Muffler Acoustics - Application to Automotive Exhaust Noise Control
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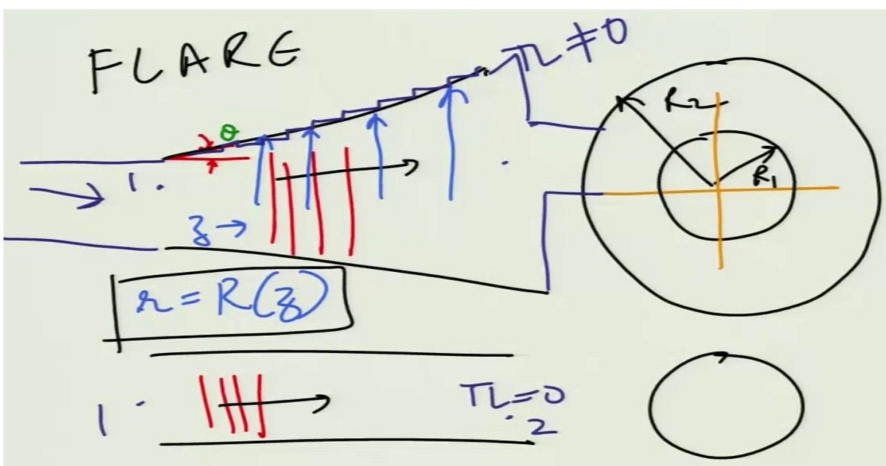
Lecture - 26
Wave propagation in gradually varying area ducts: Webster's horn equation

Welcome to week 6 of the NPTEL course on Muffler Acoustics. This is the first lecture of week 6 and the entire week 6 and possibly the initial parts of week 7 will be studying a topic which is known as Acoustic propagation in ducts of gradually varying cross-sectional area.

So, we are still considering 1D waves only that is the waves go along the or propagate along the axis, but then now the difference is that the cross-sectional area as you can see it is varying.



So, if you have a duct say of a circular cross-section and whose radius is something like this here are say it is R_1 here. At this point you project it become it has a much larger cross-section say R_2 radius. So, the question then is that why do we use why do we need to study you know ducts of gradually varying cross-section area?

Why do we need to consider wave propagation in such structures and what is the practical importance and those questions might be arising in any readers and in any students mind.

Before we actually begin embark upon the a little bit tedious derivation because obviously, here the difference is that you know compared to a uniform duct of a; duct of a uniform cross-section and if you consider wave propagation as we have been doing in such ducts and this duct the associated maths even for the one dimensional test case becomes a bit complicated because of the inclusion of the presence of additional terms.

So, now, before I answer the question regarding the practical significance let us first also briefly revisit or recall an important thing which is called you know reactive mufflers. What do reactive mufflers do? As I have probably noted in my 3rd or 4th weeks lecture probably 3rd weeks lecture mufflers are kind of categorized into two parts – primarily reactive mufflers and the dissipative mufflers.

So, talking about reactive mufflers they work on the principle of reflecting a significant portion of the acoustic power back upstream from where which back to the source and allowing only a limited portion of the acoustic power to propagate downstream. At least for a signal for a particular given frequency range, this may not be true for all frequencies because in some frequencies all almost the entire acoustic power is transmitted throughout downstream.

So, but we will be basically focusing on the frequency range in general in which at least some part of the acoustic power is transmitted back upstream and less amount of power is transmitted downstream. So, the whole idea is basically to reflect or basically reflect a reflect significant part of power back and that is possible we saw by the simplest possible element that is your the sudden area discontinuity.

You know this thing. So, when a part of the wave comes here some part is reflected back; some part is reflected back and less amount is transmitted downstream. So, basically what happens then, is that these sudden area discontinuities form the basis of the working of reactive mufflers. So, the idea is to cause a significant impedance mismatch.

Even for the case of extended inlet and outlet chambers that we just saw in the last lecture before we ended the last lecture you know by putting zero impedance here just at this interfaces resonator, and some finite impedance here. We saw that the waves choose to go towards the path of least resistance or least impedance. So, they go in the side branch and less amount of acoustic power is transmitted towards the downstream.

You know the entire idea behind all such things is to continuously reflect the acoustic power or divert the path of acoustic power from the main propagation direction. So, one you know couple of possible alternatives we saw so far was the use or the existence of sudden area discontinuity like the one I just drew and extensions at the inlet and outlet and so on.

But, what happens a question arises that what happens when you have such a thing? You know the area is gradually varying. So, you see there are lots of interesting mathematical concepts we can study by considering such systems. You know one thing is that the area is gradually varying. So, you know one can imagine the radius is gradually varying of course, with the actual distance.

So, now what it means well mathematically or one would like to think is that you know they are small small discontinuities the well this technically not a discontinuity it can be probably thought of as a you know duct with notional discontinuity something like this; discontinuities which is mimicking the horn and all that.

So, they are small very infinitesimal small discontinuities that occur or that they are present along the section and gradually from a small section we are able to get a much larger cross-section. So, this is called a segmentation approach we will talk about that in detail, but the idea of gradually varying area changes is to basically in a way kind of induce a continuous reflection of acoustic power rather than introducing a sudden impedance change.

So, it is pretty interesting because you know not only is it mathematically interesting because we are alloying a continuous variation of the impedance by actually reducing or increasing the area continuously. Remember, the characteristic impedance, what was the expression? It was

$$Y_0 = \frac{C_0}{S(z)}$$

$$Y_0(z) = \frac{C_0}{\pi R^2(z)}$$

$$R(z) = R_0 + \alpha z$$

What really it means is that you know we are having a continuous change of impedance across the axial distance z.

So, how is it affecting the wave propagation? How is the governing equation changing in the addition of such things and we saw that the transmission loss of a simple tube uniform tube this was 0 because across this point and this point 1 and 2 because there was no area change. The wave whatever the wave is propagating in this direction there is no reflections there no area changes to reflected back.

However, it is quite intuitive to think that for a such a thing you know gradually varying area the transmission loss or the basically the acoustic power that is reflected back from you know as the wave propagates from 1 to 2 there will be some nonzero transmission loss for such a gradually varying area, forget about introducing sudden area discontinuities like an inlet and outlet port.

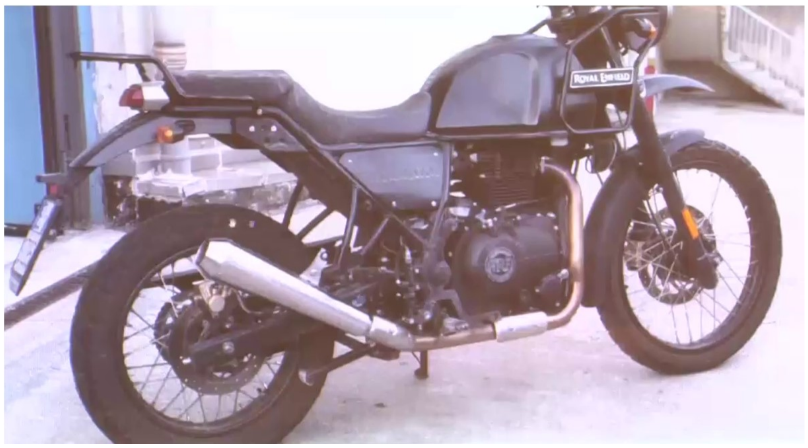
Even for a simple thing like a HORN, this is the first time I am introducing this term on transmission loss across here the end with a smaller cross-section and the one with the bigger cross-section would be nonzero. So, that is because of gradually continuous reflection small in bits and pieces small amount of acoustic power is always reflected back. Now, and the cumulative effect shows up in the transmission loss computations.

Now, apart from mathematical mathematically being very intriguing and very you know interesting to look at. What is the physical significance and why do we need to use it? Firstly, because you know in these the courses on mufflers they basically which are devices to silence the noise from IC engines whether it is two-wheelers or four-wheelers or some other application.

So, we always have limited space it may so happen that you know towards the say suppose if you connect this to a to the exhaust of a, IC engine. So, the exhaust gas is coming like this and it is going like this and you introduce a sudden area change here. So, sorry. So, introduce something like this here.

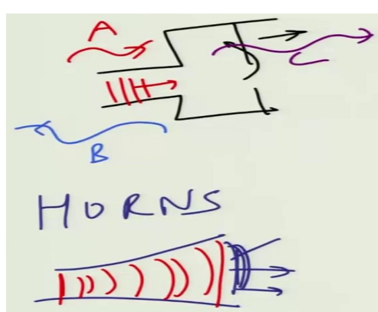
So, the real structure will obviously, be much more complicated, but the point I am trying to make is that at some parts of the engines you may not have that much space, the other part you may have much larger space. So, why not you know get optimal profile and based on the space available, manufacturing constraints, the cost of fabricating such a muffler which obviously, is more than a uniform pipe.

And, importantly you can do a lot of because of the increased space available as we go towards the outlet one would have basically one can do a lot many things inside the muffler.



So, one motivation that I would like to give you for the first time in this course is showing you a nice photograph that of course, I got it from the internet. You can have a look at such photographs you know they are widely available for different two-wheelers. Two-wheelers use things what are known as conical mufflers. So, this is exactly in tandem with in sync with what we are discussing in this course.

So, these are conical mufflers the waves it has lesser area towards the inlet pipe or the inlet side and a bit larger area towards the outlet pipe. So, the acoustic power whatever is coming here is spread over a larger area. So, exactly the same thing happens for HORNS. In fact, horns are where the first applications in which the wave propagation was studied.



So, basically you know the horn what it does, it has a flare. It has a certain flare like I said and the entire idea of horns is that you know what it does is basically we are trying to maximize the acoustic power distribution from its open end. So, it is basically is a device so, that you know if some important announcement was where to be made in the ancient times or very olden days and even in modern times we have lots of you know such devices for public addressing systems and all that.

So, the idea of the basic fundamental underlying this is to you know at an outlet side or where the waves are going propagating to the atmosphere, the pressure should be the acoustic pressure should be in phase with the particle velocity. So, even at the lower frequency lower end of the useful frequency spectrum the pressure acoustic pressure $\widetilde{p}$ & $\widetilde{U}$ acoustic particle velocity along the axis of the duct.

So, p and u are in sync or are in the nearly in the same phase. So, if that is happening if that is if this condition is met then maximum amount of acoustic power can be transmitted towards the atmosphere and we can there will be more effective in radiating sound.

So, the HORN basically flares to a larger mouth that is this part in such a manner that the shape of the wave fronts that propagate here, they are not substantially altered. They the sound energy is radiated from the horn mouth with pressure which is substantially in phase with the particle velocity. When all these conditions are met maximum power is transmitted like I said.

And, another thing that I want to quickly add is that you know what of layer does is basically split or gradually reduce the intensity of the waves because the same acoustic power is distributed over a much a significantly larger area. So, while acoustic total acoustic power is conserved the intensity reduces because the area over which it is spread has increased.

So, that is one important feature of a horn to basically create a equally energetic acoustic pulse or acoustic waves, but of lower pressure amplitude and lower velocity because it is distributed over a larger area. So, now, with this background what we will do probably horns; obviously, where a classical topic that was discussed. With this we will go to the classical equation which governs the propagation of one-dimensional wave.

So, let me draw a fresh figure let me draw it a completely new figure something like this is the horn and what we and this is the say the axis ok. So, the waves basically they propagate along the axis. So, the waves propagate along the axis along the direction that depends only on the z-coordinate. So, waves essentially 1D in nature.

So, we are basically going to have a look at the corresponding equations that govern the wave propagations in a greater detail and see how we get an extra additional term just because we have a flare or the gradually varying cross-section area.

So, interestingly you know for a special case of a conical horn you see that the spherical wave equation just like you have one-dimensional equation for a uniform tube we have spherical wave equation one-dimensional spherical wave equation there is no dependence on theta and phi coordinates that is there is only dependence of radius r from the origin.

So, such equations can also govern you know spherical waves in conical pipes. So, the form of the sphere one-dimensional spherical wave equation is the same as that of the equation obtained by the Webster horn equation for the conical pipe. So, that is a special case we will soon come to it we will do some special cases like you know exponential horns and conical horns and few cases for which the analytical solutions are well known.

And, then go for a more advanced mathematical modeling in this week like segmentation approach and possibly approach perhaps solve a few problems for the transmission laws. And, then probably end up this weeks or this particular topics presentation by introducing things like some perforated elements, although the detailed analysis will be done in the later ensuing week's lectures.

So, let us begin now with the thing what is known as **WEBSTER'S HORN EQUATION**.

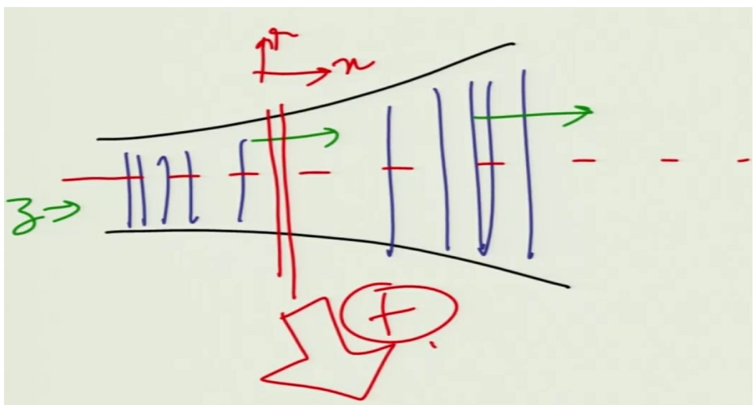
$$\frac{\partial^2 \tilde{p}}{\partial x^2} + \frac{1}{S}\frac{dS}{dx}\frac{\partial \tilde{p}}{\partial x} - \frac{1}{C_0^2}\frac{\partial^2 p}{\partial t^2} = 0$$

$$S = S(x)$$

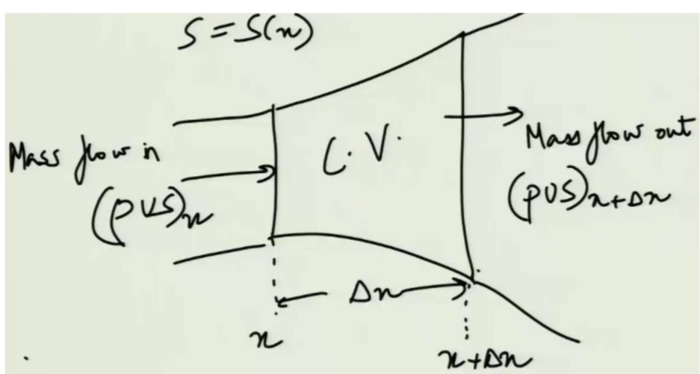
So, basically for a gradually varying area duct so, if $\tilde{p}$ is still the acoustic pressure. So, we will soon see that after a little bit tedious derivation the form that we obtain would be the following.

Well, if you consider a wave equation will be like this the difference being that this will be replaced here by del. Well, we will see how we get to this equation. This will be our objective and S here can be any function of S or z we see that.

So, obviously, when S is constant that is if you have a uniform pipe what is S? S this is a constant say alpha something some constant. So, d dx of S is 0. So, then in. So, because this term is 0, this one the entire term goes away and we get the well known equation for the one-dimensional tube. So, that is a special case, but right now well our focus should be to derive this equation which we will do in a while.



So, what we will do now is that will consider a control volume something like this which is kind of representative of the duct of gradually varying cross-section area will basically considering a small chunk of element here, but we kind of zooming it. We kind of magnifying it to get this part and so, this is a small volume actually and this goes on this.



Now, there will be definitely some amount of fluid that will be entering within the control volume. So, I will write it C V control volume and there will be some volume of fluid that will be entering.

So, density times velocity which is normal to the cross-sectional area times the cross-section area at x. So, what is net difference? Net difference of the inflow and the outflow mass is given by

$$\frac{\partial}{\partial t}(\rho S \Delta x) = (\rho U S)_x - (\rho U S)_{x+\Delta x} \; \dots \dots \qquad (1)$$

So, this is the net mass that is being added to the elemental volume or the control volume.

So, this we can relate it to the temporal change. So, what is the volume of this element S times Δx over the small length Δx, ok? So, S into Δx is the volume multiplied by density will give you the mass.

So, the temporal change of mass is nothing, but, so, that is your equation that needs to be simplified. So, temporal change of mass within the control volume is the net difference of the mass that enters the control volume and the one that leaves the control volume, ok. Let us call this equation 1. Let us simplify equation 1. So, remember S is not a function of time.

And, however, delta t can be a function of time. So, what we do basically is that we take this and divide throughout by

$$\frac{S\partial\rho}{\partial} = \frac{\rho U S|_x - \rho U S|_{x+\Delta x}}{\Delta x}$$

Now, we take the limit when,

$$Lim\, \Delta x \;\rightarrow\; 0$$

That is this element that we saw here it is basically getting smaller. We basically consider a very small element that is the length is very small. So, limit Δx is tending to 0.

This means that we can write the right hand side as let us first rearrange this term at x

$$= -\frac{\{\rho US|_{x+\Delta x} - \rho US|_x\}}{\Delta x} = -(\rho US)_x$$

So, this we can write it like rho U S with derivative with respect to x. So, suffix x means derivatives with respect to x.

So, what you get is basically

$$\frac{S\partial\rho}{\partial t} = -\frac{\partial(\rho US)}{\partial x} \qquad S = S(x)$$

$$\frac{\partial\rho}{\partial t} + \frac{1}{S}\frac{\partial}{\partial x}(\rho US) = 0$$

$$\frac{\partial\rho}{\partial t} + \frac{1}{S}\left\{\rho U\frac{\partial S}{\partial x} + S\frac{\partial\rho U}{\partial x}\right\} = 0$$

what we can do is that just rearrange a few terms here and there this thing, where S S x and now, let us open the terms within the brackets. So, when we do that we get this thing plus your S times I would still prefer putting all the terms within the bracket. So, we get this part.

So, now we can clearly see that this becomes

$$\frac{\partial p}{\partial t} + \frac{1}{S}\frac{dS}{dx}\rho U + \frac{\partial(\rho U)}{\partial x} = 0$$

$$\frac{\partial p}{\partial t} + (|ns)_x\,\rho U + \frac{\partial\rho U}{\partial x} = 0$$

$$\rho_t + \rho_0\,(|ns)_x + (\rho U)_x = 0 \qquad (2)$$

So, this happens because S here S here and S here gets cancelled. So, now when you write this above equation in even more compact form we can write it as a natural logarithm of S x times rho U plus.

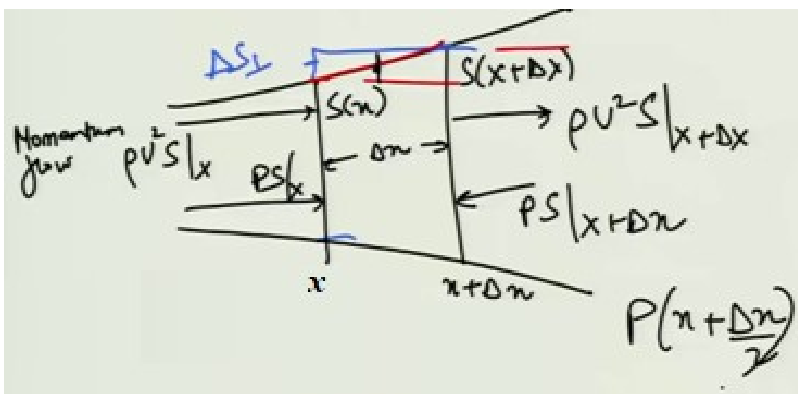
So, we get equation number let us (2). So, one corollary that can be easily seen from this equation (2), is that when, $S = S(x) = constant$, as we saw, then of course, this term goes away what I have been talking about in the last class. So, we get back your old continuity equation for a uniform duct.

But, in general case for a gradually varying area duct this will not be 0 because S is by definition a function of the actual distance x ok. Note again here that waves in a gradually varying area duct are never one dimensional they are always at least they will be 2-dimensional in the sense that they will have a motion along the z-axis or the x-axis, I am sorry and there will be some component along the r-axis.

So, but this area changes are very gradual or I would say the discontinuities are only notional if you talk in terms of segmentation approach what I talked about. So, then because the area changes so small it does not really matter if the even if there is a small transverse component they are very small compared to the actual component. So, we will focus only on the 1D part that is the axial part.

So, anyhow this was the equation (2). Let me circle this with and box this with red color.

Now, what do we need to worry about? We need to worry about the momentum equation. This was the continuity equation and **MOMENTUM EQUATION**.



So, it will turn out that momentum equation for a duct of gradually varying area section is the same as the momentum equation for a duct with a uniform cross section.

However, we will again resort back to the to the control volume approach and this in fact, is this even more needed in for deriving the momentum equation. So, before that let us first understand a couple of things that this is x, this is $x + \Delta x$, this entire thing is length is x.

Now, what is the clearly the area of the duct is increasing this is say S of x plus delta x, this is S x what is your change that we will need it which will be evident very soon why do we need it? The change is basically this part let me denote this by S delta perpendicular. You know simply subtract this entire bit from this smaller bit you get the change in the cross-sectional area.

So, we need to identify the forces that act on each of the face. We need to identify the force that acts on the face here and the forces that act here. So, now, momentum flow force due to momentum is basically $\rho\, U^2 S|_x$, but then the there is a fluid flow that is going in this direction you know $\rho\, U^2 S|_{x+\Delta x}$.

Now, let me reiterate again if some of you have missed the first few lectures that what is the unit of rho U S? It is kg by meter cube by meter by second into meter square, and if you further add meter by second so, you know this all these gets cancelled you are eventually left with

$$\frac{k_g}{M^2}\,\frac{M}{S}M^2\,\frac{M}{S} = k_g\, MS^{-2}$$

I am just trying to verify the dimensionality of this term. This term indeed has dimensions of force.

So, it is something like ρUS so, that is your, that is your mass per second into the velocity. So, that will give you the force. So, it is not just mass; it is mass per second times the velocity. So, it has the units of force dimension y. So, you get all these things basically this works out to be the force that acts on the face on the left hand side and this is the equal.

And, this is the force that acts due to the flux that is leaving on the other side due to the flux that is leaving here and we need to take a difference of the forces due to these fluxes and that is how we will probably get the terms of the momentum equation.

Now, what we basically need is the other force, obviously, the pressure force that is P. So, that is P times cross-section area at x that is the force that acts here, but then there is equal opposite force I would not say equal, but there is an opposite force

$$\mathrm{PS}|_{X\,+\,\Delta x}.$$

That was also a case when we saw a duct with a uniform cross-section area, but now we need to have an additional term that is this.

Now, remember now here comes the crucial part of the derivation. You know this is a gradually varying flare. So, you know this was the projected let us let us let me project this area. So, it will overlap with this term, do not mind. This was like this and this part is like this. So, basically we are looking at the annular area what is the difference.

So, when we consider delta S perpendicular what we actually get

$$\Delta S_1 = S(x + \Delta x) - S(x)$$

Now, this difference of the area times your the force that acts you know let us say we consider the force that acts somewhere in the middle. You consider the we consider the part the force that acts on the middle that

$$P\left(x + \frac{\Delta x}{2}\right),$$

we consider this part. So, what do we do now? Basically, what we do is that there is an additional force additional term and that

$$\Delta S_1 \, p|_{x+\frac{\Delta x}{2}}.$$

So, this is an approximation. So, what is the net body forces that are acting on the control volume? Body forces that act here and here plus your body force that x due to delta S perpendicular is nothing, but

$$F_S = p_S|_x - p_S|_{x+\Delta x} + \Delta S_2 \, p|_x$$

We get this. Now, we recognize the things that

$$p\left(x \frac{\Delta x}{2}\right) = p(x) + p'(x)\frac{\Delta x}{2} + \cdots$$

we use a Taylor series approximation and get the following. So, we get this term and then possibly what we do is sort of do some simplifications we would like to do some simplifications and substitute all this thing in here to get,

$$= PS|_x - p_S|_{x+\Delta x} + \Delta S_1 \left(p(x) + \, p'(x)\frac{\Delta x}{2}\right)$$

Let us the first retain this part, but then once you substitute this part in here and simplify. And you know sort of carry out our regular algebra. So, we will get after a little bit of algebraic simplifications the following term

$$F_S = p(x)\ S(x + \Delta x)$$

$$-p(x + \Delta x)\, S\,(x + \Delta x)$$

$$+ \frac{\Delta \mathrm{x}}{2}\, \Delta S_1 p'(x)$$

we get this. So, this is the term that arises due to the surface forces, basically surface forces that act on the different surfaces of the control volume.

So, what is the other force that acts? Basically is you know this is the net difference of the force that acts on the control volume in here, ok. So, we get this.

$$F_M = \rho U^2 S|_x - \rho U^2 S|_{x+\Delta x}$$

$$(U\rho S\Delta\Delta x)_t = F_S + F_M$$

This is the let us say the force due to the momentum let me also box it and what we get now the sum of $F_S + F_M$ should be equal to the time rate of change of the momentum. So, you see rho times S into Δx was the mass present in the control volume in the shaded control volume is it not? ρ times $S(\Delta x)$, if you multiply this by U that gives you momentum. So, if you take a temporal derivative of this.

So, after a little bit of simplification we can of course, take S out of the derivative sign because it is not a function of time and delta x we divide throughout by Δx just like we did for the continuity equation. Once we do that and then simplify

$$S(\rho V)_t = \frac{\rho U^2 S|_x - \rho U^2 S|_{x+\Delta x}}{\Delta x}$$

And I probably need to hurry a bit. So, I am just writing out certain terms. You can work out some simplifications on your

$$+S(x + \Delta x)\frac{\big(p(x) - p(x + \Delta x)\big)}{\Delta x}$$

$$+ \frac{1}{2} \Delta S_1 P'(x)$$

$$L\,im \quad \Delta x \quad \rightarrow \quad 0$$

What do we get? We will get after some simplification

$$S\,(\rho U)_t = -\,\rho U^2 S|_x - SP_x$$

What we can possibly do is that use the other equation. Now, use the continuity equation continuity equation given by

$$S\rho_t + (\rho U S)_x = 0$$

We need to use this expand this guy out and keep this aside expand this out. So, let us very quickly get about doing it. So, here you get

$$S_V\, \rho_t + \rho S U_t + pVS|_x\, U + U_x\, (\rho U S) - SP_x \ = 0$$

We transpose these terms on the right hand side and get.

Now, if we recall actually this was a simplified equation what we probably could also use is that the particular form rho U S into x is equal to 0, the other term that I wrote was expanded out version of this. So, why am I telling you this because now we can clearly identify terms. You know this term and the other term is your this term.

So, basically we can take this guy take this U guy common and then your this term and this term will cancel out because the sum of these terms is 0 as we saw. So, what are we left with? We are left with only this term, this term and this term is it not? Actually there would be plus sign here because I am kind of transposing it on this side and this will be equal to 0.

Let us nicely write it down all the things, but actually we can also divide throughout by S because S is nowhere in the derivative part. So, the simplified part is rho times I would

$$\rho(U_t + U.U_x) + P_x = 0$$

$$\rho\left(\frac{\partial U}{\partial t} + \ U\ \frac{\partial V}{\partial x}\right) + \frac{\partial p}{\partial x} = 0$$

So, p is the total pressure mind you. So, this is your momentum equation which is the same that we got for the duct that is for the duct with a uniform cross-section area.

$$U = \widetilde{U} \qquad\qquad \rho = P_0 + \tilde{p}$$

Now, obviously, if you put U to be U_0 plus $\widetilde{U}$ we can actually get the thing for a mean flow, but we will one thing that I kind of forgot to mention is that we are just considering the stationary medium here; we are not considering mean flow. So, basically U will become your $\widetilde{U}$. U will become your $\widetilde{U}$ what would what it would mean is that this equation will further simplify to the form where this is this thing plus your $\tilde{p}$. Because remember

$$\rho_0 \frac{\partial \widetilde{U}}{\partial t} + \frac{\partial \tilde{p}}{\partial x} = 0$$

and you know we are not assuming any gradients of the pressure as we go across the axial direction and we will eventually be left with this equation the linearize one. And, similarly we can do get a linearized version of the continuity equation and isentropicity relation. So, what we will do is that we will stop here it is already an extended lecture I would say.

Thanks a lot, stay tuned.