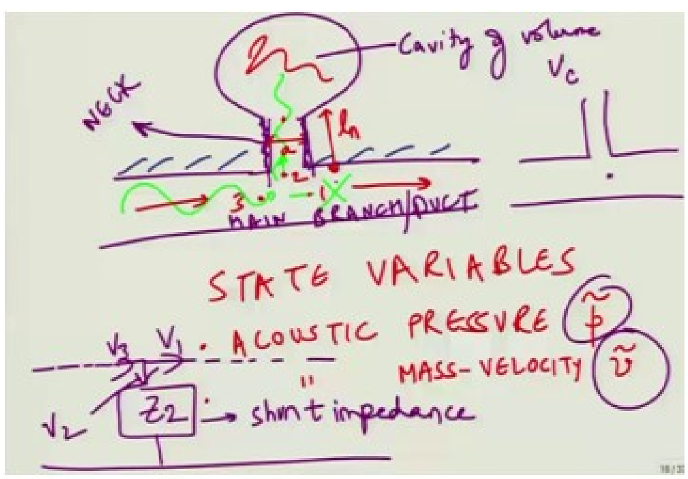
Muffler Acoustics - Application to Automotive Exhaust Noise Control
Prof. Akhilesh Mimani
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Lecture – 15
Helmholtz Resonator, Electro-Acoustic Analogy and Layout of a Typical Engine

- **HELMHOLTZ RESONATOR**
- **ELECTRICAL CIRCUIT**
- **REPRESENTATION OF A GENERAL ACOUSTIC SYSTEM**

Welcome to the last lecture, lecture 5 of week 3. So, the contents for this lecture are Helmholtz Resonator. We will discuss about properly formulating the Helmholtz resonator problem, drawing the equivalent electro acoustic circuit for that, showing that as a shunt element, we will worry about that in a while, and then the electrical representation of a general acoustic system, basically a muffler system talking about the muffler proper, tail pipe, exhaust pipe and so on.

So we will worry about all those things just in a bit. And then in the next week we are probably going to talk about the different constituent elements. This week's lecture will provide sufficient background to formally enter into the performance measure nomenclature of mufflers and constituent elements using planar wave theory and so on.



So often, what happens in systems, you have a cavity of volume V well V_c like to be consistent with our terminology. And let us say the length from here to here, I mean let us say from this point to this point is ln and let us say the radius is a ok, and this point is this point is 2, this is 3, this is 1.

So the wave goes from here to here and you would like to relate what happens to the state variables. So now, one thing you would have noticed in this course in this set of lectures, that I have been we have been introducing terms as we come across it for the first time. So, what are the state variables?

So, state variables at least in the context of muffler acoustics or general ducted system yeah, so these are your acoustic pressure variable denoted by $\tilde{p}$ and acoustic, well mass velocity is what we are working in because of conservation of mass. Let us say $\tilde{V}$. So we have this, we have this. So, the idea is to relate these two state variables;

- **STATE VARIABLES**
- **ACOUSTIC PRESSURE** $\tilde{p}$
- **ACOUSTIC MASS VELOCITY** $\tilde{V}$.

at the point 3 to the point 1.

So now in all systems, if you know what is happening with two variables; pressure and velocity, you know the entire system, because remember our equation is a second order differential equation. So we must know two things; acoustic pressure and acoustic velocity. So that is how you eliminate the amplitudes associate with the particular direction of the wave propagation and relate everything in terms of these state variables pressure and velocity.

So, that is what we measure. The acoustic pressure we can measure using specialized microphones, the ducted microphones. Anyways, so the thing is that how do we go about how do we go about relating this and why do we use such a peculiar looking structure. So, this Helmholtz resonator is actually quite popularly used in since several decades for typically for controlling certain tones or controlling noise in a very narrow band.

So, typically in air cabins, cabins of aircrafts and some other machinery where tonal noise is dominant, we use not just one, but a series of resonators Helmholtz resonator whose length this is by the way the neck or the throat and this is the cavity and this is the main branch or the main duct slash duct.

So, the idea is by effectively manipulating or controlling the different physical parameters dimensions, like the radius or the length or the neck or the throat and the cavity. Volume of the cavity which could be any irregular cavity, spherical cavities, almost spherical cavities are used or some other box kind of a structure or something, which would radiate nest noise and adjust keep adjusting the volumes and all those things in such a manner that some particular tones are eliminated.

And when you use a certain combination or a certain area of such resonators we can control not just one large number of such tones. Because tones can occur at range of different frequencies. The idea is that the when the wave is incident say somewhere in here, so if we can have the impedance at point 2 to be nearly 0; obviously, this will happen only at one frequency, it is hard to do it for all frequencies.

Quarter wave resonators; obviously are different, you have impedance going to 0 at different frequencies, but for low frequency Helmholtz resonator application for a particular frequency low frequency just at the entry of the Helmholtz resonator impedance is 0, that is at this point.

So when such a thing happens all the acoustic wave energy gets diverted here and almost no energy no acoustic wave is transmitted downstream. So, in that way we can actually control the propagation of sound downstream and direct all the sound towards the cavity.

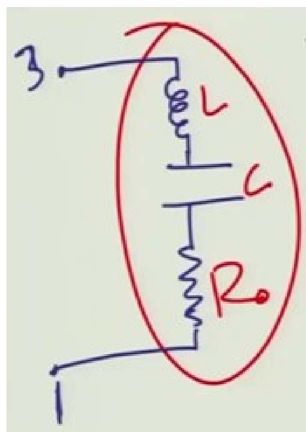
So, it is in a way of using the impedance mismatch or impedance matching principle to divert or deflect acoustic waves in a certain direction and not allowing the waves to propagate downstream, where you might have a listener.

So this is your ear, where you listen. You do not want the noise to come here. So, how do we go about it? The idea is, basically let us write down the equivalent or the effective impedance seen at the point 2.

So clearly the neck acts like a inductive element and the this thing, the capacitance of the cavity acts like a compliance. So, that is a capacitance. And then; obviously there will be some radiation losses as well because this part the neck is exposed to the big volume and to this thing.

So, it is like the duct it is acting like a cavity or certain acoustic pipe which is open at a certain end, actually at both ends, but at each end you can consider this like surrounded with a large infinite flange. Nothing is infinite, but in terms of wavelength. So, this is much larger than the dimension of the pipe. So it you can consider this thing as a flanged end here, and here also the cavity itself is a acting like a flange to the duct.

So, what happens then is that there is also some radiation losses as we saw from the immediately preceding lecture. So, you also have basically something like a resistor resistive kind of a thing and here is your state variable say well 3 and 1, and this is what we intend to do LCR.



So, what are those? Z, let us say we want to find out Z_2.

$$Z_2 = j\omega L + \frac{1}{j\omega C} + k_0$$

So this will become, if you go back to this figure this is *l*n, but due to the end correction let us

$$= j\omega \frac{leq}{S}$$

So, this S is nothing, but pi a square, assuming it to be a circular cross section. Now, you also have this term. So, what do we do about it?

$$+ \frac{1}{\frac{j\omega V_C}{C_0^2}}$$

We saw in the last class; obviously, this is in terms of the mass velocity, so that is why you are seeing slightly different form, but that is ok. And remember, what we discussed in the last lecture? We discussed about the relation impedance. So what was that?

Let me bring out the notes what we discussed in this lecture, you have this term k naught a whole square divide by 2, and you need to multiply this by y n and 2 times because you have this thing twice.

Then it becomes

$$+2Y_0 \frac{(k_0 a)^2}{2}$$

$$Y_0 = \frac{C_0}{S}$$

We need to do some simplification, some math's.

$$S = \pi a^2$$

$$R_1\ (2k_0 a) = 1 - \frac{2J(2k_0 a)}{2k_0 a}$$

$$\frac{(2k_0 a)^2}{2 \times 4} - \frac{(2k_0 a)^4}{2 \times 4^2 \times 6^2} + \cdots$$

$$\simeq \frac{(k_0 a)^2}{2}$$

But before that we will keep this expression aside or possibly what we could do, we could just sort of copy this in a more, clean manner.

$$Z_2 = \frac{j\omega l_{eq}}{S} + \frac{C_0^2}{j\omega V_C} + 2Y_n \frac{(k_0 a)}{2}$$

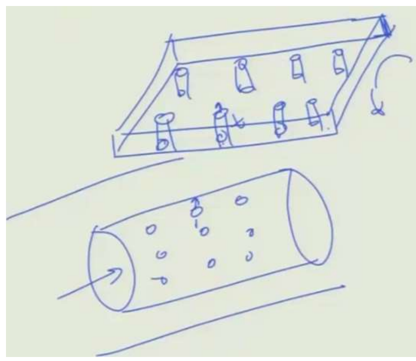
So, let me just circle it for you guys.

It could be 40 mm, 50 mm whatever it is and plus end corrections. Now, if you recall like I have been mentioning a few times, this is like a cavity an open ended tube in an infinite flange and so is this part. So, we tend to add 0.85 times the length. So, recall our last discussion.

$$X_0 = \frac{4}{\pi}\frac{(2k_0a)}{3} - \frac{(2k_0a)^2}{3\times 5} + \cdots$$

$$\simeq \frac{8k_0a}{3\pi} \simeq (0.85k_0a)$$

$$Z_{0c} = \rho_0 C_0 \frac{k^2}{2\pi} + \frac{j\omega\rho_0 S}{S^2}\left(\frac{8a}{3\pi}\right)$$



$$\rho_0 S \left(\frac{8a}{3\pi}\right) - 0.85a$$

$$\frac{kg}{m^3}m^2 . m$$

So, we had this magical number here, this one which gave us 0.85 times a. It is like a small volume of small cylindrical volume of fluid that is added to the end. It is a hypothetical cylindrical volume.

$$t + (0.85a) \times 2$$

$$t = t + 1.7a$$

So 1.7 a is the radius, and you also need to account for the thickness of the wall of the propagation tube.

So this is your tube, it will definitely have some thickness; 1 mm, 2 mm, 3 mm 2, and this thickness is important because not only from the acoustic end correction point of view, but also to figure out I mean to enforce the structural rigidity so that the breakout noise is not there, otherwise it defeats the purpose.

So, here you have your plus t or t w t wall you can think of this. So, whenever you are considering the length of the neck, it is important to consider the end correction. So 1.7 a, it may so happen depending among the radius that 1.7 a, the end this particular thing is quite comparable to l n, in that case it will be a mistake to neglect the end correction.

The point I am trying to emphasize is the importance of these values. With this background if we simplify the top most expression in brackets we will get

$$l_{eq} = ln + 1.7a + t_w$$

$$Z_2 = j\left\{\frac{\omega l_{eq}}{S} - \frac{C_0^2}{\omega V_C}\right\} + \frac{\omega^2}{\pi C_0}$$

So, that is what we are going to get. And once we simplify this term we will get $\frac{\omega^2}{\pi C_0}$ that is what you are going to get. So now, clearly things are much much more clear about what will happen now. So, these are some of the questions you can also anticipate in the exams.

These are some of the fundamental questions, what will happen at what frequencies will the Helmholtz resonator exhibit resonance? So, clearly the impedance has a real part an imaginary part a +I b or a + j b. So, this is reactance, this is resistance. This is resistance, this is reactance. So, again we are introducing some of the concepts as we go on the go.

So, one typical rules that we follow for getting the resonance frequency when you know a impedance form, which is

$$Z = a + jb = 0$$

No matter how complicated a and b look, just by

$$Reactance = 0 \rightarrow \text{Resonance frequency}$$

So, what about this case? Well, it turns out that

$$\frac{\omega l_{eq}}{S} = \frac{C_0^2}{\omega V_C}$$

$$\omega_n = C_0\sqrt{\frac{S}{l_{eq}V_C}}$$

$$f_n = \frac{C_0}{2\pi}\sqrt{\frac{S}{l_{eq}V_C}}$$

So, resonance frequency let me call this ωn. So this is what we are going to get. So at this frequency, omega or this either this or either this, at this frequency the reactive part is 0, so the impedance is the minimum there, I will not say 0 because of non-zero a part it can be non-zero. So, the almost the entire acoustic power would be going off in that direction.

So now, so this is the resonance frequency. So, the resonance frequency of resonance frequency of a HR, Helmholtz resonator is

$$\omega_n = C_0\sqrt{\frac{S}{l_{eq}V_C}} \quad low\ frequency$$

So, you need to one of the cross section area or the pipe that is your radius, volume of the cavity through some means one needs to evaluate that length and then add suitable end corrections and finally evaluate omega.

So, it can definitely help you eliminate certain tone then this formula is valid for low frequencies low frequencies. So, how do you determine that the frequency is low or not? So, this is the actually a matter of analysis for the later parts of the course, but just want to mention here that every cavity has some resonance frequency. So one can do a 3D finite element analysis get an algebraic eigenvalue problem for this guy and get the resonance frequencies for this one. And for this pipe we know.

So whatever value we are getting using this equation, let us say star that should be less than the resonance frequencies of the cavity at least. So then we are pretty much we can be confident. And, talking about the **Electro Acoustic Circuit Representation of a HR Helmholtz Resonator.**

How does it look like? So, basically it would look like, I mean the equivalent form of this one I should have drawn it here, it is probably something it looks like. So, this goes on and you have here you have your Z_2, that is what we called it right.

So, and this is like this, this is here. So, V_3, V_1 and this is V_2. So this goes on like this. So this is the shunt, a shunt impedance a shunt impedance of a typical side branch resonator.

This is also true for, I mean if you were to have something like a situation, where you have another system like a quarter wave resonator with a closed end, this electro acoustic circuit in principle would also represent this thing.

And, suppose you do not have a closed end, you have an open end, still it would do just as a resonance frequencies would change. So, one thing is clear that the acoustic pressure

$$p_3 = p_2 = p_1$$

So we will write down those relations alongside here. So, p_3 is equal to p_2 is equal to p_1.

So we need to go back and forth to this thing, p_3 acoustic pressure is same at these three points. They are just for illustrating purpose they are far apart, but they can be closed and they very close, so they are actually represent one point. And the velocity, mass, velocity whatever comes here gets split into two parts.

One that goes inside and one that goes in the downstream direction. So, basically this is one equation and

$$V_3 = V_2 + V_1$$

All these equations we will see, this one and this one will keep occurring for different configurations.

So what we could do is basically write the other part, the second equation for velocity

$$V_3 = \frac{V_2}{p_2} p_2 + V_1$$

$$=> V_3 = \frac{p_2}{Z_2} + V_1$$

Because we already use

$$p_3 = p_1$$

So, basically that is how the upstream variables here is related to the downstream variables.

We can write it in a these two, let me call this star, double star star and double star equation in a more compact manner in the matrix form popularly known as vector is equal to T matrix.

$$\begin{Bmatrix} p_3 \\ V_3 \end{Bmatrix} = [T] \begin{Bmatrix} p_1 \\ V_1 \end{Bmatrix}$$

What is T matrix? The transfer matrix. Again a new term that is introduced. So, transfer matrix is nothing you will need to play a lot with the transfer matrix it will be a bread and butter throughout the course.

So this is called, let me write it in big font, transfer matrix. Why is it called transfer matrix? Because it can using such a compact nice matrix representation we can transfer our self how we can transfer the we can describe the downstream system in terms of downstream state variables in terms of the upstream state variable, is like transferring yourself to the downstream from the upstream.

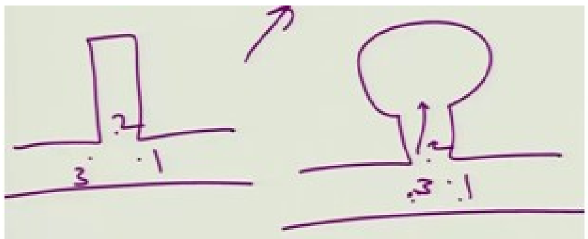
So, there is something like cascading of matrices and all those things by which you can analyze complicated muffler system, which are arranged in a sequential manner. We will worry about all those things maybe in the ensuing lectures.

$$[T]_{2\times 2} = \begin{bmatrix} 1 & 0 \\ \frac{1}{Z_2} & 1 \end{bmatrix} \begin{Bmatrix} p_1 \\ V_1 \end{Bmatrix}$$

But the point is, that this T matrix; transfer matrix which is really a 2 cross 2 matrix as a simple form in this case one. So, let us figure out this p 3 and p 1, so it becomes 1 here and 0 here for the first equation because the other thing is 0, there is no velocity dependence for this one.

So basically, here it will become Z_2 and here it will become 1. So, that is what the transfer matrix representation for a HR Helmholtz resonator, and in principle this is exactly the same form for a quarter wave resonator which acts like a side branch.

$$\begin{Bmatrix} p_3 \\ V_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{Z_2} & 1 \end{bmatrix} \begin{Bmatrix} Vp_1 \\ V_1 \end{Bmatrix}$$



So basically, whether you have your this system or you have your simple quarter wave resonator act like a side branch, they are always represented by the shunt element. The circuit for that was shown in a slide back and this is the transfer matrix relation.

So, what I would do is this represents say 3, 2, 1 3 2 and 1, P_3 and V_3. So that is what it is. It and it is also true for open edit. So any side branch element is can be represented in terms of the transfer matrix representation, which relates the upstream state variable to the downstream one.

This of course is based on a planar wave propagation and we can worry about the other things later. So this is about the Helmholtz resonator and we have introduced a lot of new concepts in this lecture.

First of all reactants going to 0, impedance being a complex quantity by setting the reactance or the imaginary part to 0 we can get the resonance frequencies. Of course, you will not just get one, you will get the sequence of resonance frequencies if you obviously consider a planar wave analysis and not just a lumped system analysis. You might have to solve transcendental equations for that or solve them numerically.

Then we introduce then we introduce things like state variables and we had a electro acoustic representation of a acoustic system. And then we introduced things like or got acquainted with a term called transfer matrix also used in electrical circuits.

There is yet another form of matrix representation used very popularly in electrical circuits and to a lesser extent in acoustics, but nevertheless important and that is your impedance matrix. So let me, since we are discussing that let me just also introduce you that part to you.

An impedance matrix, what it does is basically, it relates we need to really eliminate the second part because of the wave goes here. We want to relate this to this or relate the get a relation between the state variables at point 3 and 1.

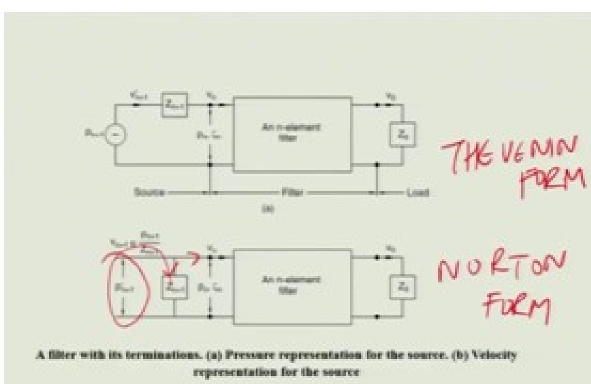
So, instead of only relating one port or one end to another end, we could do a combination of that. Meaning, we can relate pressure at these two points to the velocities mass velocities at these points.

Impedance which is a fundamental concept in acoustics that is being used here, an impedance matrix.

$$\begin{Bmatrix} p_1 \\ p_3 \end{Bmatrix} = [z]_{2\times 2} \begin{Bmatrix} V_1 \\ V_3 \end{Bmatrix}$$

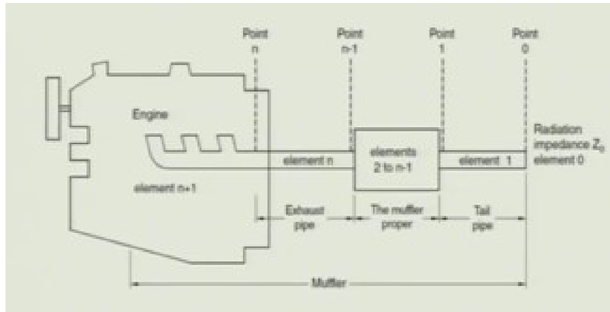
So, the idea is what will be the form of this thing.

So we will do the derivation in the next class. And before we end the presentation of this lecture or this topic, I would basically want to show you figure which basically shows the muffler thing.



A filter with its terminations. (a) Pressure representation for the source. (b) Velocity representation for the source

So, what does this filter mean? We will continue our discussion in the next weeks lecture, but as a food for thought for you guys for this week what does this do?



Typical engine exhaust system

So, any automobile system, so let us say you have the engine block somewhere here, engine block and there are so many complicated elements. This is called the exhaust pipe. It is called the exhaust pipe which basically discharges all the products of combustion through the atmosphere.

If this was not there, the muffler if it was not there it would make a lot of noise and back about a century back or maybe more when the automobile was just probably invented, an ancestor of the present day automobile is very fancy, when it was invented a century back or. So, there was no mufflers and the engine made a lot of noise and that really disturbed the people.

So, the muffler really kind of helps in controlling the noise emission because what it does it is it basically whatever engine noise comes here where I am pointing, it kind of reflects the acoustic power back or absorbs it. It is a reactive muffler or a dissipative muffler, we will talk about that later, perhaps in the next week just the beginning of the next week.

And properly formally introduce you to the different classes or different categories of muffler, but the idea is that uses block diagrammatic representation of a typical engine exhaust system to depict how does the entire thing look like.

So this is your muffler proper. You can have as much complicated as you know you can have very complicated internal structure here. You can have 2, 5 different combination of mufflers with pipes being connected in an arbitrary fashion with a to solve a certain purpose, to control noise in a certain frequency or to consider a low back pressure requirement and all that.

And this is the tail pipe which basically carries out the discharge of hot exhaust gases into the atmosphere, but only after the noise the disturbances that were transmitted in the upstream that were significantly abated or reduced by the action of this muffler. So, this then is the muffler proper and the entire course is about efficient design of that and all these lectures were leading to that.

So now, before we end this week's lecture in this lecture, we let us using this block diagram let us go back to the electro acoustic representation of that. So now, every acoustic engine is creating lot of noise. Is creating you know pressure disturbances p you can call this it has got certain amplitude, and it has got some its own frequency characteristics.

So, we can represent the complex acoustic thing as P_{n+1} and it the engine has some impedance just like atmosphere imposes certain impedance on the propagation of waves, the engine also has some certain impedance also known as source impedance.

So that is represented by the Z block impedance and this is a in line. The above is called the pressure representation and the below representation is called velocity. And this is the n element muffler which has its own impedance and Z_0 is the atmospheric impedance.

So, what happens is that in this pressure representation volume velocity or the mass velocity for the acoustic disturbances they basically V_{n+1}, it is the same as V_n here, because its going it is in the same circuit. There is a pressure drop of course, across this impedance element and then you have an action of the filter.

So, it is really a filter element. It basically if there is an impedance matching then all the acoustic power will go downstream and will be transmitted into the atmosphere. So the muffler will be useless at that frequency.

So the idea of muffler is to create maximum impedance mismatch at least one of the concepts, so that all the acoustic power or significant part of the acoustic power for a given frequency range is reflected back. And then you complete the circuit.

So this is called the pressure representation of the engine exhaust system, and the other representation is the velocity representation. Now, in this, since the engine block has certain impedance and just a while back we are discussing about the shunt impedance. So the pressure what you see here is it still the acoustic pressure of the source is the same.

What you see here is that the impedance of the engine block that is your this particular guy is moved in the shunt in the side branch, and so the velocity gets played into this part and this part, pressure is the same just like we saw for the side branch resonator and then this is the muffler and everything else is the same.

So this is called the velocity representation. So, velocity representation is called the so there are two names for this. So, the source representation or the pressure representation this is called the Thevenin form, and the other one is called the Norton form. So, this is the Norton form this is the Thevenin form.

So, we have got these two representations of the muffler. So, with this let us end this week's lecture. And I will see you in the next week with more stuff on the theory of acoustic filters, in which we will probably discuss about the simplest element. The sudden area expansions to begin with and the mathematical analysis and all those things regarding that.

So thanks for attending.