Muffler Acoustics - Application to Automotive Exhaust Noise Control

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Lecture - 27
Webster's Horn Equation (Contd.) And Exponential Ducts

Welcome to lecture 2 of week 6 on the NPTEL course on Muffler Acoustics. In this lecture, today's lecture, we just going to pick up the things that we left in the last class.

It is just going to be, today's lecture, it is just going to be about you know combining the momentum and continuity equations along with the isentropicity equation to arrive at the final form of equation, where the variable to be solved. Field variable is the acoustic pressure. We will end up with the Webster zone equation.

We look at the different simple rather simple algebraic steps now to how to arrive at the equation and then we look at the different cases. Well, perhaps only mention it and take up the cases possibly one of the cases is analysed and then rest of the cases can be analysed in the next lecture, lecture 3.

So, let us begin quickly on the process of linearization and combining combination of Euler equations and the continuity equations, ok. So, now, this was our today's lecture, lecture 2. But before that this was something that we must have a look at.

$$\rho(U_t + U.U_x) + P_x = 0$$

$$\rho\left(\frac{\partial U}{\partial t} + U\frac{\partial V}{\partial x}\right) + \frac{\partial p}{\partial x} = 0$$

$$U = \widetilde{U} \qquad\qquad \rho = P_0 + \tilde{p}$$

$$\rho_0 \frac{\partial \widetilde{U}}{\partial t} + \frac{\partial \tilde{p}}{\partial x} = 0$$

Momentum equation the same as that for the uniform duct.

$$(\rho U)_t = -\rho U^2 S|_x - SP_x$$

$$S\rho_t + (\rho US)_x = 0$$

$$S_V\,\rho_t + \cancel{\rho S}U_t + pVS|_x\,U + U_x\,(\cancel{\rho U}S) - SP_x = 0$$

But the continuity equation was a bit different, so this was the one. But we could possibly write it in a more simple manner which is one that is highlighted in red colour.

$$\frac{\partial p}{\partial t} + \frac{1}{S}\frac{dS}{dx}\,\rho U + \frac{\partial(\rho U)}{\partial x} = 0$$

$$\frac{\partial p}{\partial t} + (lns)_x\,\rho U + \frac{\partial \rho U}{\partial x} = 0$$

$$\rho_t + \rho_0\,(lns)_x + (\rho U)_x = 0$$

$$S = S(x) = constant,$$

So, let us see how we combine these equations. Let us first cleanly write it down again somewhere here.

$$\frac{\partial \tilde{p}}{\partial t} + \rho_0\,\tilde{U}\,(lns)_x + \rho_0 \tilde{U}_x = 0 \qquad (1)$$

$$\rho_0 \frac{\partial \tilde{U}}{\partial t} + \frac{\partial \tilde{p}}{\partial x} \qquad (2)$$

The momentum equation that we finally, we should be worrying about is rho naught times this thing, $\tilde{p}$times x and this is actually linearized.

$$p = \rho_0 + \tilde{p}$$

So, I would call this is $\rho_0\ \tilde{U}$. And this is of course, after linearization and dropping of the second order terms as we discussed in the beginning of the of this course. So, we get this thing. Now, this is equation, let us equations (1), (2). Note that equation 1 that I have written here is a bit different to what was written in the last class because obviously, there was term like ρ times U.

Linearization was not done until that stage and its only here that we assume

$$\rho = \rho_0 + \tilde{\rho}$$

$$U = U_0 + \tilde{U}$$

It is basically the same thing that we have been talking about so far in this course, linearization. This is what we get of course.

And in addition, of course, p the total pressure p is,

$$P = p_0 + \tilde{p}$$

So, this is what we get. Now, once we substitute this thing and this thing as well as this one in the, probably not this one, just these two equations in the continuity equation and drop the second order term.

So, based on a non-dimensionalization arguments that we presented in the week 1 lecture, that $\tilde{\rho}$ into $\tilde{U}$, the second order term or $\tilde{\rho}$ times $\tilde{U}_x$ is also a second order term regardless of the derivative. And all those you know all those arguments we can figure out that rho delta times.

Any quantity that is a product of these tilde terms regardless of the derivative being there or not you know as a second order term when we invoke such arguments we can figure out that these are second order terms can be safely ignored in the derivation. And finally, we will end up with the linearized form.

And definitely now, another thing that comes to our notice is the isentropicity hold that is entropy change is 0. So, but

$$\tilde{p} = C_0^2 \ \tilde{\rho} \quad i \ C_0^2 = \frac{\gamma p_0}{\rho_0}$$

that is small changes in the acoustic pressure is equal to small changes in the density times square of the sound speed or the disturbance of the propagation pulse, where p naught is atmospheric pressure and rho naught is the ambient density.

$$\rho_0 \tilde{U}_t + \tilde{p}_x = 0 \qquad (3)$$

So, this small signal wave equation which we are considering, this is, let me call this is equation 3. So, when you put this equation in equation 1 as usual, what do we get?

We get,

$$\frac{1}{C_0^2}\ \frac{\partial \tilde{p}}{\partial t} + \ \rho_0 U (logS)_x + \rho_0 \tilde{U}_x = 0$$

We also have our, this equation. So, as usual if you differentiate equation (2), that is this equation with respect to x and differentiate this equation with respect to t, time. So, what do we get?

$$\frac{1}{C_0^2}\ \frac{\partial^2 \tilde{p}}{\partial t^2} + \ \rho_0 (logS)_x \frac{\partial U}{\partial t} + \rho_0 \tilde{U}_{x,t}$$

$$\frac{1}{C_0^2}\ \tilde{p}_{t,t} + \ \rho_0 (logS)_x \tilde{U}_t + \rho_0 \tilde{U}_{x,t}$$

So, I probably can write it as $\tilde{U}$comma and $\tilde{p}$x; that means, twice partially differentiating with respect to the pressure. So, we get this. And we get, well equation (2), what do we do?

$$\rho_0\ \tilde{U}_{x,t} + \tilde{p}_{x,x}\ = 0$$

$$\frac{1}{C_0^2}\ \tilde{p}_{t,t} + \ \rho_0 (logS)_x \tilde{U}_t - \tilde{p}_{x,x}\ = 0$$

But then we still have this guy sitting here. So, what do we do with this guy? We again probably resort back to your rho naught times you know let us let us have a look at this thing; obviously, in a compact manner it can be written something like this. So, this can be simply put as,.

$$\tilde{p}_{x,x} + \frac{\tilde{p}_x}{\cancel{\rho_0}}\ (log\cancel{S})_x\ \rho_0 = \ \frac{1}{C_0^2}\ \tilde{p}_{t,t}$$

So, we can definitely do that. So, once we do it, we get and transpose all the terms on the right hand side. So, we do the transposition of these terms plus the substitution of U tilde is equal to - px, so once we do that we get,

$$\tilde{p}_{x,x} + (logS)_x\,\tilde{p}_x - \frac{1}{C_0^2}\,\tilde{p}_{t,t} = 0$$

So, your these two terms get cancelled and you are left with 0.

So, finally, well we arrived at the wave equation Webster's horn equation which obviously is the same as this one. So, we get this form. So, that was the equation that we talked about at the beginning of lecture 1. So, we are there with the little bit of rigorous derivation, some terms obviously, I would leave it to you guys to simplify and kind of work out term by term.

$$\frac{\partial^2\tilde{p}}{\partial x^2} + \left(\frac{\partial\tilde{p}}{\partial x}\right)\frac{1}{S(x)}\frac{dS(x)}{dx} - \frac{1}{C_0^2}\frac{\partial^2\tilde{p}}{\partial t^2} = 0 \qquad (4)$$

But as usual as a check of self-consistency it is valid for a plane wave propagation in a uniform duct as well, if you set S(x) to be constant. So, this term would then go away. In general case it would not. So, we let it be.

$$\tilde{p}(x,t) = \tilde{p}(x)e^{j\omega t}$$

And now to derive the Helmholtz equation form for a horn. So, basically what we do now insist time harmonicity. So, that is to say we assume e j omega t. So, once we substitute this guy in the equation let us say equation 4 or the much sought after equation. so we get this.

$$\frac{\partial\tilde{p}}{\partial t} = j\omega\,\tilde{p}(x) \qquad => \qquad \frac{\partial^2\tilde{p}}{\partial t^2} = -\omega^2\,\tilde{p}(x)$$

We substitute back in this equation, and realize that omega by

$$\frac{\omega}{C_0} = k_0$$

right the wave number with units per meter. This angular frequency by sound speed. Once we substitute that here. So, what do we get? We get ordinary differential equation form.

$$\frac{d^2}{dx^2} + \frac{1}{S}\frac{dS(x)}{dx}\frac{d\tilde{p}}{dx} + k_0^2\,\tilde{p} = 0$$

So, this then is the Helmholtz equation for the duct with a gradually varying area and this we can solve for a specialized cases.

So, now, you know the salient thing to be noted from the equation here, the star marked equation here, is that unless S x is known we probably cannot solve it. We need to know the form of S x. So, if obviously, the trivial case is S x being a constant cross-sectional area. So, I will probably not discuss that from now on.

You know this in this equation the coefficient of this particular guy is 1, and this is k naught square which is well constant at least with respect to the space x. But what is not constant is this guy.

So, if this particular thing is constant, so we get a equation with a constant the second order differential equation with a constant coefficients. And we can happily you know you know solve it by substituting or assuming e to the power some exponential function and solving for the value of the exponential. So, all these things we can do which would lead us to the fact that we will be getting equation whose analytical solution is known. Let us work out a few cases.

$$\frac{1}{S}\frac{dS}{dx} = \alpha$$

You know 1 by dS of this thing if this is constant. Let us say this is α. So, what do we get? We get, you know we can actually integrate it, take the limit and sort of integrate it, and get

$$\int \frac{dS}{S} = \int \alpha \ dx$$

$$logS \ = \ \alpha x - C$$

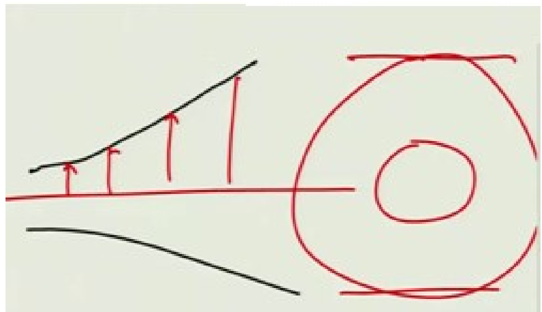
Do not worry about the arbitrary constant of integration or probably we can it will come out e to the power

$$Be^{\alpha n} = S(x)$$

Now, what does it mean?

$$
\begin{aligned}
at \quad x &= 0 \\
S(x) &= S_0 \\
B &= S_0
\end{aligned}
$$

$$S(x) = S_0 e^{\alpha x}$$



That means, what does it mean if you start somewhere here you will gradually be going like this. Let us say this is your axis, radius would increase. So, you will get a duct whose cross section gradually increases, ok. So, such a kind of a duct whose cross section increases in a exponential manner or radius also because remember obviously, from our elementary knowledge

$$S(x) = \pi\, r\, (x)^2$$

$$r(x) = \sqrt{\frac{S_0}{\pi}}\;\; e^{\frac{\pi}{2}x}$$

So, it will just another constant. Basically, the idea is that duct whose radius is gradually increasing exponentially does have an analytical solution for acoustic one-dimensional acoustic wave propagation.

And let us see what we get. So, we assume that we are getting this α here. So, we will substitute alpha in here, so to arrive at the following form which

$$\frac{d^2\tilde{p}}{dx^2} + \propto \frac{d\tilde{p}}{dx} + k_0^2\, \tilde{p} = 0$$

Now, any guesses how do we go about solving for it? Is there any guess? Note

$$\tilde{p} \;= A(x) e^{(j\omega t\, - k_0 x)}$$

which we already assumed the moment we considered a Helmholtz equation kind of a solution, j omega remember, but that is what we did here, $e^{j\omega t}$ t, so it is p_x.

But now we are assuming it to be having this form, where A can also be a function of x and we need to find out A or you know some A can be which is a function of x we can need to find out this thing with the solution of this equation.

So, how do we go about doing that? We can assume a is a function of *x* or possibly some other thing.

Let us consider

$$\tilde{p} = Ae^{-jkx}\, e^{j\omega t}$$

We need to find for k. Now, k is different from k_0, that is k is not $k_0 \neq k_0$ remember it is omega by C k some other exponential constant. So, obviously, when you substitute this guy here and actually we can sort of neglect this part because time is already taken care of.

$$-k^2 + \alpha(-jk) + k_0^2 = 0$$

$$k^2 + j\alpha\, k - k_0^2 = 0$$

So, let us see A is a constant, A is a constant here, it comes it will be taken out everywhere. So, I will not bother writing it.

$$k = -\frac{j\alpha \pm \sqrt{(+\alpha)^2 + 4\,k_0^2}}{2}$$

$$= -\ \frac{-j\alpha \pm \sqrt{-\alpha^2 + 4\,k_0^2}}{2}$$

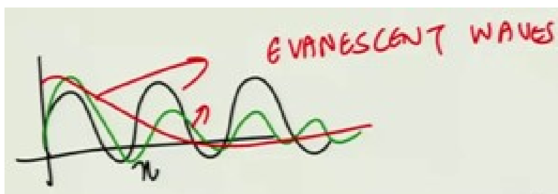
$$= \ -\frac{j\alpha}{2} \pm \sqrt{k_0^2 - \frac{\alpha^2}{4}}$$

So, we get this particular form. And we can actually simplify this thing further. We should in fact.

$$k = \pm k_0 \sqrt{1 - \left(\frac{\alpha}{2k_0}\right)^2} - j\frac{\alpha}{2}$$

We get the following thing, ok. So, what does it mean physically? So, now, clearly when alpha is 0 that is your flare angle is 0. So, remember recall the relation, when alpha assume is a constant cross section area duct. So, when then in such a case k would become $k = \pm k_0$. So, that is a normal duct of a uniform cross section area.

But obviously, we are interested in the non-trivial or the more interesting case of exponential horn for an exponential duct. So, this would basically mean that the wave number is deciding whether the wave would be propagating or not. Now clearly, we assume the solution of the form A into e^{jkx}. So, already j is there, so k has to be real, k has to be real. Why? For propagation.



If it is not real, then what will happen? j into j that is your exponential will have a decaying term; obviously, exponential the solutions cannot grow exponentially. So, that term has to be suppressed otherwise will be violating the linearized wave propagation theory. So, the signals will exponentially decay that still a physical solution, but then that would mean evanescent modes.

So, remember the discussion in week 2 of our course, there are two kind of waves, the one that propagates or normally has a sinusoidal or some kind of a wave pattern. It may have a decaying amplitude with space or it may not decay it can just propagate unattenuated. But at least it is propagating.

Other kind of wave was evanescent wave. So, it does not propagate just decay. So, you know if you have a space coordinate like this and you can have a solution like this or you may have a solution like decaying wave, but then there are solutions which do not propagate at all, which just simply decay.

So, we are interested in this guy, talking about this guy here. So, such a thing is called evanescent, evanescent waves or at least that waves which decays much faster in comparison to the other dominant term. They do not propagate at all. So, why am I telling you about all this thing now?

Because you know you see here that this particular term, if the term inside the bracket, that is this guy, if this guy is greater than 0, then obviously, there is no problem and there will be some kind of a along with the propagation thing there will be some kind of a spatial dk as well, because *jj* gets multiplied. And you will have you remember the solution the solution was minus *j*, minus plus *j* square is minus.

So, minus alpha by 2, e to the power you will get some part will be e to the power minus $\alpha/2$. Alpha being constant it will be it will signify the decaying part along with the propagating part. So, we will get something like the green kind of a solution. Spatially, for a given frequency k_0 is a frequency.

So, for a given frequency if the value encircled here if this is greater than 0, then it propagates no problems. But if it is not greater than 0. If it is equal to 0, still you will get some you know decaying solution the wave will be just about to be propagated or it will just called a cut on frequency for the wave. Remember the cut on frequency concepts that we talked about in the week 2 lectures. So, same thing applies here, but in a different context.

So, for an exponential duct, **for an exponential shaped horn cut on frequency**.

If it is less, then its evanescent. So, the idea is

$$1-\left(\frac{\alpha}{2k_0}\right)^2 \geq 0$$

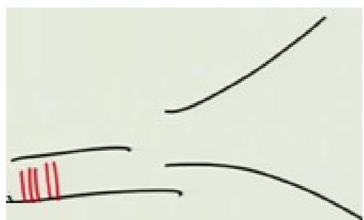
$$4k_0^2 - \alpha^2 \geq 0$$

$$k_0 \geq \pm\sqrt{\frac{\alpha}{2}} \qquad => \frac{2\pi j}{C_0} \geq \sqrt{\frac{\alpha}{2}}$$

$$f_c > \frac{C_0}{2\pi}\sqrt{\frac{\alpha}{2}}$$

So, if the frequency of excitation is greater than this number then of course, the waves will be cut on that is they will propagate. If it is less or at least equal to then it will not be propagating, there will be it will decay down exponentially. So, that is an important thing.

If we never actually encountered such a thing for a planar wave propagation in a duct of a uniform cross section area. So, no matter what frequency you are assuming, if you considering planar waves they will always propagate, right. But for a thing like an exponential horn we could get a clean relation between for the cut on frequency, let us say, let me call this f_c.



So, you need to excite if you know alpha value is typically a bit small value. So, beyond a certain frequency almost all frequencies would sort of propagate, alpha is a constant. So, below that the waves will not propagate. So, that is something new that what we are kind of seeing in this kind of a thing.

So, and then of course, then there are things like amplitude decay and the phase velocity relationships for the exponential things horn. So, amplitude d k is nothing, but you know we saw basically you know this part is there, e to the power minus alpha. So, if we assume the amplitude to be A, so you know it is basically amplitude dk would basically mean it is something like this and e to the power j times whatever part. Something like this.

$$Ae^{-\frac{\alpha}{2}x}\, e^{j(-)}e^{j\omega t}$$

$$p\alpha\ \frac{1}{\sqrt{S}}$$

So, this is the Ax function that I was talking about, it means a decaying function amplitude dk with respect to the space x. Now, similarly you can derive phase relationship between velocity and the phase difference between the velocity in the pressure, and the phase velocity, all those things we can easily do.

And we can get an amplitude of the rms acoustic pressure field, we can show that it is decaying approximately for an exponential horn it is decaying as 1/ $\sqrt{S}$. And we can do all these kind of detail derivations. But we will probably stop here.

And in the next class, what we will do is we will consider the case of a conical muffler, or a conical duct. We will derive the acoustic pressure field solution and the velocity field solution. We will focus a lot on conical shaped ducts because they are used in resonators called con CCTRs, conical concentric tube resonators.

We will analyse that in probably in week 7, towards the end of week 7 or probably towards week 8 beginning, they are used a lot in these two wheelers the photograph that I have shown you; the internal section comprises of uniform duct with the perforations. And the annular cavities, is a conical annular cavity.

So, there are lots of things that we could do. And Sx being general we can keep studying I will probably mention some of the important things like you know parabolic shape ducts, hyperbolic shape ducts, the solutions which were of which were known since a long time, it is now there have been papers published on that which report exact analytical solutions. But Sx in general is not, for general Sx the solutions are not quite known.

We can, there are lot of approximate method like WKB method and all those kind of things. So, what I will do? I will kind of introduce the conical shaped horns or mufflers in the next class, and then mention some of the cases which have an analytical solutions and some approximate techniques analytical techniques you know and then probably worry about the more important test cases using some numerical techniques which we readily can use. So, that is how this week's lecture would shape up.

So, till that time a good bye and stay tuned. I will see you in the next lecture, lecture 3 of week 6. Thanks.