Muffler Acoustics - Application to Automotive Exhaust Noise Control
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Lecture - 04
Linearization of Governing Equations, and Development of 1–D Acoustic Wave and Helmholtz Equation

Welcome back on this series of lectures on Muffler Acoustics. We stopped at the momentum equation in the last lecture 3. So, in this lecture, our goal will be to linearize the momentum continuity and momentum equation appropriately by dropping certain higher order terms, and then combining that with your equation of state, which is basically your isentropic equation, to finally arrive at the one-dimensional wave equation.

$$C_0 = \sqrt{\gamma RT}$$

$$C_0 \propto \sqrt{T}$$

$$V \ll C_0$$

$$\tilde{\rho}\ \rho - \rho_0$$

$$\frac{\tilde{\rho}}{\rho_0} \ll 1$$

$$\tilde{\rho} \ll \rho_0$$

$$\tilde{P} \ll (\rho_0\, C_0).C_0$$

So, let us first write down the continuity equation what I basically briefly mentioned. This was your continuity equation that I mentioned let me encircle it again here. So,

$$\frac{\partial \rho}{\partial t} + \rho\,\frac{\partial U}{\partial x} + U\frac{\partial p}{\partial x} = 0$$

here is the total density; similarly, U is the velocity, often in flow ducts we have a mean velocity also and small perturbation velocity.

So, here of course, we will just consider right now we just consider U as the particular city, we will consider a stationary medium that is a look in mufflers of used in automotive application in exhaust system of automobiles there is usually a flow. So, something like there is a constant flow plus there disturbances that are convected in along the direction of the flow.

So, right now to for the development of these equations the basic wave equation without flow. We just going to ignore the mean flow effect and just consider U as the acoustic particle velocity and pressure of course, is the ambient pressure plus the pressure perturbations. So, now, let us consider this continuity equation what I have encircled and go about the process of linearization.

Before we begin the linearization process the following things might be important that

$$\frac{\partial \tilde{\rho}}{\partial t} = 0 \quad \frac{\partial \rho}{\partial t} = \frac{\partial \rho_0 + \tilde{\rho}}{\partial t} = \frac{\partial \tilde{\rho}}{\partial t}$$

$$\frac{\partial \rho_0}{\partial x} = 0$$

Similarly, now when we pull out the continuity equation which is somewhere here and start expanding the terms. There is also another thing that I want to point out is that we are also considering that spatially the ambient density is not changing, that is to say we do not have a stratified medium or well, we are not considering those cases we just considering a medium in which the ambient density of air or any gas does not change. It is uniform density gas with respect to space.

Now, with this when we pull out the continuity equation and use these relations what we eventually get is something,

$$\tilde{\rho}_t + U \cdot \tilde{\rho}_x + \rho_0 Ux + \tilde{\rho}\ Ux = 0$$

So, now, let us underline and identify terms which are first order and which are sort of second order. Now, that is the process of linearization.

So, we immediately recognize that intuition says that $\tilde{\rho}_t$ that is the first order term and so is this term, whereas, these two terms are second order. The reasons can be justified by noting that well $\tilde{\rho} \ll \rho_0$ smaller than the ambient density and your velocity is $U \ll C_0$smaller than the sound speed or in other words,

$$\frac{\tilde{p}}{\tilde{\rho}} \ll 1$$

Now, there might be a concern that if $\tilde{\rho}$ is small, its temporal derivative might not be small. Similarly, if U is small its spatial derivative U_x need not be small. It is a well justified concern and that can be elevated by considering you know more physical argument when like you know in the beginning lectures we talked about the disturbance is propagating at sound speed C_0.

So, if we consider the disturbance,

$$(\delta\rho)_t = \tilde{\rho}_t \simeq C_0 \, \tilde{\rho}_x$$

if this propagates if you take a temporal derivative of this, so this is your rho t approximately since its acoustic disturbance is small speed for small speed equations this would approximately. So, basically what it means is that if we were to basically substitute this relation in the above equation let us see what we get.

So, here we substitute

$$C_0\tilde{\rho}_x + U\tilde{\rho}_x + \rho_0 \, Ux + \tilde{\rho} \, Ux = 0$$

So, this is an important substitution that we made and let us compare the terms. So, clearly, we if we take well this term and this term $\tilde{\rho}_x$ suffix *x* that is derivative with respect to *x*.

So, this term if we take common and C_0 and U and we also made an assumptionthat.

$$U \ll C_0$$

So, clearly this is the dominant term and this term sort of can be ignored and similarly in these two terms if,

$$Ux \, (\rho_0 + \tilde{\rho})$$

So, we will see clearly that this term is a smaller order term and only this term would survive. So, basically what it means is that if you pull out this equation only this particular term would survive,

$$\tilde{\rho}_t + \rho_0 \; Ux = 0 \qquad (12)$$

which is. So, this is what we are going to get.

Another way of looking at this linearization process for the continuity equation is basically to non-dimensionalize a variable. So, suppose if we introduce variable

$$\delta\rho^+ = \frac{\tilde{\rho}}{\rho_0}, \; U^+ = \frac{U}{C_0}, \qquad Z = \frac{t}{t_C}, \qquad z = \frac{x}{C_0 t}$$

With this non-dimensionalization choice what we end up we can put this continuity equation given in here in the following form

$$\delta\rho_\tau{}^+ \; U^+\delta\rho_z{}^+ + \; U_z^+ + \delta\rho^+ U_z^+ = 0$$

So, now as we have already assumed that this quantity has to be much smaller because rho tilde by rho naught is much much smaller. So, what it means is that the quadratic terms that is these terms are much smaller than the linear terms that regardless of the differentiation.

Basically, this resolves the concern that you know in the continuity equation which was presented here the linearized we basically dropped the terms without the consideration of the fact that in the first term here we are differentiating with respect to time t well in these terms we are differentiating with respect to the space x.

So, what if the density acoustic density were to vary rapidly with distance but slowly with time? So, now, that is the reason that we could probably introduce some non-dimensionalization constant and which is your given by this set of variables and what we see immediately that $\delta\rho_\tau{}^+$, this is a linear term and similarly U_z^+ this is also linear term and $U_z^+ + \delta\rho^+ U_z^+ =$ this is clearly a quadratic term and so, is this one.

So, this one so, so, this is the second order term, these two and the first order first order. So, basically that is how we retain this and when we put that back into the dimensionalization constant we get back the linearized version of continuity equation. So, this equation is very important. I would number this as (12) linearized continuity equation.

$$\rho(U_t + U \cdot U_x) + U(\rho_t + (\rho U)x) = -\frac{\partial P}{\partial x}$$

$$\rho(U_t + U \cdot U_x) = -\frac{\partial P}{\partial x} (6)$$

$$\rho \frac{\Delta U}{\Delta t} = -\frac{\partial P}{\partial x} \qquad (7)$$

Similarly, let us pull out the momentum equation now momentum equation which is this. So, let me write it down.

$$\rho(U_t + U \cdot U_x) = -\frac{\partial P}{\partial x}\frac{\partial P}{\partial x} = 0$$

$$(\rho_0 + \tilde{\rho})(U_t + U \cdot U_x) = \frac{\partial \tilde{P}}{\partial x}$$

So, basically what happens now is that if we expand the left-hand side out completely, we would get

$$\rho_0 U_t + \rho_0 U \cdot U_x + \tilde{\rho} U_t + \tilde{\rho} U \cdot U_x + \tilde{P}x = 0$$

Now, clearly let us identify the different orders of term this is your first order term clearly and so, is this one $\rho_0$is ambient quantity and you just differentiating the velocity with respect to time t and here you have your first order quantities and this term $\tilde{p}$ into Ut this is a second order term.

So, I am putting two lines underneath it and so, is the convective term U x into U which I am also putting double line and this quantity is actually a triple order quantity. So, this can be obviously, can be completely ignored.

So, based on the same arguments that I presented for the continuity equation what we could do possibly is get rid of the second order terms initially and then the third order terms as well. Ending up with this one, which is nothing but. So, I am going to call this as equation (13) right equation (13) and this was equation (12).

$$\frac{\tilde{p}}{p_0} = \frac{\gamma \delta \rho}{\rho_0} + \frac{\gamma(\gamma-1)}{2!}\left(\frac{\delta \rho}{\rho_0}\right)^2 + HOT$$

$$\frac{\tilde{p}}{\delta\rho} = \frac{\gamma p_0}{\rho_0}$$

$$\frac{\tilde{p}}{\tilde{\rho}} = \frac{\gamma p_0}{\rho_0} = C^2 (10)$$

$$\tilde{p} = C_0^2 \tilde{\rho} \qquad (11)$$

So, now what we need to do is that this relation equation number (11) and substitute that in equation number 12 and 13 that is to say we will now formally replace all the density variable in terms of the pressure perturbation variables. So, let us begin the stuff for the continuity equation and which basically means is that your continuity equation is equation number (12).

$$\rho_0 U_t + \tilde{P}x = 0 \qquad (12)$$

$$\rho_0 \frac{\partial U}{\partial t} + \frac{\partial \tilde{p}}{\partial x} = 0 \qquad (13)$$

$$\tilde{p} = C_0 \tilde{\rho}$$

$$\tilde{p}_t = C\,{}_0^2\, \tilde{\rho}_t = \frac{\tilde{p}_t}{C\,{}_0^2} = \tilde{\rho}_t$$

So, this we will put back in the continuity equation,

$$\frac{1}{C\,{}_0^2} \frac{\partial \tilde{p}}{\partial t} + \rho_0\, Ux = 0 \qquad (14)$$

So, this is our equation number say (14) and let us bring out our equation number (13) which is we do not need to do any simplifications here in terms of substituting for density. We can just use it let me write it down again here,

$$\rho_0 \frac{\partial U}{\partial t} + \frac{\partial \tilde{p}}{\partial x} = 0 \qquad (15)$$

So, we need to now eliminate. So, I am I would just put it like this we need to eliminate U and put everything in terms of p. So, what do we do? We differentiate equation (15) which is actually the same as equation 13, I just replaced it. I probably not want to do that. I would just differentiate this with respect to the space that is x. To get this, this is square and again differentiate this with respect to time t to get say equations(12)and(14) and this would be (15). Subtracting 15 from 14 we get.

So, we end up with a familiar equation. You can also write this as like this just putting it like this,

$$\frac{1}{C_0}\frac{\partial^2 \tilde{p}}{\partial t^2} - \frac{\partial \tilde{p}}{\partial x^2} = 0$$

$$\frac{\partial^2 \tilde{p}}{\partial x^2} - \frac{1}{C_0^2}\frac{\partial^2 \tilde{p}}{\partial t^2} = 0$$

So, finally, we arrived at the one-dimensional wave equation the hyperbolic equation and now, this is the wave equation because here you have that with respect to time t it like I said like we discussed in the first couple of lectures it models or governs the propagation of pulses dt of pulses or some transient waves, the free waves also if you insist harmonicity you end up with Helmholtz equation and then you can get the force response.

So, now suppose,

$$p = p(x,t) = p(x)e^{j\omega t}$$

where now you have a complex exponential that is cos omega t plus j times sin omega t. So, if you do that, what do you end up with? If you put this equation here, you actually end up with an ODE rather than a PDE Ordinary Differential Equation. You end up with a term called

$$\frac{d^2 \tilde{p}}{dx^2} + \left(\frac{\omega}{C_0}\right)^2 \ p = 0$$

So, omega by C, now we are gradually one by one introducing important notations that we will use throughout the course in muffler acoustics and also in acoustics generally,

$$\frac{\omega}{C_0} = k_0 \quad m^{-1}$$

the wave number k naught is the ratio of the angular frequency divided by the sound speed, where k naught has a unit of per meter because this has unit of per second, this has unit of meter per second; this, this goes away. So, it has units of

$$\frac{s^{-1}}{ms^{-1}} \quad f$$

It is like the spatial frequency just like f or the frequency is how things vary at a particular point or location with time. Similarly, your k naught tells you if you freeze time how does the wave vary in space it is like the spatial frequency.

So, basically this

$$\frac{d^2p}{dx^2} + k_0^2\, p = 0$$

So, this is the Helmholtz equation Helmholtz equation one-dimensional.

Obviously, for 3-dimensional things which we will discuss in the next weeks lecture, we have additional terms the complication arises just by intuition of additional terms. So, right now we just focus on the one-dimensional aspect.

So, with this I think we will stop the lecture for this for this class and resume in the next class where we will take up the cases for the forced response. We will particularly take up the things that we are going to do for forced response like a piston excitation, we will talk about different boundary conditions like rigid wall boundary conditions, open end boundary conditions, talk about the general solution in terms of harmonic waves.

Which is something like I will just give an idea of what we are heading towards something like solutions of the

$$p(x,t) = (Ae^{-jk_0x} + Be^{+jk_0x})e^{j\omega t}$$

So, we will see we can still have waves that propagate along the positive x-direction and the negative x-direction and how they satisfy the Helmholtz equation.

These equations are central to muffler acoustics at least the plane dimensional test case, a plane dimensional frequency propagation wave propagation and we will talk about that in the next lecture.

So, thanks for attending.