonance peak. Other configuration that we have is this end side inlet and a side outlet port and this is going to sort of do well until the 0 to 0 2 to 0 to 0 or 0 0 0 mode, at which frequency is going to fail. Now, note that if for such a side inlet in a side outlet port.

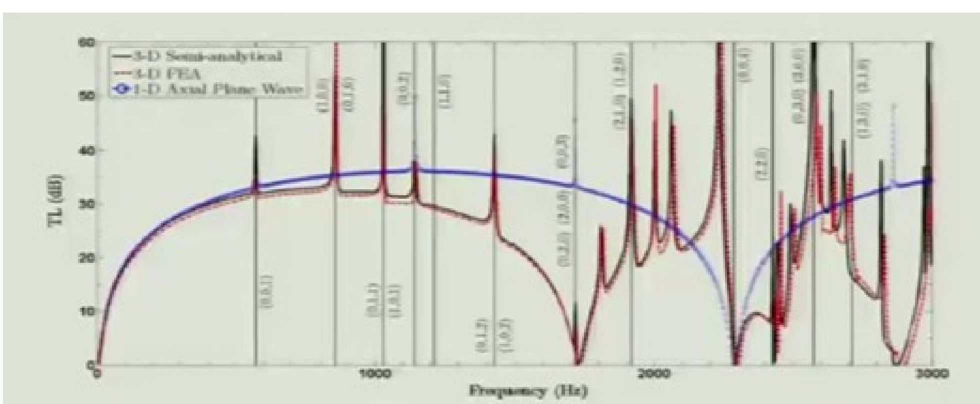


Figure 10: Broadband TL performance of the cross-flow configuration shown in Fig.9 having the following chamber and port dimension: B = H = 200mm, L = 300mm, d_0 =

40mm and side port location given by x_1= 0.5B, y_1= H,z_1 = 0.5L, x_2= B, y_2= 0.5H,z_2 = 0.75L- comparison of the 3-D semi-analytical approach used in this work, the 3-D analytical approach used by Ih [2], 3-D FEA and the 1-D axial wave theory.

One of the ports is located you know at z is equal to L/ 4; other is located L/2 and the you know and on this phase, it is located at you know at right at the mid mid-point of this thing and that is at y is equal to y is equal to H / 2 and similarly, here for this spot, it is located at v/ 2 somewhere here. So, ah. So, basically what we get is. So, basically what we get is that for such a configuration you know we as for a side inlet and side outlet configuration, if this is at 90 degrees you can eliminate you know the plane wave will obviously fail.

But you can sort of by cleverly locating the port not just at l / 4 and l / 2, but also ensuring a 90 degree angle between them and this port being located at centre of the phase, corresponding phase. You can definitely eliminate the troughs say by 0 1 1 or 1 0 1 mode or these modes and obviously, 0 1 2, 1 1 0 0 mode and you can one can do a lot of interesting combinations ok. So, and then, you have single inlet double outlet system.

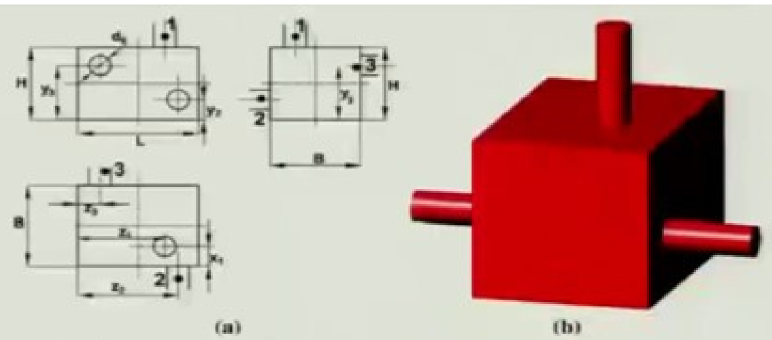


Fig. 11 A 3-port cross-flow configuration of a rectangular expansion chamber having a side-outlet port (marked as 1) located on the B − L face and two side ports (marked as 2 and 3) located on opposite H − L faces: a Orthogonal projections and b 3-D view

So, what we are going to do in the next class is that we are going to derive the we are going to in the next final couple of lectures for the next class combined lectures; combined couple of lectures for the next class, we are going to kind of analyse circular cylindrical muffler ok.

A circular cylindrical muffler with arbitrary location of inlet and outlet ports and using a Green's function representation and then, you know integrating it to get the get the response to a Piston driven model and maybe do some algebra and I will probably run some code for a circular chamber and show you some interesting results that we can get.

So, till that time stay tuned and I will see you very shortly.

Thanks.