Muffler Acoustics - Application to Automotive Exhaust Noise Control

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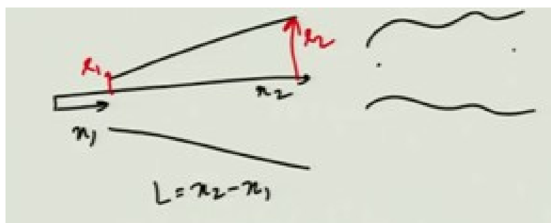
Lecture - 29
TL Analysis for Conical Muffler Configurations (MATLAB)

Welcome to lecture 4 of week 6 on NPTEL course on Muffler Acoustics. In this class we are going to talk about derivation of Transfer Matrices for Conical Mufflers, transfer matrix for conical mufflers and or conical sections I would say, something like this, where you probably know x_1 and x_2 and you know r_1 and r_2 and L would be x_2 - x_1.



Let us derive the transfer matrix, but before we do that it might just be a good idea to talk about transfer matrix for any arbitrary thing in terms of the generalized functions. So, let us assume that we have our function

$$\tilde{p}(x) - G_{11}(x) \cdot C_1 + G_{12}(x)C_2$$

$$\tilde{V}(x) - G_{21}(x) \cdot C_1 + G_{22}(x)C_2$$

$$\begin{Bmatrix} \tilde{p}(x) \\ \tilde{V}(x) \end{Bmatrix} = \begin{bmatrix} G_{11}(x) & G_{12}(x) \\ G_{21}(x) & G_{22}(x) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix}$$

Now what we probably can do is that we need to eliminate the constant C_1 and C_2 and I express everything the state variables p_1 and V_1 mass velocities with those at p_2 and V_2. So, this actually can be written in a compact form. Let us call this thing as a state vector S of x.

$$=> \ \{S(x)\} = [G]\{C\}$$

So, what will this mean?

$$\{S(0)\} = [G(0)]\{C\}$$

$$=> \ \{C\} = [G(0)]^{-1}\{S(0)\}$$

$$\{S(l)\} = [G(l)][G(0)]^{-1}\{S(0)\}$$

Now C can be substituted back in this equation to yield S x S of x the vector is given by your G times x matrix. Remember this was the function of x although not written here explicitly, and S of 0 like this.

And let us evaluate this as

$$[G(0)]\ [G(l)]^{-1}\{S(l)\} = \{S(0)\}$$

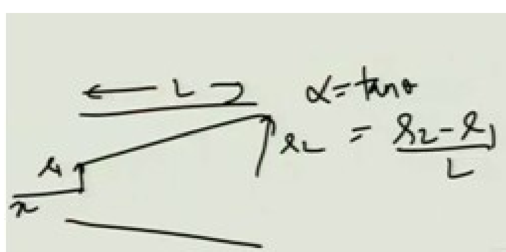
So, basically we have related the state variables at

$$\underset{x=0}{\{S(0)\}} = \underbrace{[G(0)]\ [G(l)]^{-1}}_{[T]}\ \underset{x=l}{\{S(l)\}}$$

$$[T] = [G(0)][G(l)]^{-1}$$

So, the idea is that how do we use this generalized expression for the transfer matrix we write the transfer matrix for the conical chamber, so conical section I would say. So this is like this; *T* matrix is nothing, but *G* of 0 into *G* of *l* whole inverse.

This form would be particularly useful in the functions that we talked about here; these ones, all these are a bit tedious or algebraically complicated. When G_{11} and G_{12} for the conical section that we saw here.



$$\frac{d^2q}{dx^2} + k_0^2\, q = 0 \qquad where \quad q = q(x)$$

$$=> \qquad q\,(x) = C_1\, e^{-jk_0x} + C_2\, e^{jk_0x}$$

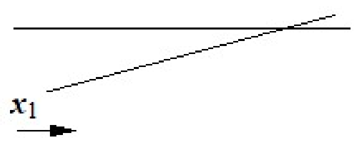
$$=> \quad \tilde{p}(x) = \frac{1}{x}\,\left(C_1\, e^{-jk_0x} + C_2\, e^{jk_0x}\right)$$

This does not appear to be too complicated, but when you talk about the expression for the mass velocity there are few extra terms here, so that becomes a bit tedious.

$$\rho_0 \frac{\partial \tilde{U}}{\partial t} = -\frac{\partial \tilde{p}}{\partial x}$$

$$\rho_0\; j\omega\tilde{U} = -\frac{\partial}{\partial x}\left\{\frac{1}{x} C_1\, e^{-jk_0x} + \frac{1}{x}\, C_2\, e^{-jk_0x}\right\}$$

$$\rho_0\, \tilde{U}(x) = \frac{j}{\omega\rho_0 x}\left\{\left(-jk_0 - \frac{1}{z}\right) C_1 e^{-jk_0x} + \left(jk_0 - \frac{1}{z}\right) C_2 e^{jk_0x}\right\}$$



So, what I will do? I will not probably go through the entire derivation of the transfer matrix for this thing, rather I would just write down the transfer matrix for the conical duct which has the following form.

Transfer Matrix for a Conical Duct a Conical duct

Where,

$$\begin{Bmatrix} \tilde{p}(x_1) \\ \tilde{V}(x_1) \end{Bmatrix} = \begin{bmatrix} \frac{x_2}{x_1}\; cosk_0 l - \frac{1}{k_0 x_1} \sin k_0 l & jY_2 \frac{x_2}{x_1}\; \sin k_0 l \\ \frac{j}{Y_2}\, \frac{x_1}{x_2}\left(1 + \frac{1}{k_0^2\, x_1 x_2} \sin k_0 l\right) & \frac{1}{k_0 x_2} \sin k_0 l \\ \frac{-j}{k_0 x_2 Y_2}\left(1 - \frac{x_1}{x_2}\right) \cos k_0 l & + \frac{x_1}{x_2} \cos k_0 l \end{bmatrix} \begin{Bmatrix} \tilde{p}(x_2) \\ \tilde{V}(x_2) \end{Bmatrix}$$

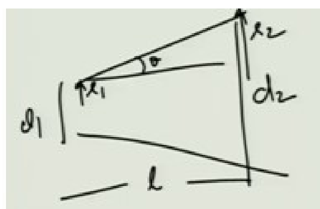
So, remember there is no mean flow that is considered here, and the next thing would be a little bit more involved. So, well what I would do is that I will put this thing here.

$$l = x_2 - x_1$$

So, this is related to the variables at x 2. So this; obviously, has a transfer matrix form like this; tilde, tilde, tilde and tilde. So, these two terms; so what are those? So, those are We get this and then what do we get?

So, this is pretty tedious, algebraically this is a little tedious and what do we do for the other term that is the term that is present here.

So, it is a pretty tedious expression compared to the case for a uniform. From an engineering point of view, rather than working in terms of the hypothetical apex and all that sort of a thing, its a bit more convenient to talk in terms of the radius at the two sections; r_1 r_2 or d_1 and d_2 and the length l.



So, where,

$$\tan\theta = \frac{r_2 - r_1}{l} \; and,$$

$$= \frac{d_2 - d_1}{2l}$$

So, let me call

$$M = 2\tan\theta = \frac{d_2 - d_1}{l} \; and$$

let me call this as the slope M. Let me have a look at this.

What do we get here? p at let us say, when we use these variables rather than x_1 and x_2 and we further make an assumption that, well not assumption is basically your some relation,

$$x_1 = \frac{d_1}{M}$$

$$x_2 = \frac{d_2}{M}$$

$$Y_2 = Y(l) = Y_1 \left(\frac{d_1}{d_2}\right)^2$$

$$d_2 - d_1 \;\; M = 0$$

The idea is, that if we use such a variable representation we will get; we will get the following. So, this would become d_2 by d_1, where d_1 is a smaller part remember and

$$\begin{Bmatrix} \tilde{p}(0) \\ \tilde{V}(0) \end{Bmatrix} = \begin{bmatrix} \frac{d_2}{d_1} \; cos k_0 l - \frac{M}{k_0 d_1} \sin k_0 l & jY_1 \frac{d_2}{d_1} \sin k_0 l \\ \frac{j}{Y_1} \frac{d_1}{d_2} \left(1 + \frac{M^2}{k_0^2 \, d_1 d_2} \sin k_0 l\right) & \frac{M}{k_0 d_2} \sin k_0 l \\ \frac{-jM}{k_0 d_0 Y_1} \frac{d_2}{d_1} \left(1 - \frac{d_2}{d_1}\right) \cos k_0 l & + \frac{d_1}{d_2} \cos k_0 l \end{bmatrix} \begin{Bmatrix} \tilde{p}(l) \\ \tilde{V}(l) \end{Bmatrix}$$

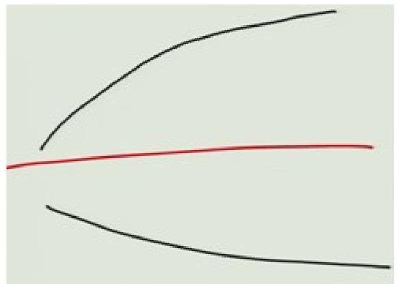
$$M = 0$$

We get this a little tedious looking expression, although not very difficult, and so this would be your l and, so we will get this particular thing.

So, this is the transfer matrix for a conical duct. Now, here we have your m the slope or the twice the slope, when m is 0 its very easy to see that you know $d_2 = d_1$ when M = 0 right. So, in that case this d_2 is d_2 by d_1 is 1, and this term would go away and so would all the terms pertaining here, here and this would be unity.

So, this would I would cancel out and you will get the familiar things and similarly this is also 1, this will go away and this will be 1. So, you get back your transfer matrix for a uniform duct, and that is a check of self consistency what we have been always been insisting upon.

Similarly, one can derive transfer matrix exact transfer matrix for a exponential duct with which is a bit more simpler than the conical one and then you know exact analytical solutions do exist for a duct which is of a you know parabolic and hyperbolic shape. So what it basically it means is that, if we have a duct whose cross section is varying in terms of a parabolic kind of a thing or hyperbolic kind of a thing, that is **Parabolic duck.**

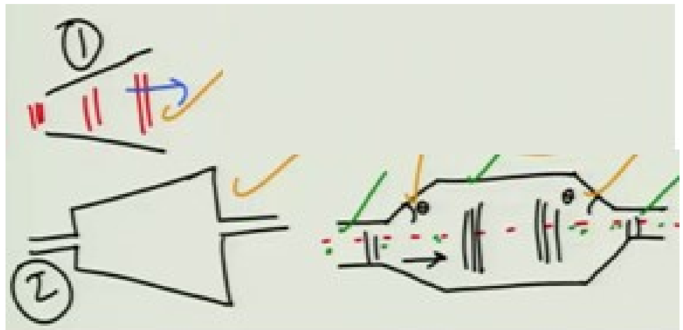


So, it is something like this. This is just a schematic or a rough figure. So, you have this kind of a thing for a parabolic shaped duct and how does the diverging or a converging **Hyperbolic Duct**.

So, it looks like a diverging thing. Similar to your exponential thing, but the way the profile varies is a bit different. So, the reason that I brought up this topic is because such systems, they have exact analytical solutions although a bit complicated in terms of the Bessel function series or second function. So, I would suggest that interested readers to find out about the parabolic duct and the hyperbolic duct solutions for the one dimensional wave equation. It is algebraically very tedious although exact solutions exist?

So, we might just be wondering why we even do such things where obviously, you know computational tools are available nowadays. And actually that will be the focus of our next lecture when we talk about the segmentation approach or a approach which is a kind of a numerical scheme to analyze ducts in which the cross section area gradually varies.

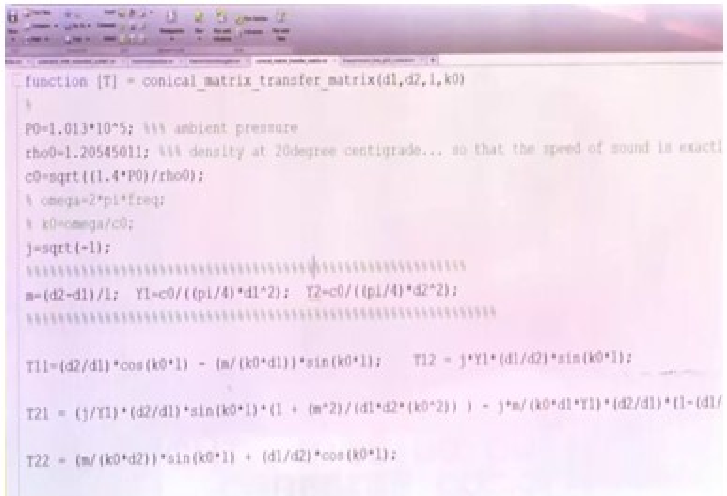
So, what we will do now is that, we will focus our attention back on the transfer matrix here and worry about the transmission loss calculations for such a thing. So what we will do is that we will analyze the following configuration. Let me write it let me make these things in a compact manner somewhere here only.



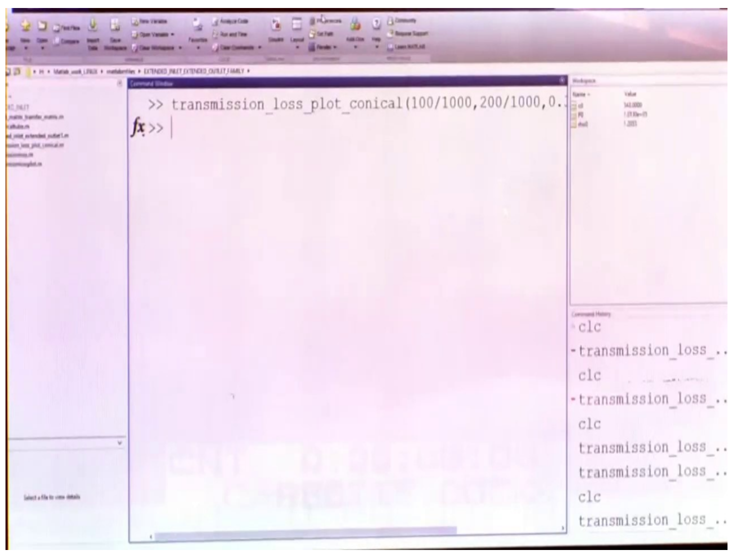
We will consider just a duct here and whatever waves are coming which is going here in this direction. And the other configuration we will be worrying about is sudden expansion and so this is 1 this is 2. Often we have things like this. Where this angle is given here. So, what we could do for such more complicated test cases is the you know make use of the transfer matrix between say this section and this section this section and this one, this one and this one then this then this one and this finally, from here to here.

So, it becomes as simple as cascading or sequentially multiplying the transfer matrix for a given element. So, these are clearly uniform sections where I have marked with the orange thing, these are diverging or a varying section the converging or diverging nature. So, one can figure out the transmission loss of such a composite kind of a configuration or a simple thing.

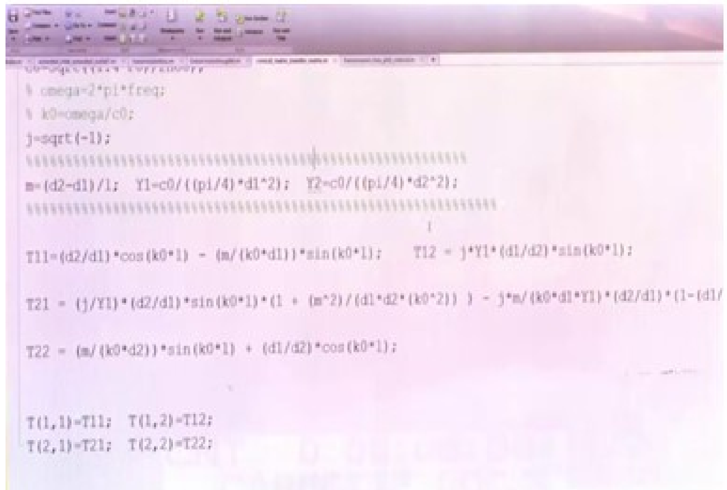
So what we are going to do in the remainder of this class is to derive, perhaps the transmission laws for this system and this system and make some general remarks and in the next class we will do segmentation. So, let me open up the MATLAB for you.



As usual this is the MATLAB command screen.



And the usual practice in MATLAB is to write function files that is the most efficient way. So, what I have done quickly now is that, all the parameters that I talked about if you look at the screen here, this one all this big matrix that we discussed in a while back. So, what I have done is that I have used this form of this form of a matrix and I have just simply written down the four pole parameters.



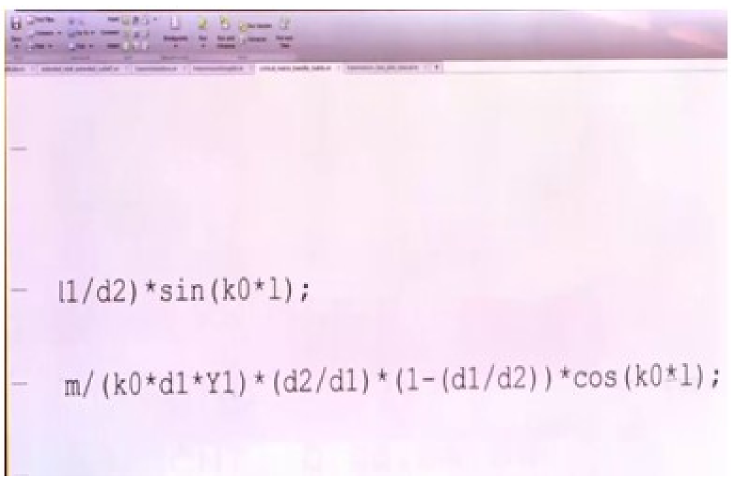
So, T_{11} is basically the one that is highlighted here. So, not sure if you guys can see this properly or not. So what I am going to do very quickly is that zoom it to the maximum allowable limit.

So, perhaps this would be much better. So, T_{11} is your sorry, is this parameter. So, notice that the function starts with the, this is basically a module. So, here I am assuming a little bit background on MATLAB programming or any programming for that matter. So, we would need certain variables. So, variables pertaining to the computation of sound speed is what is being declared at the top and then the sound speeds is given by square root of gamma P_0 divided by ρ_0.

And then k naught is the variable wave number that is made available at the beginning of the program, that is the calling function, the function that invokes this particular module already has access to k_0. So, we have k_0. So, very quickly now, T_{11} is basically d_2/d_1 into cos k_0l. So, d_2 remember was the diameter at the section 2 and d_1 was the diameter at section 1, so and l is basically x_2 - x_1 that is the length of the axial length of the chamber not the inclined length axial length.

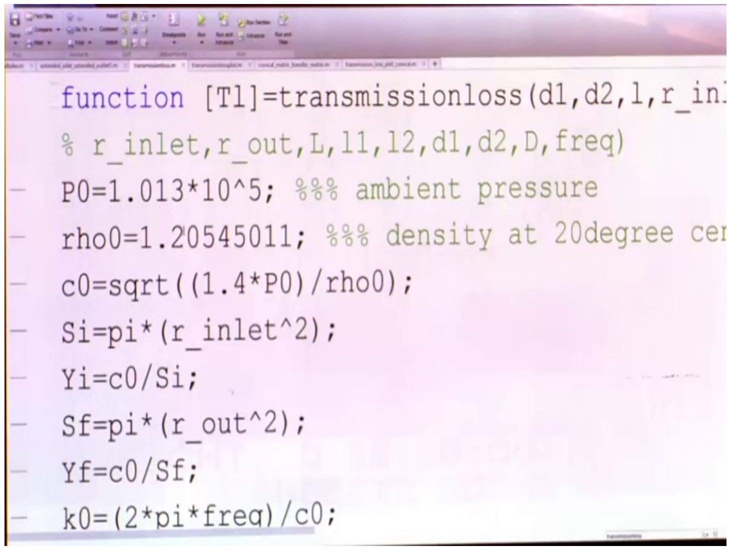
So, this is what it was and then m was, m was also defined I guess somewhere here that is basically d_2 - d_1 / l and the characteristic impedance at the respective section; section 1 and 2 are given by 1 and 2. Ordinarily I would write it, I would kind of comment it and that is the advice that I would give to you students as well, but it is just written in a less time the score, so there were no comments.

But, anyways T_{11} is the first affordable parameters that like we discussed and T_{12} is your $j \times Y_1$ that is characteristic impedance at section 1 into the ratio of d_1 / d_2 into sin k_0* l, it was that thing. And now, T_{21} j into Y_1 into d_2 / d_1 into sin k_0*l and all the terms that follow.

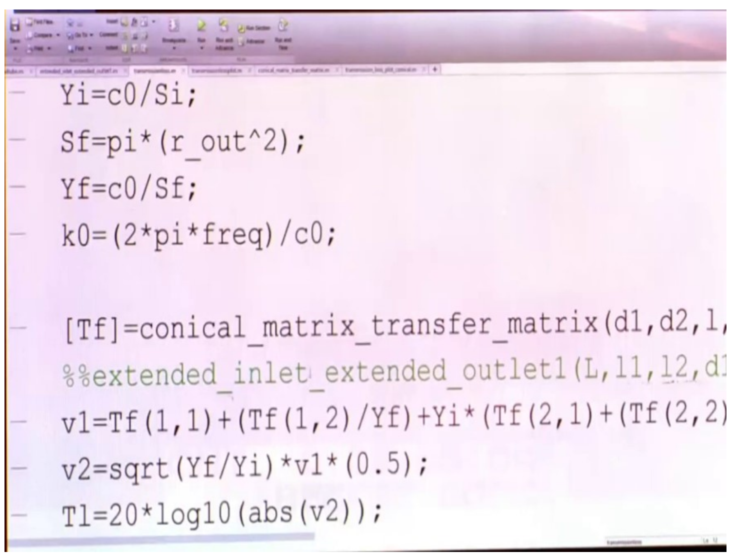


So, what I suggest is that you guys do some practice on your own in the sense that the derivation look at the notes, look at the screen carefully and copy the this thing and also you can refer to the monograph by Munjal, Acoustics of Ducts and Mufflers. On the second chapter these expressions are given and I guess you would also be able to derive on your own.

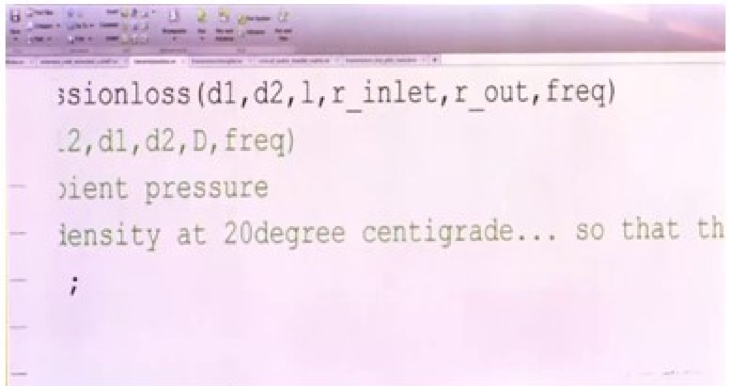
So, anyhow these are the four pole parameters what I have mentioned here and they are nicely written into a matrix form. They are just variables at this stage, but now they have put in a matrix form and this matrix is then passed on to the function that invokes this particular routine sub function. So, this T variable is the matrix which is for a given frequency and contains all the four pole parameters information. Now have a look at this transmission loss.



So, I kind of modified the code that I used for extended inlet and outlet element.



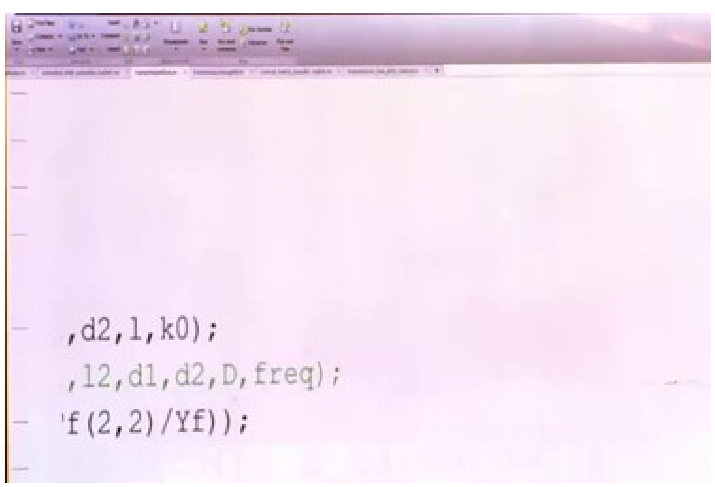
That is why some things are commented out. So, these are the things that I would need at the calling.



So, notice one crucial thing that I would like to point out is r inlet and r outlet. So, what it means is that you know, if you recall this section, let me get back to the screen.

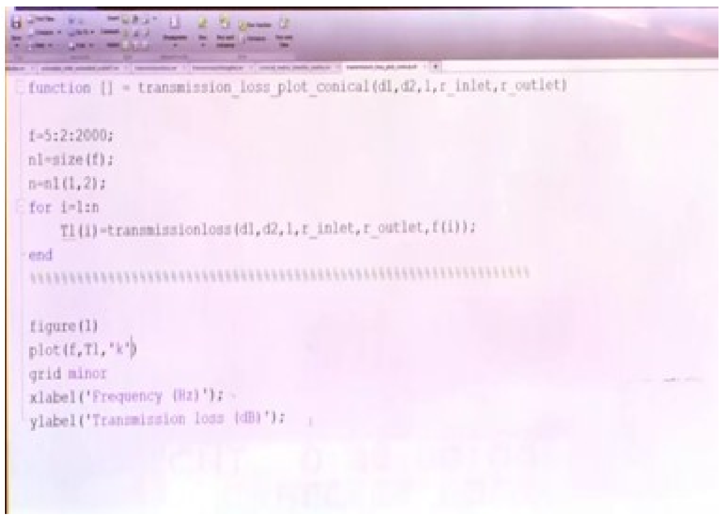
This was your d_1, this one and this was your d_2. Now, once we have this, you know this r underscore inlet and r underscore out they need not be the same as r_1 and r_2. So, it can be something like this, like I mentioned here. If this is r inlet and this is r outlet and, but; however, if they are the same that is to say if r inlet and that is your if you have a duct like this, you know then r inlet and r_1 would be the same and r outlet and r_2 will be the same value.

So, basically by cleverly choosing these values we can analyze both the configuration using the same code. So, this is the area of the inlet pipe and characteristic impedance, and similarly for the other one. And conical matrix transfer matrix has given is basically the function is invoked.

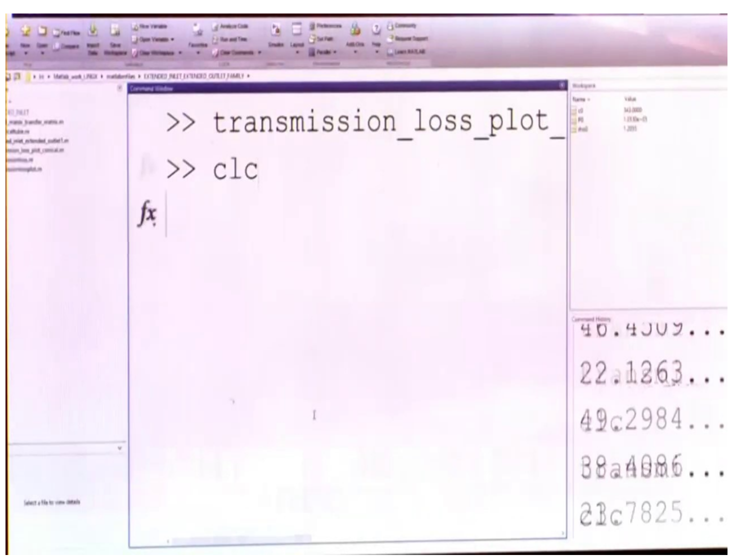


And now comes the computation of the parameters transmission loss value. So, this is something that we discussed in last weeks lecture. So, transmission loss in terms of the transfer matrix parameters basically transfer matrix parameter. So, these are basically this one. Now notice that Yi and Yf, that is your nothing, but these ones. So, this is what is multiplied Yi is multiplied throughout.

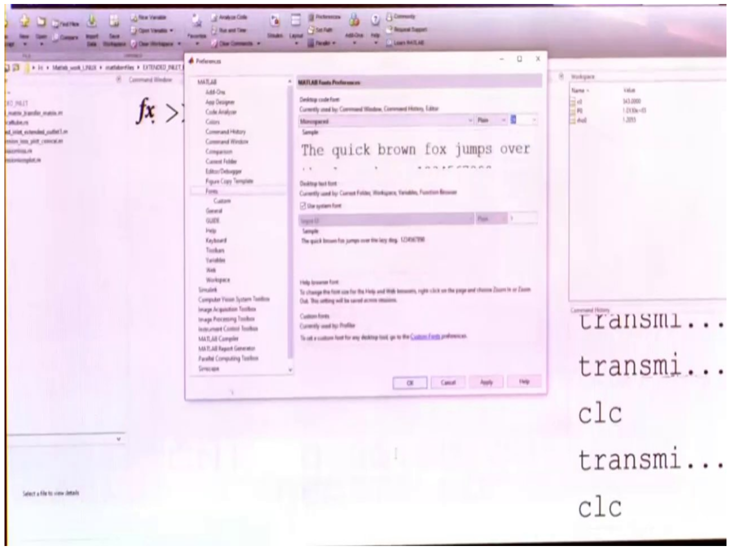
Now, clearly when Yi and Yf are is the same, then this is just unity this will cancel out. So we can do a whole lot of things. And then finally, these parameters and transmission laws then this functions is invoked by the main calling routine.

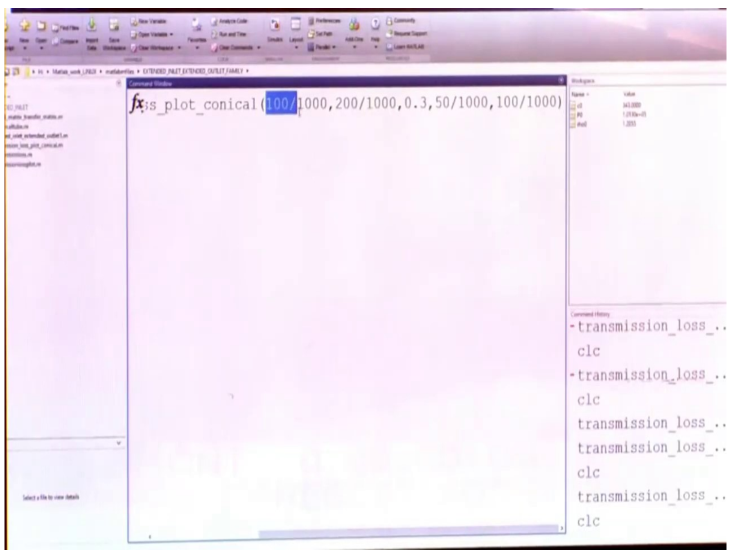


Transmission loss plot conical duct for conical. So, what I am going to do is that minimize this.



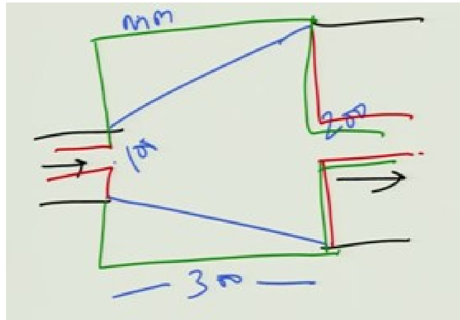
And perhaps you know get back to the old thing say 24 font and.



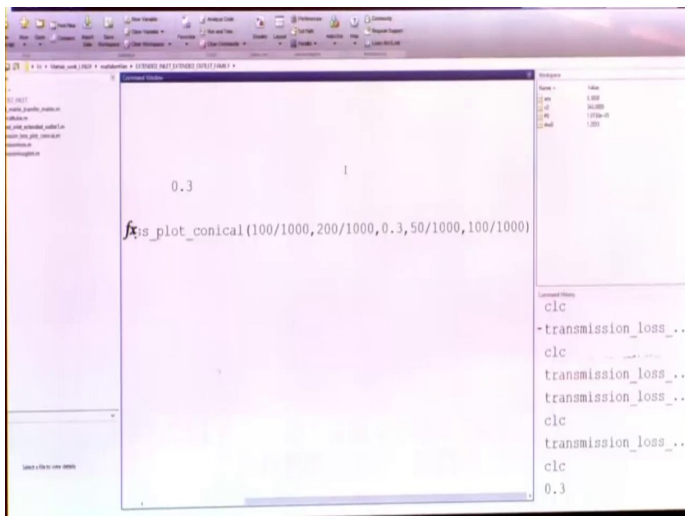


So, let us see configuration which say the inlet pipe is 100, that is basically let us analyze the configuration in which let us have a look at the screen here.

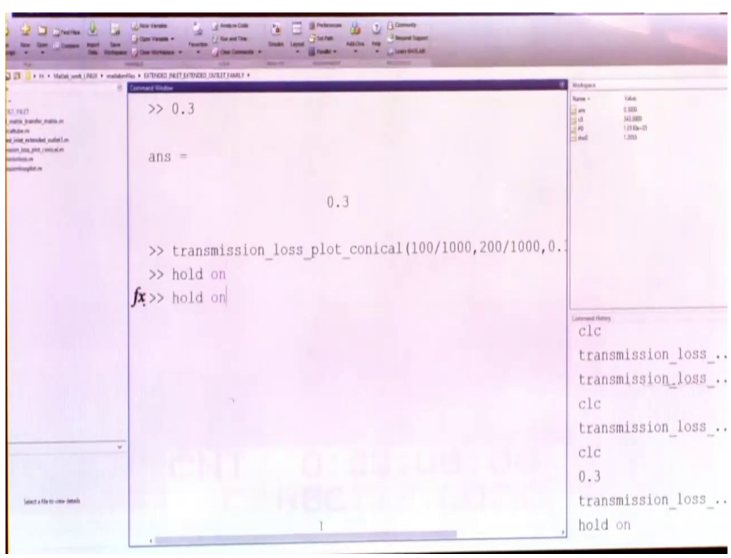
So, we are analyzing a configuration where this diameter is 100 mm all dimensions in mm here what I am writing on the screen, but I have divided by 1000 to make it an meter there. So, this is 200 mm, this is your 300 mm length and so that is what is highlighted here, 200 mm and 100 mm length here not sure if you can see and 0.3 is the is the length and 50 mm and 100 mm are the inlet an outer diameters in which case we are considering 50 and 100 we are actually considering a configuration like this.

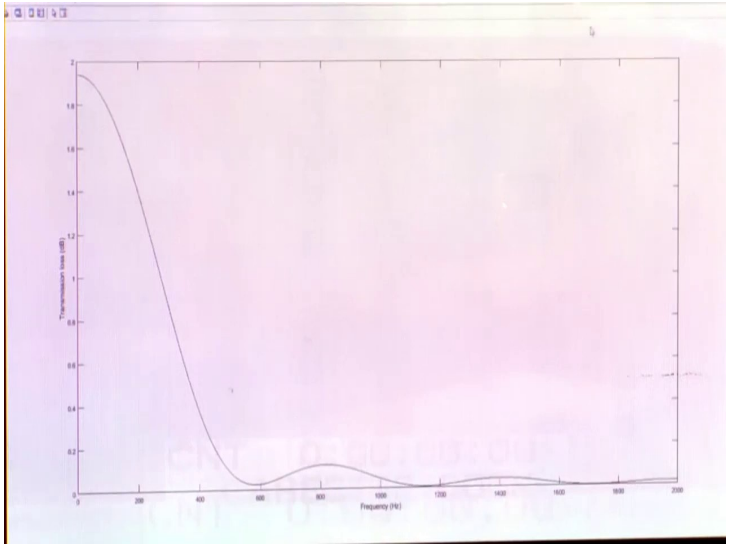


Exactly it. The waves come here so we are just trying to study the effect of flare on the transmission loss.



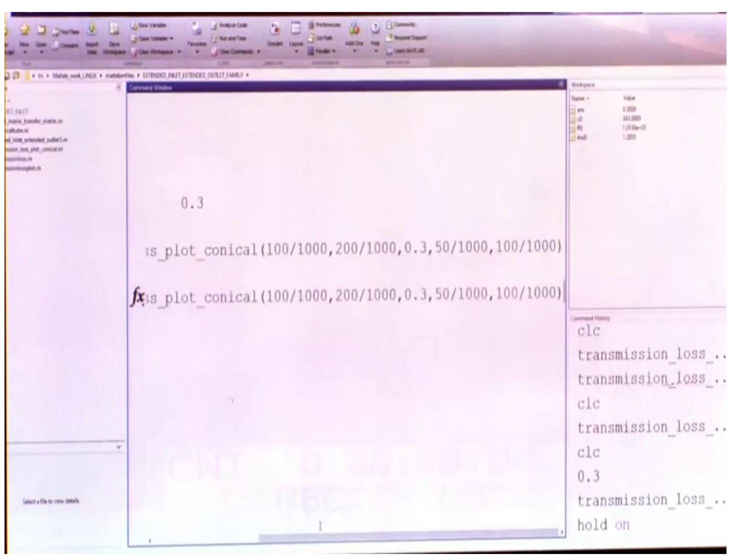
So, once I hit enter, what do I get? I get this.



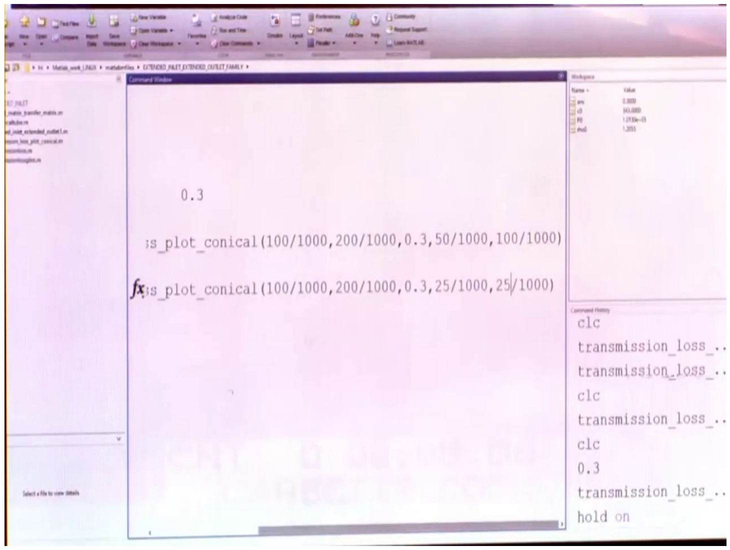


So, basically its transmission loss is nonzero. That is important to know it is very very negligible. Basically, such a flare would introduce all was negligible transmission loss generally throughout the frequency domain by itself it is very inefficient, so it has to be supplemented by sudden area expansion. Those are much more effective than gradual area this is what I am trying to say.

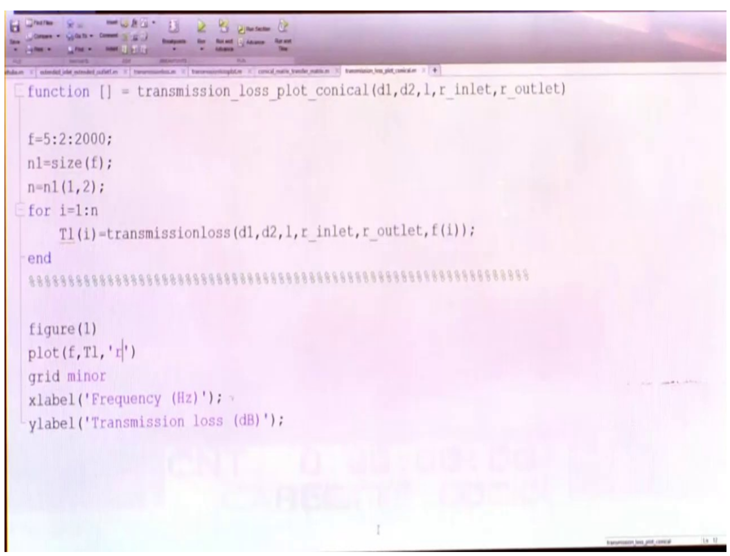
So, it has a peak or it starts it is gives a maximum value at 0 frequency, but you know then there kinds of it kind of has a maximum, but the die is down very rapidly. Now I will, what I will do is, I will hold this thing on. I will do hold on just to demonstrate what sudden expansions can do for you.



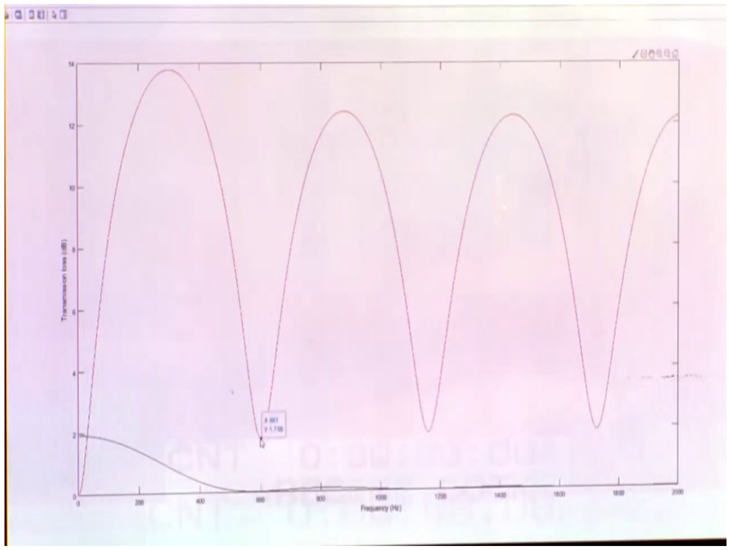
So, you know 100 mm. Now, let us have a look at the screen again and let us analyze the following configuration, you know this one. So, here at the inlet and outer section the chamber has diameter 100 mm and 200 mm respectively, but inlet and outlet pipe diameter they can be much smaller.



So, you know they can be something like 50 mm. So, 25 mm is the radius at the inlet and also is the outlet as in the usual case. And let us use the somewhat different color, let us use the red color ok.

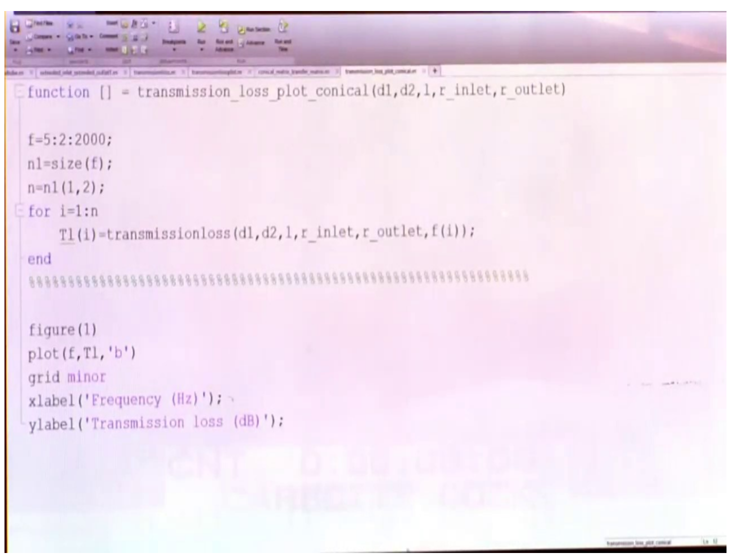


So, we get the following.

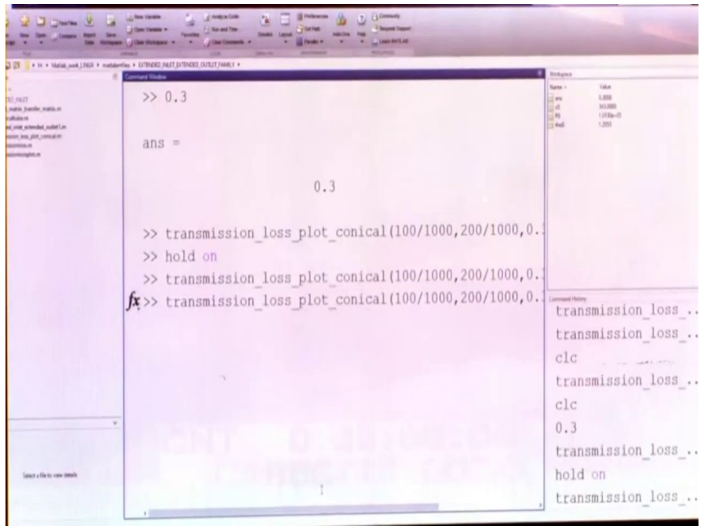


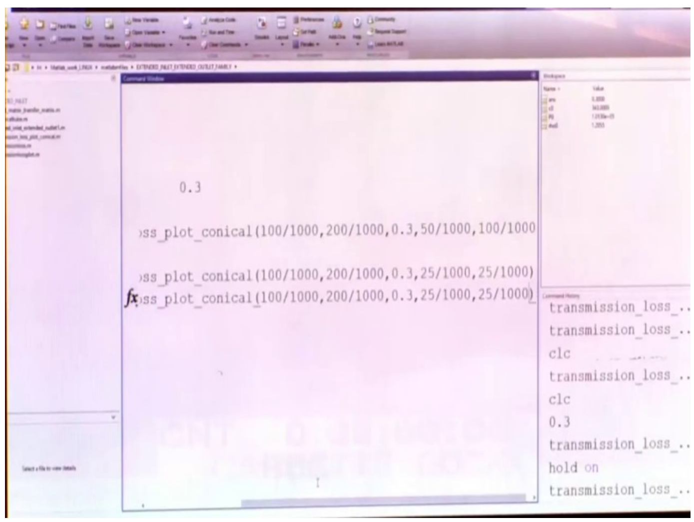
So, you see the dramatic increase in the transmission loss. There is no comparison actually. So, sudden expansion the conventional chamber produces a much better performance, but in fact, at low frequency it does go to 0, but then that is kind of completely offset by high transmission loss once the performance picks up.

But and another thing that to be notice that transmission loss never quite gets to 0 even at the resonance frequency is where I am pointing here and then you are here and so on. So, this is quite different from the simple expansion chamber, just to make the matters even more clear.

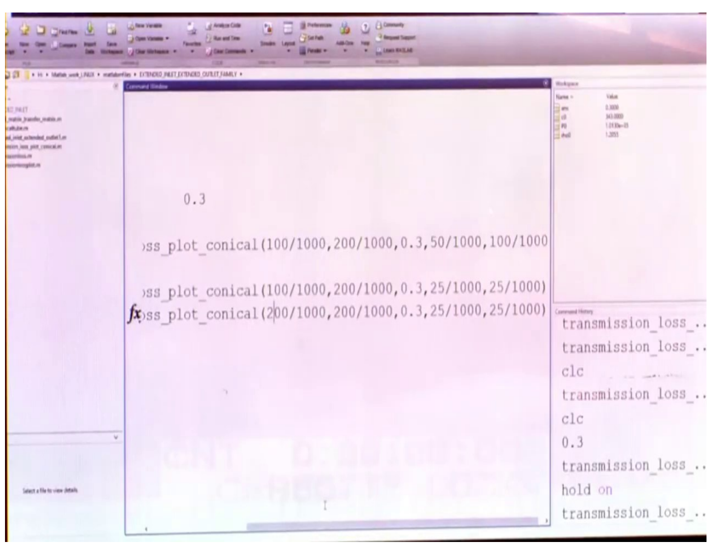


If I put a use a blue color just to distinguish between the next configuration that I am going to present.

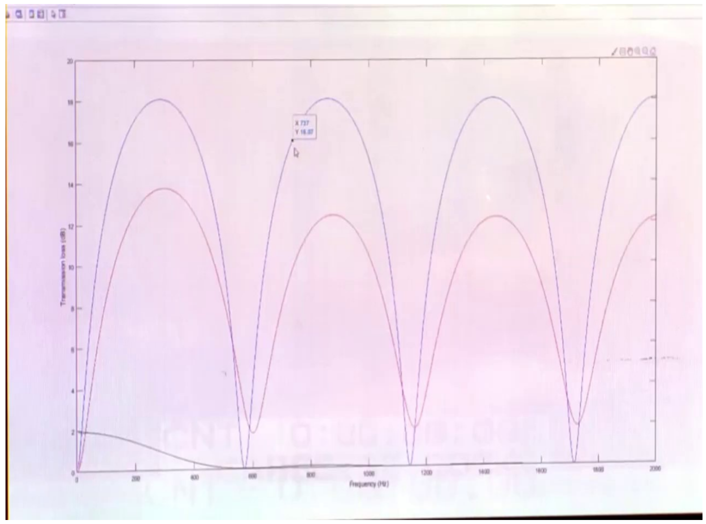




And let me say, this is diameter.

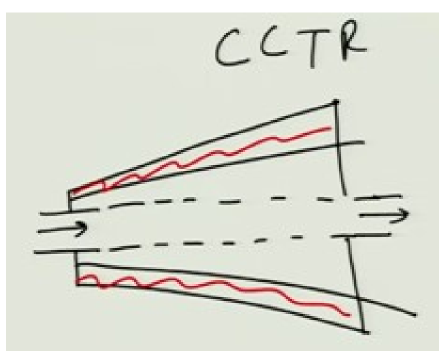


Let me call this is 200 mm thing.



So, you see so this was the uniform simple expansion chamber. This was something like let us get to the screen and it was something like, it was like this. So, you see that produces a better performance, but then it also has dips at a certain frequency. So conical chamber gradually varying area chambers conical being one of the cases pretty much has advantage that it can get rid of the traps at certain frequencies.

So, you know, a clever way is basically to combine this into a composite structure like the one that I discussed and actually one can also use a conical, you there are things like **CCTRs; Conical Concentrative Resonators**. Let me just draw although I cannot analyze it right now.



So, there are things like the configuration which I used in two wheelers so mufflers which are like this, you know they can be fully perforated, the length can be fully perforated or they can be partially perforated. So, that is a very interesting actually area of study. A person can actually do some good research using based on higher multidimensional wave propagation in such a system specially when the flow is there of course.

We are not considering any flow here right now, but the point I am trying to make as is that this these are called **CCTR; Conical Concentric Tube Resonator**. We will worry about that in the next week, but just as a teaser of a result and actually you can line it with different things and do holes or whole lot of things.

So, there are lots of avenues for interesting work on this conical chamber, because this is most likely, most commercially, most popularly used conical shapes rather than other gradually varying area shapes. So, we will end the discussion for this class now, but in the next class when we meet, what we are going to do is that we are going to study some configurations which basically cannot be solved using analytical things that we are discussed in this class.

I mean, so for example, if you have an extended inlet and outlet, but with the outside being a conical surface then what are you going to do. Analytical solution probably does not exist. So, one has to resort to a numerical thing like a segmentation approach or a matricent approach.

And then there are short chambers. So, short chamber means the length l by d ratio l being the length and d, d being the diameter l divide by d ratio is much smaller than 1. They typically about 0.2, 0.3 or so. So, for such chambers the axial way propagation, that is probably not the way to go ahead.

What we need to do is to change the direction of a propagation along the diameter or perhaps the major axis of the chamber. So, there are lots of interesting things. We will take up a couple of cases which can be which are to be analyzed using numerical techniques; like segmentation approach in the next lecture. Till that time goodbye and stay tuned. I will see you in the next class.