Muffler Acoustics - Application to Automotive Exhaust Noise Control
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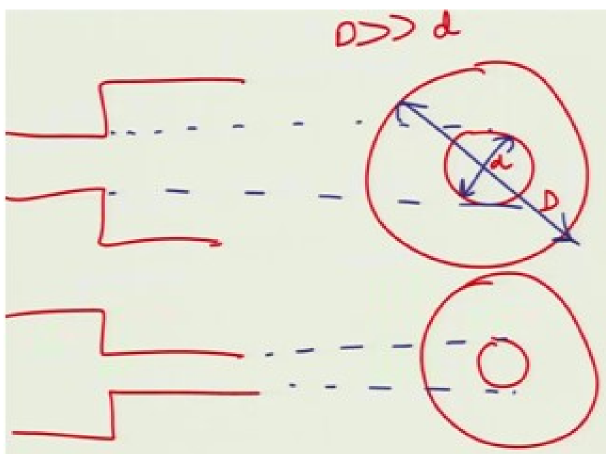
Lecture – 19
Sudden area discontinuity (Continued)

Welcome to lecture 4 of week 4. In this Muffler Acoustics Course in the NPTEL series of lectures. Well, we could not quite complete the sudden area discontinuity in the last lecture; lecture 3 of this week, but we will probably take it up today in this lecture and let us get back to our diagnosing towards the end. So, you know sudden expansion or sudden contraction that is what constitutes a **Sudden Area Discontinuity**.

$$\tilde{p}_1 = \tilde{p}_2$$

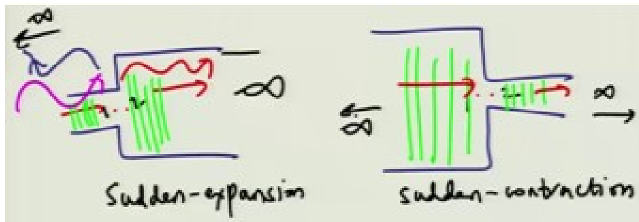
$$\tilde{V}_1 = \tilde{V}_2$$



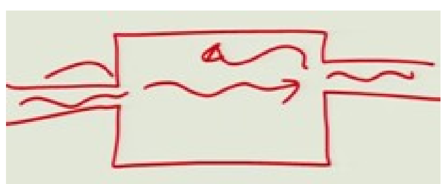
So, you see the Schematics here. So, suppose you know you have a pipe like this and if you from the side view and if you look at from the other side is probably like this. So, the idea is that so this is like the diameter of a smaller pipe and this is the diameter of the bigger pipe.

So, basically you know, whenever the wave comes it gets a lot of area to expand into and suddenly, so there is a sudden increase in the diameter. So, if this diameter is say d this is capital D. So, D is typically much much greater than d. You can have the other Scenario of course, where you have sudden contraction in which case this, this would probably be the same, something like something like this.

So, again your basically the idea is D is much much greater than small d. What happens here, the wave either gets sudden area a lot of area to expand into or it suddenly it is forced to contract or propagate through a much smaller area. So, both of them under both situation what we have is basically a significant part of the acoustic wave that is carried in the forward direction gets is forced to be reflected back into the system.



So, if you have a wave coming like here, significant part will be propagated will be forced to retreat and less part will be propagating downstream. We will see how it works and the same logic applies here. So how do we do the maths behind it and why are this important of course?

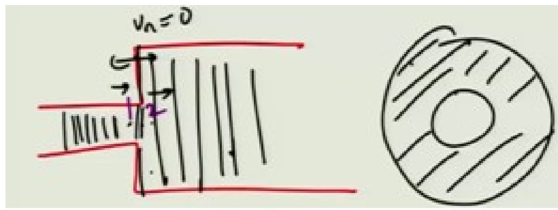


So, if you consider a muffler that is basically a simple expansion chamber muffler, is called simple expansion because there is a sudden area discontinuity and sudden area expansion, sudden area contraction and it is called simple really because it is nothing but a empty chamber with no lining nothing, rigid wall chamber so that breakout noise is minimized from the this noise that goes in this direction that is not allowed.

So, basically the structural vibration noise is prevented and all the structure does is basically when the wave arrives here, lot of part is reflected back and some part is propagating downstream. Again it hit it meets a discontinuity area, again some part is reflected back, some part is transmitted and there is some kind of a standing wave pattern that is formed here.

So apart from really introducing basically fitting inlet and outlet ports of diameter much smaller than the chamber or the shell diameter there is nothing much to it. So, it is that is why it is a simple or as simple as it can get.

We will build up on the simplicity or the add more complexity to the simple expansion chamber, but before we do that, let us mathematically model the sudden expansion of sudden contraction. Fortunately, for both without any flow equations that are valid for this one is also valid for this one.



So, let us see how we go about doing it. So, you know sudden expansion. So if we have a point somewhere here 1 and point 2. So now, we are assuming a **1 Dimensional Or Planar Wave Propagation.** So, what is the consequence of such an assumption?

As we can recall from our very first lectures of this series, when we have a wave whose phase is constant over the cross sectional area; that is if you are in the small diameter tube across the cross section the pressure as the velocity particle velocity normal to the plane they are all constant no matter at which point you are there in the section.

So, similarly the same logic applies here. So, the idea is that when the wave expands into such a well expansion chamber, basically the wave planar wavefronts remain planar. That is, it is if it is planar here it is also planar here. Of course, this is just an assumption and we will see that well this is valid only below a certain frequency and even called the cut on frequency as you can probably recall from weeks 2 lecture.

But even in the low frequency range, the higher order modes might affect the performance due to what we call at evanescent modes. But we will not worry about that part just now. What we will probably do is for now the points, let me use another color for the point listing this is say let me use well purple color 1 2. So, the pressure right at the interface then is given by

$$\tilde{p}_1 = \tilde{p}_2$$

So, the pressure planar wave pressure is equal to the planar wave pressure here. So now, why is this happening? Because we are actually insisting on the field continuity conditions that is to say, over this cross section whatever the pressure is there the same pressure is there at this point, but then since it is a planar wave front.

If we insist that the pressure right here is the same as the pressure just in the point in the chamber then by the plane wave assumption this is the same pressure is carried over on the annular surface as well. So the idea is, they remain plane even when they expand or suddenly they contract. The other important bit or the part is U_1, the particle acoustic particle velocity is constant at 1 and 2.

Obviously, they are you are considering one section that forms one end of the tube, that is a smaller tube and other end a part of the chamber. So of course, they are referring to the same points the acoustic particle velocities have to be constant; that is U_1 right here in the direction like this and this is the direction this.

They have to be the same because you are referring to the same point. Now, as we were discussing previously, the annular surface area is assumed to be rigid that is to say it is a rigid walled annular surface. So normal velocity is 0. So you have U_n is equal to 0 here. With normal velocity over the annular area 0 because the end plates of the chamber they are not vibrating, they just remain stationary.

So, now $\widetilde{U}_1 = \widetilde{U}_2$ and basically over the port say the inlet port inlet port. But $\widetilde{U}_2$, probably I should be writing this as U_2 is equal to U_1, 2 being the chamber over the inlet port that is only over this thing this small part. And, $\widetilde{U}_2 = 0$ over the annular area that is over this thing.

So now, what does it mean? if we were to integrate, so we have

$$\rho_0 \widetilde{U}_2 \, S_p = \rho_0 \widetilde{U}_1 \, S_p$$

$$\rho_0 \widetilde{U}_2 \, S_p = \tilde{V}_1$$

$$\rho_0 \, \widetilde{U}_2 \left(S_c - S_p\right) = 0$$

Why? Because you multiply both sides by ρ_0 S_2. S well rho naught into let us say the cross sectional area of the chamber is S_c. So, what is the annular area? This area is S_c - S_p

port. So when you multiply both the sides by this one, you get this. Now we will add this and this. So what do we get?

$$\rho_0\, \widetilde{U}_2 S_c = \tilde{V}_1 \;\; => \; \tilde{V}_2 = \tilde{V}_1$$

let us say this is we take this as common $\widetilde{U}_1 = \widetilde{U}_2\,(r, \theta)$ being a function let us say; where r is the radial direction theta is the azimuthal direction.

So, we are eventually left with, when you add this two; S_p this one and this term they get cancelled and we are left with only this one and this is equal to$\tilde{V}_1$. So this implies. So this is what we get.

This is what we get. $\tilde{V}_1 = \tilde{V}_1$. So, basically the mass velocity is the s constant. If you have this kind of a thing then p two is equal to p_1 that is given and the field continuity conditions are more fundamental rather than the mass velocity conditions. So basically, mass velocity follows from the enforcing the field continuity conditions. So, when you enforce such a thing for a rigid end plate it can be easily shown that V_2 is equal to V_1 and p_2 is equal to p_1.

So, of course in a way you can say this also follows from the continuity equation, because from elementary fluid mechanics knowledge if you have a port like this and you have sudden expansion. So whatever flow comes here the same quantity goes here from the conservation of mass.

$$\rho_1\, S_1\, U_1 = \rho_2\, S_2\, U_2$$

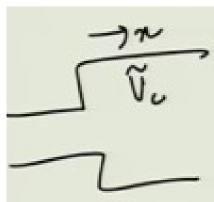
$$\rho_0\, S_1\, \widetilde{U}_1 = \rho_0\, S_2\, U_2$$

$$\tilde{V}_1 = \tilde{V}_2$$

So, from the continuity equation you can also see that, but if you want to have a more rigorous mathematical analysis of why this is valid, no matter even if you disregard the higher order modes. If the frequency of excitation is a is less than a certain frequency, though those modes are cut off that is to say they do not propagate.

Only beyond a certain frequency they become cut on or they start propagating. So basically what happens? As long as we are well within the first cut on frequency, that is to say no higher order modes are propagating, only the planar waves are propagating. So, higher order modes as we recall them from the last from lectures 3 to 5 of week 2.

They have variation across the cross sectional area. So now, if we integrate it across the cross section area the higher order modes in the chamber and basically add them up, we will see that they the in the integral sense.



So, suppose if you have your U, let us say the normal velocity is given by U and U within the chamber U c. So if you have your chamber like this. So, this

$$\int_C \tilde{U}_C\,(r_1, \theta, n) r ds\, d\theta$$

whatever you want to say. So the idea is that for high order mode, if you were to integrate it across the cross sectional area that is this thing.

So, only the fundamental mode will survive the higher order modes will not survive. They will have the entire product over the chamber cross section area will be 0, only the planar wave mode will survive. Because the acoustic pressure field or the acoustic velocity fields can be expressed as sum of infinite summation of modes orthogonal mode. So, from orthogonality, we can prove that only the fundamental or plane wave mode will survive.

That is the reason why we have your continuity of mass velocity. The same logic actually applies for your sudden contraction also. Here also you have your condition

$$\tilde{p}_1 = \tilde{p}_2$$

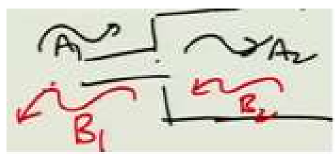
$$\tilde{V}_1 = \tilde{V}_2$$

And velocity mass velocity in the chamber is equal to the mass velocity in the outlet port ok.

So basically, we have this kind of thing. Now what is the mathematical implication of such a thing? Finally, we can write down the equation

Regardless of whether they are, this is the chamber, this is the port or vice versa. This will always hold good. So now, if we were to fix the coordinate system somewhere here, well, here you have your A_1 and fix the coordinate system here at x is equal to 0 here. So,

$$A_1 e^{-jk_0 L} + B_1 e^{jk_0 L}$$



So, this will be

$$A_1 + B_1 = A_2 + B_2$$

So, this is A1 and the wave that goes in this direction reflected back is called p 1 and this is and the waves that are transmitted back is called B_2 and this is A_2; so A_1 A_2 B_1 B_2. However, we are considering a certain area discounting. So we are technically assuming that this part, actually this part is going to infinity. And this and also this part is going to infinity. It is coming from infinity, it is going to infinity.

So it is in so it is basically we are looking we are interested only at what the interface does without worrying about the lengths of the expansion part, that is this part or this part. We have we are not worried about that. So, we are considering only your interface condition and letting each thing to infinity ok.

What will happen?

$$A_1 + B_1 = A_2 \qquad (1)$$

but then your since it is going to infinity nothing comes back from infinity. So, your this thing will go will be 0. There will be because nothing comes back from the from infinity, there is no reflections back.

So, you get one condition like this. Other condition is mass velocity. So remember, mass velocity was nothing but

$$\frac{A_1 - B_1}{Y_1} = \frac{A_2}{Y_2} \qquad (2)$$

and is e to the x complex exponential evaluate to be unity because that is where you are fixing the coordinate system. So, you get another set of equation here.

So, what do we do with this? So, to quantify basically the transmission loss as a result of the incident wave and the waves that is transmitted downstream. So, basically what we know is that A_1. We know A_1 and we need to figure out we need to figure out A_2 and B_1 from equations 1 and 2. And then let us use these equations to figure out B_1 and A_2 as a function of

$$B_1 = f(A_1)$$

$$A_2 = f(A_1)$$

So, how do we get about doing that? We first multiply equation (2) throughout by Y_1. So, we get this.

$$A_1 - B_1 = \frac{Y_1}{Y_2} A_2$$

$$A_1 - B_1 = A_2$$

So now if you simply add this one this fellow and this fellow, so what are we going to get?

$$2A_1 = A_2 \left(1 + \frac{Y_1}{Y_2}\right)$$

We are going to get this.

$$=> \qquad \frac{2A_1\, Y_2}{Y_1 + Y_2} = A_2$$

$$=> \qquad A_2 = \frac{2Y_2}{Y_2 + Y_1} A_1 \qquad (3)$$

We are going to get this. We have to keep this equation a side, we will call this equation 3. So, what about equation for B_1? B_1 then is, we just need to subtract A_1 from A_2.

$$B_1 = A_2 - A_1$$

$$= \frac{2Y_2}{Y_1 + Y_2} A_1 - A_1$$

$$= \frac{2Y_2 - Y_1 - Y_2}{Y_1 + Y_2} A_1$$

So, or let me write down both the expressions for A_1 and B_1 a little more cleanly.

$$B_1 = \frac{Y_2 - Y_1}{Y_1 + Y_2} A_1$$

$$A_2 = \frac{2Y_2}{Y_1 + Y_2} A_1$$

This is what we are going to get. So what does it mean? So, the reflected wave or the wave that is reflected back into the constitute element which from which it came; that is the reflected wave where I am pointing, this one B_1 is this and A_2 is this and this is A_1.

So, let us analyze by taking different values of Y_2 and Y_1, what happens to this value; of course when

$$Y_1 = \frac{C_0}{S_1}$$

$$Y_2 = \frac{C_0}{S_2}$$

So, when Y_1 and Y_2 are the same; that is the characteristic impedance of the inlet ports as well as the chamber are same.

What does it physically mean? There is no area discontinuity in such a case. So, basically the port becomes equal to the chamber. The diameters are the same. So, if

$$Y_1 = Y_2, \quad B_1 = 0$$

and

$$A_2 = A_1$$

That means, the waves whatever they come they go through, I mean they pass through completely without suffering any reflections or any phase change anything at all, because there is no area discontinuity in such a case. So, that is one extreme.

Another extreme is basically, you know when you have a very very small port basically Y_1 is very very small compared to Y_2. So under such a situation you can probably ignore Y_1 in the numerator. So you will basically get B_1 is equal to A_1; that is the entire wave is reflected back and A_2, will become equal to what you can ignore Y_1 and it becomes 2 times A_1. So, A_2 will become 2 times A_1 and B_1 is equal to A_1.

So, there is a complete reversal of or complete reflection of the waves that come in this thing. But, basically these are all extreme cases. What we are interested in some is in some finite values. Basically, what happens for practical system?

Say for example, typically you know for automotive exhaust systems the inlet pipe is typically about $d = 40\ to\ 55\ mm$ for big mufflers and diameter or the chamber is something $D = 300\ mm$ or something like that. So, that is what we are going to get.

So now, what does it mean? So, if we were to derive a general expression for that, what does it mean? So you get, basically let us derive the acoustic power that is incident on the muffler. So, basically if you have something like this kind of system, the power that is incident at the point 1, that is the wave the acoustic power that is carried by the wave front A_1.

So what is that? It is incident power expression, if you recall from our last lectures incident. So, this is 1, this is 2. So, this is incident at point 1. So this is A_1, mod A_1 and this is transmission is that one that goes here.

$$W_{inc} = \frac{|A_1|^2}{2\rho_0 Y_1}$$

So that is nothing but mod. If you recall, we just found out what Y_2 is in terms of A_1. So, this becomes then

$$W_{trans} = \frac{|A_2|^2}{2\rho_0 Y_2} = \frac{4Y^2}{2\rho_0 (Y_1 + Y_2)^2} |A_1|^2$$

Similarly, we can find out the power that is reflected back.

$$W_{reflected} = \frac{|B_1|^2}{2\rho_0 Y_1}$$

So, we need to be correct with this thing. So actually, I guess I figure I kind of forgot this term here, apologies. So, then basically this will get cancelled, this will get cancelled and we will see how all these thing works. And let us quickly derive the expression for this one.

$$= \left(\frac{Y_2 - Y_1}{Y_2 + Y_1}\right)^2 \frac{|A_1|^2}{2\rho_0 Y_1}$$

Based on our definition of transmission loss, TL is defined as nothing but

$$TL = 10\, log_{10} \left| \frac{W_{inc}}{W_{trans}} \right|$$

of incident power to transmitted power. So, we know now the transmitted power and the power that is acoustic power that is reflected. So, T L then,

So, here you are there the square thing and this is A_1 mod. So, this of course, gets cancelled. So thus

$$TL = 10\, log_{10} \left| \frac{\frac{\cancel{A_1}}{\cancel{2}\rho_0 Y_1}}{\frac{4Y_2}{\cancel{2}\rho_0 (Y_1+Y_2)^2} \cancel{|A_1|}} \right|$$

So, if you simplify further this T L is

$$TL = 10\, log_{10} \left| \frac{(Y_1 + Y_2)^2}{4Y_1 Y_2} \right|$$

Of course, Y_1 plus Y_2 whole square divided by 4 Y_1 Y_2. So, this can further be written in terms of the areas. So, S_1 and S_2 they are always positive, so we need not write mod. Now this is something very interesting. So why is this interesting? Because transmission loss of a, you know all these lecture I was talking about transmission loss being a function of frequency.

Well, it was implied in any case. That is because in some frequencies the transmission loss or the acoustic performance of a muffler is good and some frequencies it is not. But here you see there is no dependence at all on the frequency. So, why is that? Well, because it is a kind of semi-infinite elements and it is a characteristic feature of us just a simple area discontinuity in which we assume only planar wave propagation.

If there is no reflection there is only unidirectional wave and; obviously, some part is reflected back, but there is no standing wave that is formed in the from the downstream element. So, as a result I mean the maths works out in such a manner that the transmission loss of a simple expansion or a sudden area expansion or a sudden area contraction just analyze in isolation will be independent of the frequency and it will be basically,

$$TL = 10\, log_{10} \frac{(S_1 + S_2)^2}{4S_1 S_2}$$

So, actually this will be the same for sudden area sudden expansion and sudden contraction, sudden expansion or sudden contraction they are just the same. So, we can; obviously, take extreme cases and keep analyzing that. But we can actually draw some reduce some very interesting thing.

For example, I talked about some numbers. So, like the port area being saved. Let us say $d = 50$ mm and diameter of the shell becoming $D = 350$ mm, really large mufflers for automobiles. So, the diameter ratio is 7 and area ratio chamber to port is

$$\frac{D}{d} = 7 \quad \frac{S_C}{S_p} = 449.$$

So, what happens here is that, if you were to simplify this thing further you will get

$$TL = 10\ log_{10} \frac{S_1^2 + S_2^2 + 2S_1S_2}{4S_1S_2}$$

$$TL = 10\ log_{10} \left[\frac{1}{4}\left(\frac{S_1}{S_2} + \frac{S_2}{S_1} + 2\right)\right]$$

Now, we figured out S_1 by S_2. So let us say this is 49, this this becomes 1 by 49.

$$= 10\ log_{10} \left[\frac{1}{4}\left(49 + \cancel{\frac{1}{49}} + 2\right)\right]$$

We can probably sort of ignore this part this bit. So, we can basically figure out how much it would be, log of 13.

$$= 10\ log_{10} \left[\frac{S_1}{S_2}\right] \simeq 10 log_{10}\ 13$$

So, we can actually do some calculations and figure out what is log of 13. So, turns out that it is

$$TL = 10 \times 1.1139$$

So that

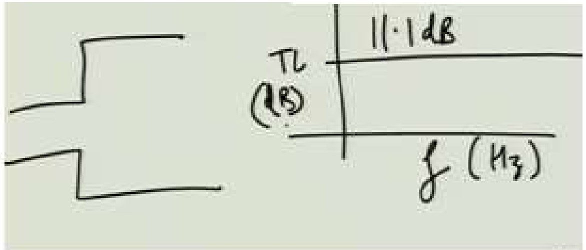
$$= 11.13.9dB$$

$$\simeq \ 11.1dB$$

So if you have a sudden expansion like this, it will produce a constant frequency attenuation which is a constant graph of roughly 11.1 dB for expansion ratio.

$$\frac{D}{d} = 7$$

$$M = 49$$



So, we can actually do some this is transmission loss in dB and this is frequency in hertz and this is in dB. So, we can actually let us let me go a little further and put some and basically kind of simplify this in terms of the area ratio. So if we want to call the expansion chamber

$$\frac{D}{d} = M \; ; \frac{d}{D} = \frac{1}{M}$$

$$\frac{D^2}{d^2} = M^2 \; ; \frac{d^2}{D^2} = \frac{1}{M^2}$$

So, what is the area ratio? This is so the usually if M is large, this is much much smaller. So we can sort of ignore this thing. So if we were to put this value in here,

$$TL = 10log_{10}\left[\frac{1}{4}\left(M^2 + \cancel{\frac{1}{M^2}} + 2\right)\right]$$

$$= 10log_{10}\left[\frac{M^2+2}{4}\right]$$

$$M = \frac{D}{d}$$

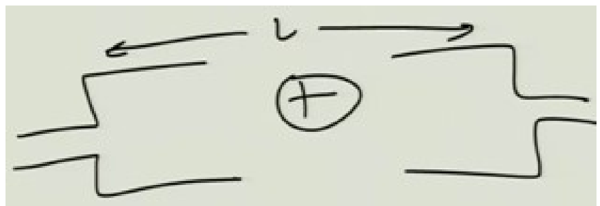
Well, I guess the sudden contraction will also figure will have the same match. And I probably leave it for the students to figure out what happens for a sudden contraction.

Still you will you will get identically the same expression where I have put a tick mark. So, this was the transmission loss analysis for a sudden expansion or sudden contraction, assuming only planar wave propagation. We are not done any lumped system approximation, because it is not necessary also; planar wave will suffice it is easy to use and all that.

And we can also see that using simple maths will see that another one final thing that I want to mention here, that incident energy is equal to transmitted part plus the reflected part.

$$W_{inc} = W_{trans} + W_{rfl}$$

$$\frac{(A_1)^2}{Y_1} = \frac{(A_2)^2}{Y_2} + \frac{(B_1)^2}{Y_1}$$



So, 1 by 2 ρ_0 factor gets cancelled. And once you substitute A_2 and B_1 in terms of A_1 you can kind of verify this identity here. So, with this I will stop the lecture stop this lecture 4 for week 4 and in the next class we will worry about the simple expansion chamber; that is when this part is combined with this part and you now consider the effects of finite length L.

What is going to happen to the transmission loss? Is it going to be increase, decreased or is it going to first of all it is going to vary whether it is going to vary this frequency or not. All those things we are going to see in a detailed manner in the next lecture; lecture 5 of week 4. Thank you stay tuned.