Muffler Acoustics-Application to Automotive Exhaust Noise Control
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Lecture - 36 and 37
Concentric Tube Resonator: Partially Perforated Pipe or Airway (MATLAB)

Welcome to week-8 of our NPTEL course on Muffler Acoustics. So, in the since the last week, we have basically move towards move towards the business end of the things you know really stepping into analysis of you know commercially used mufflers which are perforated mufflers.

So, after developing some of the background, some of you know although very briefly the background required for dealing with such stuff where mean flow is present like air acoustic state variables and you know introduction to perforated elements, what we will do in this in this weeks lecture a majority of the part of the lecture will be detailed analysis of **CTRs, Concentric Tube Resonators.**

CTR, concentric tube resonators which is either fully perforated that is basically it is a counter part of a simple expansion chamber concentric simple expansion chamber. And otherwise, it can have only a partially perforated airway or a central duct through which air passes and it is only partially perforated from either sides.

So, this serves to you know develop some sort of a quarter wave resonators or neck extensions at the inlet as well as at the outlet ok. So, the purpose of such a thing is basically so that the resonators that are formed at the inlet and outlet you know at the resonance frequencies of such quarter wave resonators, the attenuation produced is maximum and that would completely annihilate or you know completely overcome the troughs that would form if such extensions were not there.

It is exactly the same as extent inlet and outlet thing where you know the first the peak due to the quarter wave resonate are the inlet is able to nullify or cancel the trough that would occur at the first resonance, whereas the peak at the due to the second or the quarter wave resonator at the outlet you know it is able to nullify the second resonance and so on.

So, we pretty much apply the same technique, but again like I have been mentioning time and again it is basically the perforates that are very crucial and they help in guiding the flow. And we need to develop they act like a bridge really. So, although extent inlet and outlet have been studied a lot relatively lesser focus is there on CTRs with neck extensions.

So, in this lecture, we will go through a detailed development of the transfer matrices for such a thing incorporating the effects of mean flow, and you know you probably go to MATLAB to help us with some of the parametric studies where mean flow is involved. And we can also possibly take into account the effect of end corrections at least for the stationary media ok.

$$\frac{d}{dz}\{\dot{X}\} = [B^{-1}]_{4\times4}\,\{A\}_{\,4\times4}$$

$$\{X\} = \begin{Bmatrix} \tilde{p}_1 \\ \rho_0 C_0 \widetilde{U}_1 \\ \tilde{p}_t \\ \rho_0 C_0 \widetilde{U}_2 \end{Bmatrix}$$

$$A = \begin{bmatrix} -jk_0 - \frac{4}{d_1\zeta} & 0 & \frac{4}{d_1\zeta} & 0 \\ 0 & -jk_0 & 0 & 0 \\ \frac{4d_1}{(d_2^2-d_1^2)\zeta} & 0 & \frac{-jk_0-4d_1}{(d_2^2-d_1^2)\zeta} & 0 \\ 0 & 0 & 0 & -jk_0 \end{bmatrix}_{4\times4} \begin{matrix} C_1 \\ M \\ C_2 \\ M_2 \end{matrix} \begin{Bmatrix} p_1 \\ \rho C_0 U_1 \\ p_2 \\ \rho_0 C_0 U_2 \end{Bmatrix}$$

$$B = \begin{bmatrix} M_0 & 1 & 0 & 0 \\ 1 & M_0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{matrix} p_1 \\ \rho_0 C_0 U_1 \\ p_2 \\ \rho_0 C_0 U_2 \end{matrix}$$

$$\frac{d}{dz}\{X\} = [B^{-1}]_{4\times1}\,[A]_{4\times4}\,\{X\}_{4\times1}$$

$$\{X\} = \begin{Bmatrix} \tilde{p}_1 \\ \rho_0 C_0 \tilde{U}_1 \\ p_2 \\ \rho_0 C_0 \tilde{U}_2 \end{Bmatrix}$$

In the last class, I guessed we let me go to the last slide. So, we stopped somewhere here ok, B inverse you know B will be when Mach number is 0 that is basically you know your this case Mach number when it is 0, it will not be an identity matrix because of the particular arrangement of the state variables that is alright. But, in any case, even when M naught is 0, this is not you know inverse it will be; it will be 1. So, there is no problem in that. When it will be not 1, then we have to invert this matrix.

So, you know for argument sake let us call this B inverse A you know let us call this

$$\frac{d\{X\}}{dz} = [C]_{4\times4}\{X\}$$

This is what we get. We get this part. So, the question now arises how do we solve it? So, typically what is known as a generalized decoupling method and other things in the literature is basically you know the same thing can be solved in using just one line command in MATLAB using a exponential function, matrix exponential function.

So, basically you know let me do some simple linear algebra things. So, e to the power you we already know the function, is not it?

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

So, if a matrix is now raised to the power you know e to the power A. A can be some matrix for argument sake, we will keep this as c matrix ok. So, this is given by defined as

$$e^C \equiv I + [C] + \frac{[C]^2}{2!} + \frac{[C]^3}{3!} + \cdots$$

And how do we compute that? There is a separate matter you need to find out the eigen values and eigen values matrix when it is written side by side inverse of that into the you know it is basically you know part of the linear algebra. So, I will not go into the fact that how this guy is computed. It is defined like that or it is defined like that. But in MATLAB what we just need to do is that

$$D = expm\ (C) \Big| \ [X] = e^{[C]z} \Big| \{Q\}$$

So it will basically shoot out you some matrix D is equal to expm c. So, it will compute this using some algorithm that is there in it.

Why is this useful to us? Now, because let me take a small digression here. Let us say we have a simple first order ODE you know. So, what is the solution of this ok?

$$\frac{dy}{dx} = \text{ay}$$

$$y = \alpha\ e^{ax}$$

The α is the constant which needs to be determined from the one boundary condition. Because you know if you put this thing LHS and right hand side will be the same where alpha is a arbitrary constant.

Now, similarly, if we were to you know propose a solution or the form of this equation you know X vector is nothing but your say you know e to the power c matrix times z. Remember we need to integrate with respect to z. So, here we have this. And the constant let me call that constant you know some number let say Q ok. Q is a basically a vector of unknown constants that is basically,

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

So, this is the solution.

Why is this a solution? Because it is easy to see that I mean if we were to substitute this solution into this one LHS and RHS would be the same. You will eventually get back c matrix times this thing because suppose if you differentiate we expand this guy using this formulae, and differentiate with respect to z. So, we get

$$I + [C]z + \frac{[C]^2 z^2}{2!} + \cdots$$

So, if we differentiate, this term will be 0. And what will we get is basically,

$$= [C] + \frac{[C]^2 z}{1!} + \frac{[C]^3 z^2}{3!} + \cdots$$

$$= [C] \left\{ I + [C]z + \frac{[C]^2 z^2}{2!} \dots \right\}$$

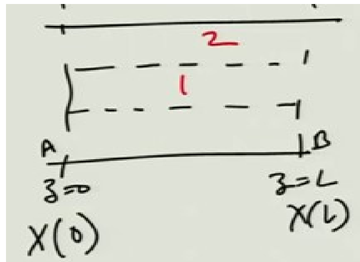
So, and eventually we can just take c common, and realize and this will be z here, and realize that z square I am sorry for that that we are eventually left with this thing c square by z square by 2 factorial and so on. So, let me move to the other page. So, we will be getting

$$[C]\, e^{[C]z}\{Q\} = [C]\{X\}$$

So, if you differentiate this, so this is nothing but this underlying terms only from here to here this is nothing but c matrix times you know X because that is what we proposed, is not it? So, this is X. So, basically if we differentiate this guy with respect to z, we will get back this solution, and this happens to be this particular guy this happens to be a RHS.

So, in short what I am trying to say is that this solution this, this particular form of the solution always satisfies our differential equation ODE a set of linear first order differential equations. So, if we do the matrix exponential, we will get this. And in MATLAB like I said all we need to do is like it will you know pop out some matrix D is equal to exponential expm, not exp just exp m c. So, you can have fun with this thing. And Q is a constant, we needs to be determined.

So, let us say we have this solution. So, we will move on from here. Now, our job now is to determine. So, let us go back let me draw it again for you anyways. So, this is a perforated bridge section, is not it? And we have the duct 1 here and 2 here. Annular part let say we want we are fixing a coordinate system z is equal to here, at this point and z is equal to L ok.



So, now what do we do? We are fixing a coordinate system at these points mentioned here right point A and B, z = 0 and z = L. Our first job now is to determine the constant Q in terms of the state variable; state variable at we will call simply this as X(0) and X(L) ok. So, we know the solution in the form this thing.

$$[X]\, e^{[C]x0}$$

which is you know your identity matrix.

$$[X] = [I]\{Q\}$$

$$\{X\}_{z\infty} = \{Q\}$$

$$\left\{\frac{X}{z}\right\} = e^{[C]z}\,\{X(0)\}$$

$$\left\{\frac{X}{L}\right\} e^{[C]L}\,\{X(0)\}$$

So, I will simply write this as identity matrix times the vector {Q}. Now we want to relate things at z is equal to L. So, what we need to do is that at z is equal to L, this becomes e to the power *C* is L we will simply multiply the matrix *C* each and every element of the matrix *C* with L, and multiplied here this is your vector here alright.

So, eventually you know what do we get? So, you know if you recall transfer matrix representation; obviously, you know relates things at the upstream with those at the downstream. So, we need to invert it. we relate things with those at z is equal to L . What happens

$$\{X(0)\} = \left[e^{[C]L}\right]^{-1}\{X(L)\}$$

So, we will sort of get this. So, this can obviously, So, we take this inside this guy inside and ok. So, we do get that.

$$\{X(0)\} = e^{[C]L}\{X(L)\}$$

So, we get this representation ok. This becomes very simple now. Just in MATLAB you would just probably do

$$expm\ (-(C)L)$$

So, we can just multiply the number L, length L, or the perforated section to each and every element of the c matrix that is basically B inverse A. And put a minus sign and just you know give this command. So, you know this what this whatever this guy is you know whatever this thing we can for just for our convention, we can just for our simplicity what we can do is that we can say this is your let

$$\{X(0)\}_{4\times1} = [T]_{4\times4}\ \{X(L)\}_{4\times1}$$

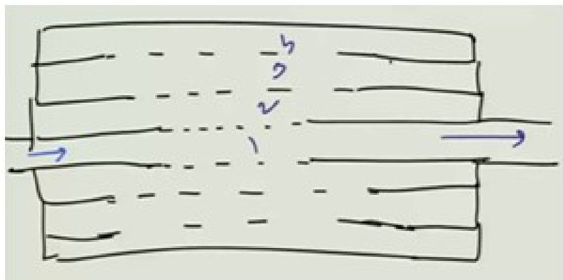
Let us expand this out and you know let us

$$\begin{Bmatrix} \tilde{p}_1 \\ \rho_0 C_0 \widetilde{U}_1 \\ p_2 \\ \rho_0 C_0 \widetilde{U}_2 \end{Bmatrix}_{z=0} = \left[\begin{array}{c|c|c|c} T_{11} & T_{12} & T_{13} & T_{14} \\ \hline T_{21} & T_{22} & T_{23} & T_{24} \\ \hline T_{31} & T_{32} & T_{33} & T_{34} \\ \hline T_{41} & T_{42} & T_{43} & T_{44} \end{array}\right] \begin{Bmatrix} \tilde{p}_1 \\ \rho_0 C_0 \widetilde{U}_1 \\ p_2 \\ \rho_0 C_0 \widetilde{U}_2 \end{Bmatrix}_{z=L}$$

So, let this is an expanded out form of this guy. And here we will have this is at z = 0, and $\rho_0 C_0 \widetilde{U}_1$ z = L. So, this is your fully expanded out form ok. So, for simplicity, let me put this dotted lines ok.

Now, where do we go from here? So, what needs to be done at this stage? So, we have got the solution at least symbolically, we would need to use computer algebra otherwise hand calculations even for a simple two dock interacting system can get very messy.

So, you know often you have multiple interacting that is in principle you know you its possible to get things like this kind of a thing you have multiple you know perforated duct something like this, and you have like this. And then you finally, have this kind of thing. It forms basically you know your multiple extensions, so multiple neck extensions are there in principle you can have any. So, you can have systems like this, and the flow comes in from here.

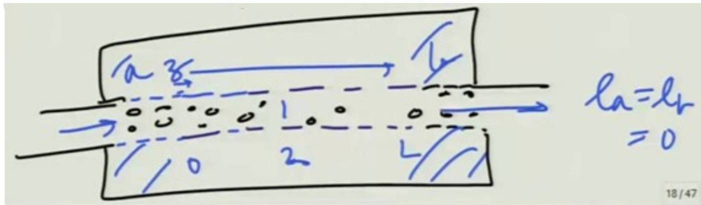


And let me just sort of complete this figure and similar thing would sort of exist here we have this kind of a thing ok. So, you can have multiple such things. Imagine if you were to do all this by hand calculations alright, things would become just impossible you know solving like this is like 1, 2, 3, 4, so each duct has two variables. So, you know it is something like you know 8, 4, 4 into 2 that is 8, 8 cross 8 matrix.

So, you know it becomes almost impossible to deal with such a coupled 8 cross 8 system or the 8th order differential equation if we were to combine all of the variables into 1. So, you know it becomes quite messy and then that is the reason that it is best to work I would prefer working in terms of a set of coupled linear ODEs and keep it first order.

So, you know MATLAB, but any other algorithm would do it pretty efficiently for you even if we do not rely on this thing, you can always write algorithms to you know find out matrix exponential. So, the idea is definitely use some computational algorithm rather than you know seeking analytical solutions because perforated ducts really require that especially when you have mean flow.

So, I will keep this figure you know just like that, and sort of move on. So, you know I would like to remind ourselves back that this is the system that we originally intended to study, we intended to study this ok, we intended to study this aright. So, how do we go about this? Now, we have got this system, and we would like to impose the boundary conditions. Now, it is the time to do that.



Now, clearly you know as you can see from the figure that this is the partially perforated section. So, at z is equal to 0, at z is equal to L, we have this. So, what about the annular cavities 1 and 2 or let me 1 and 2 is the use for this duck. So, let me use some other term we will call it a and b.

So, if for the portion a that is the cavity formed here at the extension at the inlet if the length l a and l b are 0 you know then it obviously, reduces the case to the case of a fully perforated expansion chamber fully perforated concentric tube expansion chamber is something like this. It is fully thing.

So, let me let me just draw this thing again for you guys. So, it is something like this, apologies for the color. We have holes everywhere perforates right from the beginning all the way until the end. So, things would become sort of fairly simple. Let me derive you know this sort of a thing for you guys and then we will perhaps do some you know get back to this equation again and in involved in neck extensions ok.

So, we will do that. So, what happens here really, so you know U_2 is the stuff where we need to apply the boundary conditions remember this is the duct. So, you know really what we have is that to $U_2 = 0 \;\; at \; z = 0 \; and \; z = L$ because your you know your this thing is kind of squeezing or basically you have kind of extended the duct perforated section throughout the chamber. So, your z really z is equal to 0 starts from one of the faces, and z is equal to L is at the other end face alright.

So, you have this sort of a thing. What it means for the transfer matrix that we have derived here let me use the red color. So, U_2 at z is equal to L is 0 as we have just noted here this is the condition ok. So, what it would mean is that we do not have to deal you know this is multiplied by this, this with this, this with this, but then this guy is 0. So, you know we can get rid of this particular column ok because we do not have to do with deal with this.

Similarly, U_2 is also 0 at z is equal to 0. So, what it would mean is that this into this plus this into this plus this into this is 0, but we would like to sort of relate you know these two guys with these two guys. So, although we have eliminated U_2, but we still have T_{13} multiplied with p_2, and T_{23} multiplied with p_2 also. And we have one more relation p_2 is related p_2 at z = 0 is related to the other variables in the downstream using the third this thing.

But in the process we also found out another relations you know T_{41} multiplied by p_1; T_{42} multiplied by this $\rho_0 C_0 \widetilde{U}_1$; T43 multiplied by p_2 linear combination of this is equal to 0. From that we can get some equations for p_2 I mean express p_2 in terms of these two quantities, and then basically put that p_2 in the first two equations and simplify things.

Let us sort of get started with it that we will put

$$T_{41}\,\tilde{p}_1 + T_{42}\,\rho_0\,C_0\widetilde{U}_1 + \,T_{43}\,\tilde{p}_2|_{z=L} = 0$$

So, what this would mean is that $\tilde{p}_2$ is your nothing but

$$=> \qquad \tilde{p}_2|_{z=L} = -\frac{\left(T_{41}\,\tilde{p}_1 + T_{42}\,\rho_0\,C_0\widetilde{U}_1|_L\right)}{T_{43}} \qquad (1)$$

So, this happens and T_{43} ok. So, this is there, and so we get this guy. And this is like I said at z = L ok.

So, now, we will keep this equation. Let us call this equation (1). We will keep this equation (1) aside. And in the other chunk of equations you know now we need to focus really on this guy, this entire thing is multiplied by this thing. So, basically p_2 will come in here.

So, let us do the maths. Let me go to another slide. So, where I put

$$\begin{Bmatrix} \tilde{p}_1 \\ \rho_0 C_0 \tilde{V}_1 \end{Bmatrix}_{z=0} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \end{bmatrix}_{2\times 3} \begin{Bmatrix} \tilde{p}_1 \\ \rho_0 C_0 \tilde{U}_1 \\ \tilde{p}_2 \end{Bmatrix}_{z=L}^{3\times 1}$$

as something like this is going to be a bit messy. So, I am kind of forewarning you about the things. So, be a bit patient with the simplifications. What I would strongly recommend to all of the students here in the class specially the ones who are certain to take up the final exam and doing assignments is that please work out the algebra on a piece of paper.

And you know the using the long hand approach that is very important; otherwise you will probably never be able to understand or appreciate things you know, otherwise it is just like this. So, this is z = L. And of course, I am sorry there is a mistake. So, I should have written this is the last. So, this is p_2 my mistake sorry.

Now, you know using equation (1) we need to eliminate p_2 this is of course, at z = L. So, p_2 thing will be put in here. So, let us sort of do that its going take the entire thing. So, I am not going to bother.

$$= \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \end{bmatrix} \begin{Bmatrix} \tilde{p}_1 \\ \rho C_0 \tilde{U}_1 \\ -\frac{T_{41}}{T_{43}} \tilde{p}_1 - \frac{T_{42}}{T_{43}} \rho C_0 \tilde{U}_1 \end{Bmatrix}$$

And now it is the time to substitute all these things in here.

So, I am not writing at z = L it is sort of very well understood, so minus T_{42} by T_{43} at T_{41} this thing. And so we get this thing. So, what this thing is there? Now, we have really this equation ok. So, now, we need to do term by term multiplication of each of the parts and rearrange the things.

So, what I am going to do is that I am just going to go like this and start to multiply,

$$T_{11}\tilde{p}_1 + T_{12}\, \rho C_0 \tilde{U}_1 - \frac{T_{13} T_{41} \tilde{p}_1}{T_{43}} - \frac{T_{13} T_{42}}{T_{43}} \rho C_0 \tilde{U}_1$$

And now multiply this chunk with the entire thing. So, this is going to be the part that requires a little bit of patience and a lot of concentration. So, my request again is that please do these derivations for yourself on a piece of paper, maintain a separate notebook those who are at least taking exams and are serious about this one.

So, let us first simplify this particular first row. So, this would or rather let me just sort of write down the second row itself, and it is then we will simplify all of that together.

$$T_{21}\tilde{p}_1 + T_{22}\,\rho C_0 \widetilde{U}_1 - \frac{T_{23}T_{41}\tilde{p}_1}{T_{43}} - \frac{T_{23}T_{42}}{T_{43}}\rho C_0 \widetilde{U}_1$$

Now, we can comfortably combine these guys into the following. So, we get

$$\left(T_{11} - \frac{T_{13}T_{41}}{T_{43}}\right)\tilde{p}_1 + \left(T_{12} - \frac{T_{13}T_{42}}{T_{43}}\right)\rho C_0 \widetilde{U}_1$$

So, this is the 1st equation. The other one is

$$\left(T_{21} - \frac{T_{23}T_{41}}{T_{43}}\right)\tilde{p}_1 + \left(T_{22} - \frac{T_{23}T_{42}}{T_{43}}\right)\rho C_0 \widetilde{U}_1$$

So, this is your second row. So, now I guess we are in a position to relate things that the upstream of a CTR with those at the downstream, so ok. So, we get this ok. And here of course, we get p_1 at z = L ok. This is how it looks like. Now, the big transfer matrix is finally reduced to a 2 × 2 matrix, where we have related things what goes on at this point to this point where your z is really stretched, z becomes here at 0 and z = L here.

$$\begin{Bmatrix} \tilde{p}_1 \\ \rho C_0 U_1 \end{Bmatrix}_{z=0} = \begin{bmatrix} T_{11} - \frac{T_{13}T_{41}}{T_{43}} & T_{12} - \frac{T_{13}T_{42}}{T_{43}} \\ T_{21} - \frac{T_{23}T_{41}}{T_{43}} & T_{22} - \frac{T_{23}T_{42}}{T_{43}} \end{bmatrix}_{2\times 2} \begin{Bmatrix} \tilde{p}_1 \\ \rho C_0 \widetilde{U}_1 \end{Bmatrix}_{z=L}$$

So, it is a fully perforated pipe, and we have related things just here to that somewhere here z is equal to 1 to z is equal to L. And so this is how it looks like in terms of flow. Well, flow is there implicitly because you know you if you recall we are calling this entire guy [B^{-1}] inverse [A] as C and e to the power C is z that is your T matrix. And so all these manipulations will eventually you know lead you to this form. So, now since mean flow is there and as we can see you know your diameter of the inlet and outlet is also the same. So, the transmission loss expression would really basically you know sort of result back to the one that is used for classical state variables because you remember you have a term,

$$\frac{1+M_i}{1+M_0}\begin{pmatrix}\cdot\\\cdot\end{pmatrix}=[T^1]\begin{pmatrix}\cdot\\\cdot\end{pmatrix}$$

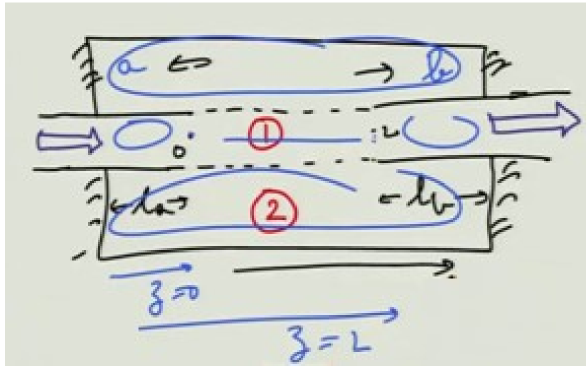
Means inlet and outlet or maybe the other way around, the main thing is that these things will cancel out because they are the same.

So, eventually in terms of the final matrix, you can call this matrix as T1 matrix. So, you know you can call this entire thing is T_1. So, T_1 matrix which relates things here. So, the transmission loss will be completely decided by this particular matrix and it you can use the expressions for the classical state variables as well in this case.

So, we have analyzed at least symbolically a fully perforated concentric tube resonator. What we now have to do is basically the way we will go about doing the derivations from now is that we will sort of get the things for the extensions when you have extensions at the inlet and outlet. So, there what are the boundary conditions and how do we go about that ok.

So, let me sort of do the derivation again for you, but with sort of different boundary conditions. So, it is going to be much more tedious than this case because now you have your other impedance expressions in terms of cot or tan and all that sort of a thing. We will probably put that later, but the fund a is the same, the main idea fundamentals are the same.

So, let us do that. So, what we are going to do now is analyze basically figure out the transfer matrix for a concentric tube resonator, but with extensions at the inlet and outlet. So, we probably we are looking for a solution of this sort, but only thing is that now we have our neck extensions also there.



So, let me draw clean rather clean image of this, and somewhere like in here, something like this ok. So, the flow goes something like this and these here ok. So, what we want is the stuff between this is z = 0, z = L. Like I said this part is a, b. So, it is quite clear.

Now, that you know this is duct 1, duct 2 alright. And this can be circle or elliptical in shape whatever it is. There are couple of things that you know sort of needs to be clarified right at the outset is that you know looking into the cavity you know velocity direction is taken positive along the along this direction that is along the direction that I am pointing somewhere here.

So, looking into the cavity like this is basically what you will do is put the impedance

$$\frac{\tilde{p}_2(0)}{-U_2(0)} = -j\rho_0 C_0 \cot k_0 la$$

but here you will have a minus sign because it is looking into the cavity. So, this is equal to your $-j\rho_0 C_0$ note that it is not y naught because you have we are kind of changing the acoustic velocity variable. It is not a mass velocity any longer; it is a non or velocity variable as the same dimension as pressure. So, it is cot k naught l a. So, this length is l a ok.

So, this length is l a and l b. So, we have this sort of a thing. And

$$\frac{\tilde{p}_2(L)}{-U_2(L)} = -j\rho_0 C_0 \cot k_0 lb$$

where lb is the length like I mentioned. So, this of course, we are looking into the cavity like this alright.

So, with this boundary condition not here we do not have U_2 and U_2 at 0 equal to 0 neither we have U_2 at z is equal to L 0. Although I must mention here that these plates these end plates, they are rigid so that much we know of. What does this mean? That is why we have this cot a naught expression. But now using let us say the boundary condition 1 and 2, we would like to eliminate some variables that is your p_2 and rho naught C_0 U_2 at z = 0, and z = L, we would want to eliminate that. And let us see how we sort of go about our business. So, well we have this. So, well in terms of p_2 at z at z is equal to 0, we have this thing. So, what we really

$$\begin{Bmatrix} \tilde{p}_1 \\ \rho_0 C_0 \widetilde{U}_1 \\ p_2 \\ \dfrac{p_2}{j\rho_0 C_0 \cot k_0 la} \end{Bmatrix}_{z=0} = \begin{bmatrix} T_{11} & T_{12} & T_{13} & -T_{14} \\ T_{21} & T_{22} & T_{23} & -T_{24} \\ T_{31} & T_{32} & T_{33} & -T_{34} \\ T_{41} & T_{42} & T_{43} & -T_{44} \end{bmatrix} \begin{Bmatrix} \tilde{p}_1 \\ \rho_0 C_0 \widetilde{U}_1 \\ p_2 \\ \dfrac{p_2}{\rho_0 C_0 \cot k_0 lb} \end{Bmatrix}_{z=0}$$

I guess, but you also have a minus sign sitting in here. So, this would be plus alright. And this entire thing can be written something like this.

Note that the transfer matrix parameters themselves would not really change is just that the boundary conditions are different, so eventually the final matrix T_1 matrix that will change. So, we have this $\tilde{p}_1$ $\rho_0 C_0 \widetilde{U}_1$ $\tilde{p}_2$.

Now, again here we need to do the same trick you know rho naught C naught U 2 goes on the numerator, and this guy comes down. So, this would be $p_2 / \rho_0 C_0 \cot k_0 lb$, and here will be minus. So, we have this sort of a thing.

So, what we can possibly do is that you have a minus sitting in here ok. So, if we take the minus here, and minus here, here, here, then we can possibly get rid of the terms in here, is not it? So, this is $\cot k_0 lb$ sorry for my handwriting, but for it is. So, we have done then well so far now what we really need to do is as usual we need to relate these two guys with these two guys. And in the process, eliminate p_2 somewhere here. Anyways we have reduced now the things to only three variables. So, we need to wisely use the equation.

So, as usual we start from the this row, so this, so this one into this, this into this, T_{13} into p_2 and T_1 T minus 14 and 2 again. So, p_2 can be taken common in this set of equations and also in this one. And then we have sort of another relation. We can choose whatever we want in terms of we have actually variables here. So, we have two more equations. Let us sort of cleverly do it. It is going to be a little bit involved algebra. So, please bear with me in the derivation.

$$\begin{Bmatrix} \tilde{p}_1 \\ \rho_0 C_0 \tilde{U}_1 \end{Bmatrix} \left[\begin{array}{c|c|c} T_{11} & T_{12} & T_{13} + T_{14} j tan k_0 lb \\ \hline T_{21} & T_{22} & T_{23} + T_{24} j tan k_0 la \end{array}\right]_{2\times 3} \begin{Bmatrix} \tilde{p}_1 \\ \rho_0 C_0 \tilde{U}_1 \\ \tilde{p}_2 \end{Bmatrix}_{z=L}$$

So, I am just writing 0 here with the understanding that this is z, z is equal to 0. T_{11} into p_1 and T_{12}. And now what we have to do is T_{13} into p_2 and minus T_{14} into p_2 $\rho_0 C_0$cot $k_0$this thing.

So, we can now really combine these two things. So, T13 and minus T14 $\rho_0 C_0$ minus T_{14} into p_2. So, p_2 is common $\rho_0 C_0$ and cot becomes tan cot becomes tan ok, cot $k_0 l$b ok. And this guy becomes at z = L alright. So, and here it is the same old business, and here also divide by $\rho_0 C_0$ what we will sort of get is p_2 is common.

So, cot goes in the numerator cot goes in the numerator. So, here we will have tan, but I guess I probably making a small mistake. I realize now sorry it is a small mistake. So, here I think I kind of did something very elementary mistake. This cannot be this thing. So, here we have this and similarly we have p_2. My sincere apologies for the small error, but we will sort it out now in the, this equation.

So, basically what happens is $\rho_0 C_0$ still we would want $\rho_0 C_0$. So, basically we have minus minus plus, so the only thing is that this guy would not be there, we will just have a j sitting in here, is not it? We will just have a j sitting in here. So, $\rho_0 C_0$ is multiplied by U_2 at z = 0. This comes in denominator j. So, $j cot k_0 l$, and it goes in the numerator.

So, it becomes it becomes really tan later on. And here also we can simply sort of get rid of this guy. And there is obviously, a j sitting in here which I also forgot my apologies again, but I will make comments minus sign is taken that is correct. And what it would really mean is that coming back from this equation minus t one four times p 2 and this thing.

So, cot you know goes in the numerator it becomes tan, and j is multiplied in the numerator denominator becomes minus j ok. So, it becomes so and obviously, we need to get rid of this guy is not quite correct. So, going back and checking things minus T14 times this. So, cot becomes tan as we can clearly see here and T14 is multiplied by tan and p2 is taken out common j is a denominator j multiplied j. So, it becomes minus j because minus j square is there. So, this is all good.

Similarly, for the 2nd equation is also alright. So, we have this my sincere apologies once again for the elementary mistake here, but this is what we get. So, things are looking good, they are looking sort of a bit more manageable. This is your 2 cross 3, this is 3 cross 1 matrix. So, all good. Now, we still have a bit more work to do because this is mind you at z = 0, and this is at z = L. So, these this variable is different from this one; in the sense that we are relating things at different sections.

So, what do we do now basically let us sort of write it down and you know we would basically eliminate you know kind of p_2 at z = 0 from using either of the relations. So, let us use the third row thing.

$$p_2|_{z=0} = \left(T_{31}\ p_1 + T_{32}\ \rho_0 C_0 \widetilde{U}_1 + (T_{33} + jT_{34}\, j tan k_0 lb\ )p_2|_{z=L}\right)$$

I am just dropping the tilde sign for now, it is more convenient to do. So, actually it will become actually plus here plus here because j is multiplied j square minus, minus minus is cancelled. So, you get basically just j, $tan k_0 lb$ or something like that. So, here also we get pretty much the same thing, but what we get is also j and this is j where j is root over minus 1 ok. And this is this thing. And this guy is sitting here basically what it would mean is that T_{34} times j $tan k_0 lb$ that needs to be added to this guy to get us p_2.

So, this is all at z = L. So, we have this sort of relation. Now, we can directly use this guy put it here I mean put the right hand side, I mean this thing this entire thing here, and expand it out and then we can completely eliminate, but be prepared for a rather nasty algebraic simplification. So, it does require some work. So, what we would do is write down the set of equations.

So, p 2 tilde at z is equal to 0 we can just directly copy this guy

$$T_{31}p_1 + T_{32}\,\rho_0 C_0 U_1$$

$$+\frac{(T_{33} + j\tan k_0 lb\, T_{34})\tilde{p}_2}{j\cot k_0 la}$$

So, this thing is divided by $j\cot k_0 la$ ok, So, we just simply copied p_2 what we got from this equation here into this guy in the numerator and just divide by $j\cot k_0 la$. So, there is no problem with the minus sign here.

And once we do that, we would just keep it aside for the time being, and go back perhaps to this equation and expand this out completely. So, now all these variables mind you at z = L ok. And we have to you know patiently term by term multiply,

$$= T_{41}p_1 + \rho_0 C_0 U_1 T_{42}$$

And T_{43} + j times T_{43} you know here it is the same thing minus is there. So, multiplied by j square minus minus cancelled, $jtan\ k_0 lb$ times T_{44} is what you get.

And P_2 is again taken common ok.

$$+(T_{43} + jtan\ k_0 lb\ T_{44})p_2$$

So, we get this. This should be abundantly clear. So, now, and I think, I would like to drop off this term because it is understood that all the variables in this equation are at z is equal to L section. So, how about I sort of just multiply throughout with $j\cot k_0 la$, and then go about simplifying things. So, with the idea that you know this thing p_2 has to be expressed in terms of this thing. So, we could do that. So, basically what it would mean let me do it on another piece of another slide. So, the idea is that

$$\tilde{p}_2 = \frac{(-jtank_0 la\, T_{31} - T_{41})\tilde{p}_1}{X'}$$

And similarly, your the other term that

$$+\rho_0 C_0 U_1\ \frac{(-jtank_0 la\, T_{32} - T_{42})}{X'} \qquad (2)$$

So, we get this p_2 thing, is not it? We get this p_2 thing. So, this entire thing this entire expression let me call this equation (2) is being it has to be substituted in place of this thing ok. And then we again regroup terms in terms of p_1 and $\rho_0 C_0 U_1$.

So, it becomes you know as you can see its already getting quite complicated we it is a good idea maybe to involve you know computer algebra like for example, maple or something like that to simplify matters or at least just write a MATLAB code, and let the code do all the simplifications for a given frequency. So, we could do that. And eventually what we would do I mean once we get p_2 in terms of the big expression that we just discussed now we regroup.

And at the end of the day, we will get a simplified well, notionally it is simplified analysis is basically,

$$\begin{Bmatrix} \tilde{p}_1 \\ \rho_0 C_0 U_1 \end{Bmatrix}_{z=0} \begin{bmatrix} T'_{11} & T'_{12} \\ T'_{21} & T'_{22} \end{bmatrix} \begin{Bmatrix} \tilde{p}_1 \\ \rho_0 C_0 U_1 \end{Bmatrix}_{z=L}$$

So, T1 matrix is different from here T just the T matrix, it is this thing $\rho_0 C_0 U_1$ at z = L, and this is z = 0 ok. So, at the end of the day, what we have done we have related things just at the point here see the point at you know 0 the point at L.

Taking into account all the complex things that happen in this annular region and the propagation that coupling due to this thing. Obviously, these two regions are uniform, so it would not matter much. So, it would not matter at all apart from change in phase. So, here to here we have related the transfer matrix. And using once we get this T matrix, we can sort of do a lot of things we can find out the transmission loss expression, I use this team equation for transmission loss as that used for the classical state variables to get the stuff.

Now, you know what really we have to do from now is that if you have more complicated things like cross flow expansion chamber and all that, I will in the next class I will probably or maybe next to next class these are lectures 1 and 2 of week-8. So, in the lecture 4, perhaps I would talk about other expansion elements what I am going to do tomorrow I mean in the lecture-3 is you know evaluate do some MATLAB demonstration.

Basically I will and actually before I do that, I will talk about some present some popularly used perforate impedance expression that is available in the literature. A very brief historical review or probably just mentioning some important ones you know starting from the work by Ingard and Munjal and Salamat and other guys Elnady, and they have been lot of expression.

So, I will kind of present those expressions to you and the situations in which they have to be used only for the case of well grazing flow that is what we are dealing here. Let us remind ourselves we are dealing with only grazing flow. So, I will present those expressions. And some of them actually involve mean flow, some of them do not really involve mean flow, but they have other effects.

Well, once we do that, we would go to a MATLAB code and type in the appropriate perforate impedance expression ς what we have been figuring it out in the T matrix and which is in inherently presented and incorporates all the boundary conditions, and possibly analyze your fully perforated CTR with and without flow. And also present the analysis of CTR with extensions at the inlet and outlet neck extensions are extended inlet and outlet the perforated bridge without flow.

But, in this case, I had also introduced some empirical expressions for end corrections which is popular, and then possibly do some mean flow parametric studies I guess, but there is lots to be done even after doing that because this is one of the basic elements. So, you know let us get sorted out with this first.

And possibly in tomorrow's lecture that is lecture 3 and also involving perhaps lecture-4, may be lecture-3 and 4 would be combined and then we will see what we will do in the lecture 5 for this week. So, till that time stay tuned, and I will see you in the next class.