Turbulent Combustion : Theory and Modelling
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Lecture-37
Turbulence (contd…)

Welcome back, Let us continue the discussion on the turbulence scaling. So, we are looking at the effect of buoyancy and we derived the governing equations like contnuity momentum and temperature equation and we have seen the extra term which comes because of due to the effect of buoyancy.

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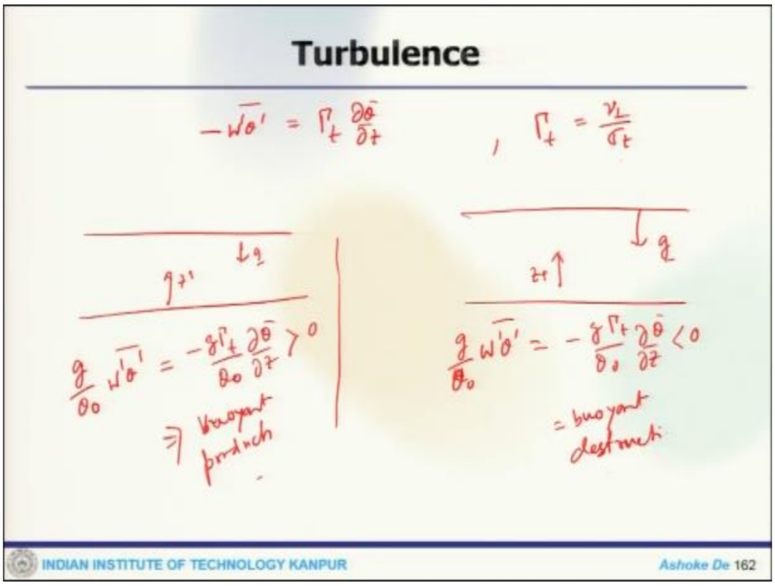

Now, we are looking at the buoyancy production and destraction. This is the one if you have a parallel plate and then this part is where you get buoyant production and this is the buoyant destruction. Now, we can look at for this plane channel flow. This is a plane channel flow we can look at the buoyancy verses shear, so buoyancy versus shear. So now, turbulant kinetic energy budget for this fully developed plane channel flow with equal heat flux.

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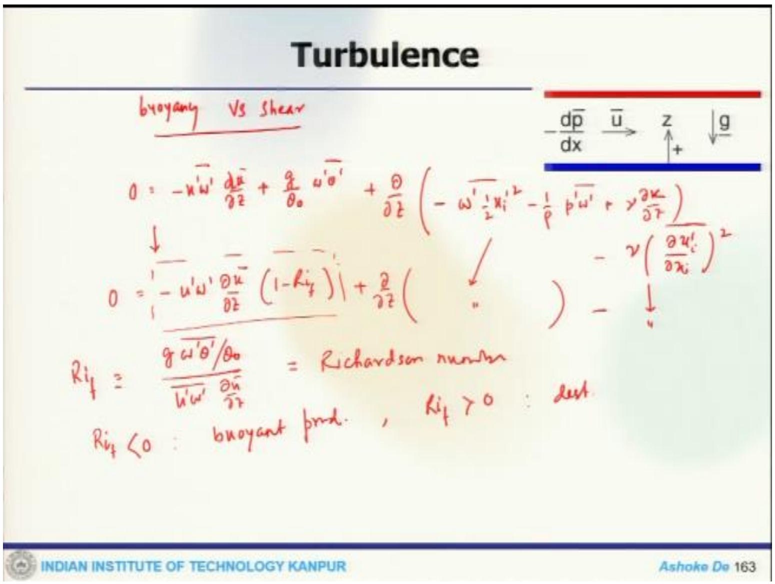

And bottom and top we get like

$$0 = -u'\overline{w'}\frac{\partial \bar{u}}{\partial z} + \frac{g}{\theta_o} w'\overline{\theta'} + \frac{\partial}{\partial z}\left(-w'\overline{\tfrac{1}{2}u_i'^2} - \frac{1}{\rho}p'\overline{w'} + \upsilon\frac{\partial x}{\partial z}\right) - \upsilon\left(\frac{\partial u_i'}{\partial x}\right)^2$$

So, this we can rewrite in a different from like

$$0 = -u'\overline{w'}\frac{\partial \bar{u}}{\partial z}\left(1 - R_{if}\right) + \frac{\partial}{\partial z}\left(-w'\overline{\tfrac{1}{2}u_i'^2} - \frac{1}{\rho}p'\overline{w'} + \upsilon\frac{\partial x}{\partial z}\right) - \upsilon\left(\frac{\partial u_i'}{\partial x}\right)^2$$

So, this particular term one can see is in this is where

$$R_{if} = \frac{\dfrac{g w'\overline{\theta'}}{\theta_o}}{\overline{u'w'}\dfrac{\partial \bar{u}}{\partial z}} = \text{Richardson Number}$$

So, one can also categorize if R_{if} less than 0. Then we get buoyant production and if R_{if} greater than 0 you get destruction, buoyant destruction. Now net turbulent kinetic energy production if R_{if} greater than 1 which is the upper bound of that. Now, you can see this effect of buoyancy on overlap region.

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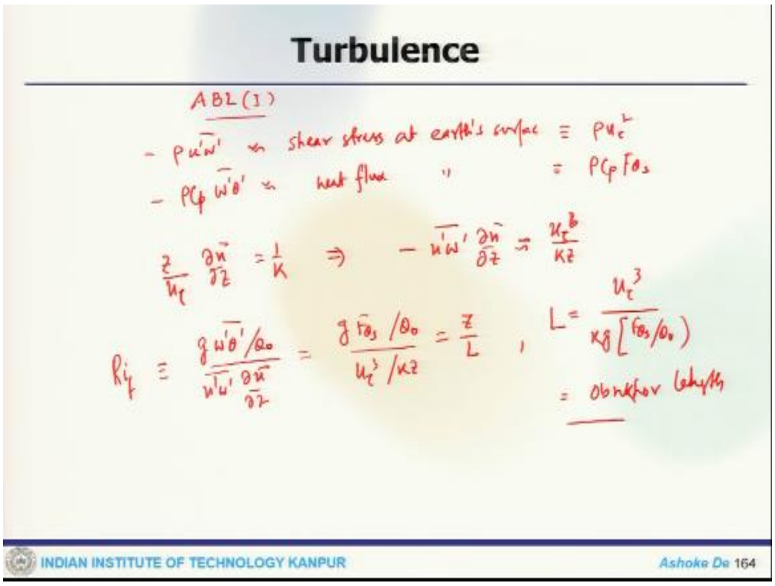

So, you can think about this is an atmospheric boundary layer, where in the overlap region is the region between your inner layer and outer layer. So, the turbulence shear stress can be approximated at shear stress, basically this is shear stress at earth's surface which can be

$$-\rho u'\overline{w'} \approx Shear\ stress\ at\ earth's\ surface \equiv \rho u_\tau^2$$

$$-\rho C_p \overline{w'}\theta' \approx Heat\ flux\ at\ earth's\ surface \equiv \rho C_p F\theta_s$$

Now, if the buoyant effect on mean velocities small which is a neutral boundary layer case then the mean velocity in overlap region one can write

$$\frac{z}{u_\tau}\frac{\partial \bar{u}}{\partial z} = \frac{1}{\kappa} \Rightarrow -u'\overline{w'}\frac{\partial \bar{u}}{\partial z} \approx \frac{u_\tau^2}{\kappa z}$$

by u tau del u by del z equals to 1 by kappa.

Now, in this case,the flux Richardson Number in overlap regin will be equal to

$$R_{if} = \frac{g\overline{w'}\theta'/\theta_o}{u'\overline{w'}\frac{\partial \bar{u}}{\partial z}} = \frac{gF_{\theta_s}}{u_\tau^3/\kappa z} = \frac{z}{L}$$

where

$$L = \frac{u_\tau^3}{\kappa g[F_{\theta_s}/\theta_v]}$$

which is also called obukhov lengh. So that's how you can get these things.

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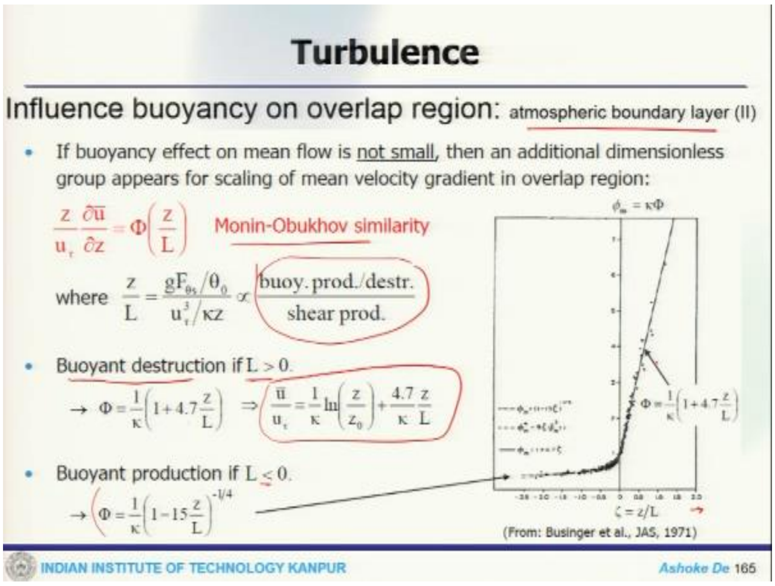

Now, similarly, one can look at the buoyancy effect on overlap region which is an atmospheric boundary layer II. Now, if the buoyant effect on mean flow is not small so, previous case we said it is small then an additional dimension less group appears. So, which is like

$$\frac{z}{u_\tau}\frac{\partial \bar{u}}{\partial z} = \Phi\left(\frac{z}{L}\right)$$

this is how are z by L essenitally the buoyant production or destruction to the shear production so, that is the ratio is the z by L.

Now, if L greater than 0 it is an buoyant destruction because your phi becomes

$$\Phi = \frac{1}{\kappa}\left(1 + 4.7\frac{z}{L}\right)$$

where using that you can get the mean velocity profile like that and if L less than 0 this is a production. So one can plot this, in this axis is ratio z/L and look at these function and see how this actually varies. Now, another situation which may appear is the neutral atmospheric boundary layer where you can have this limit phi by z/L would be 1/kappa in overlap region. Now, typically atmospheric boundary layer this length scale is quite high.

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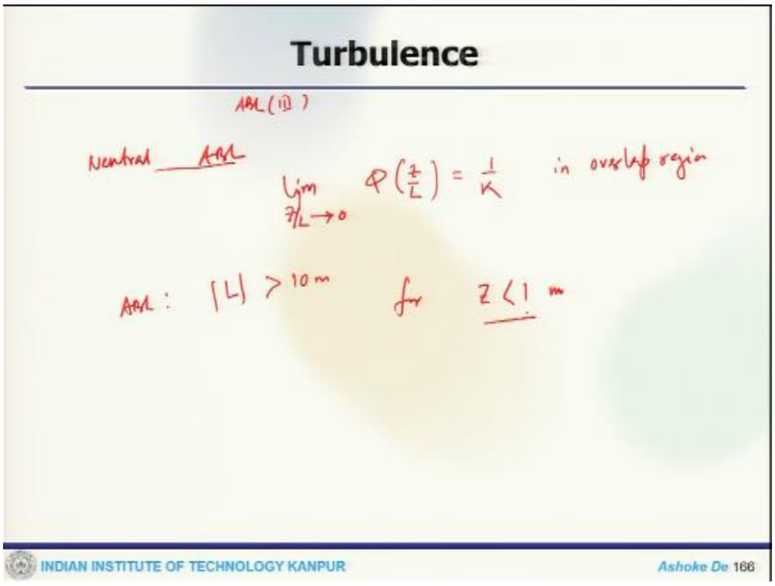

So, the atmospheric conditions are approximately neutral for z less than 1 meter so, if that is the situation one can. Now we go recap our statistical description where we had this RANS equations. This is continuity, this is momentum and this is Reynolds system. Now here we got 1 continuity equation, 3 momentum equation or 3 component of the moment equation so, total we get 4 equations but we got here total 10 unknowns like P, u_i, u_i u_j prime physically 3 velocity component 1 pressure component 4 plus 6 reynolds stress component.

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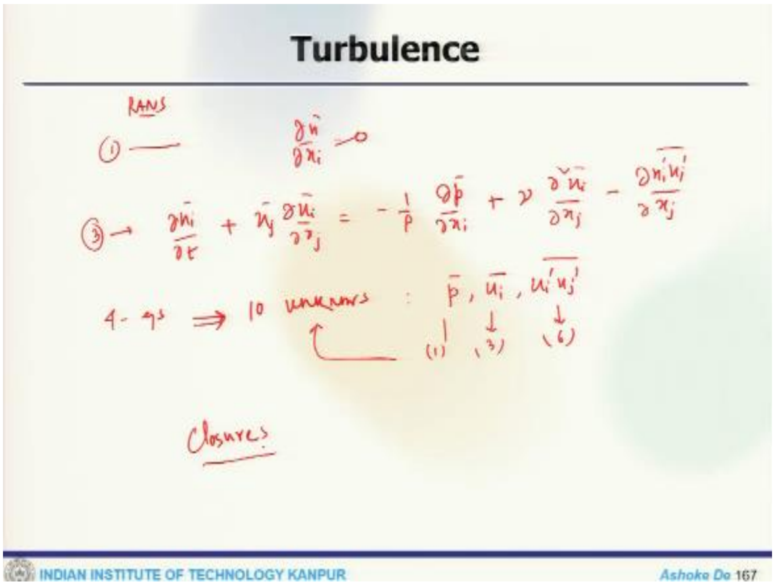

This is 6 component, this is 3 and this is 1. So, total it lead to 10 unknowns. So, we need closures otherwise, we cannot solve this system of equation. Now, if we rewrite the Reynolds stress term that is

$$-\rho\overline{u_i'}u_j' = -\frac{2}{3}\rho_k\delta_{ij} + \rho(-\overline{u_i'}u_j' + \frac{2}{3}k\delta_{ij})$$

these term isotropic part and this is the deviatric part. So, this is essentially your turbulent shear stress and this is turbulent pressure. Now, using the turbulent viscosity hypothesis.

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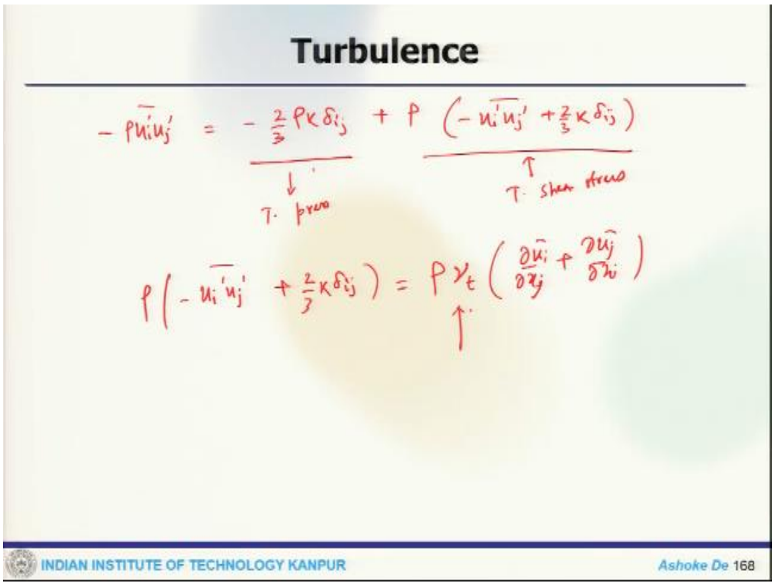

So we have close this term like this deviatric part or turbulence stress part

$$\rho\left(-\overline{u_i'}u_j' + \frac{2}{3}k\delta_{ij}\right) = \rho\nu_t\left(\frac{\partial\overline{u_i}}{\partial x_j} + \frac{\partial\overline{u_j}}{\partial x_i}\right)$$

ν_t here is the turbulent eddy viscosity. So, if you recall the scaling of that thing, so, this is an velocity profile and if this is the size of the eddy, the velocity scale would be u(l). So, turbulent viscosity has the dimension that are velocity into landscape. So, for ν_t, we can say that it would be a order of u_o and l_o and this u_o and l_o for scale of large eddies.

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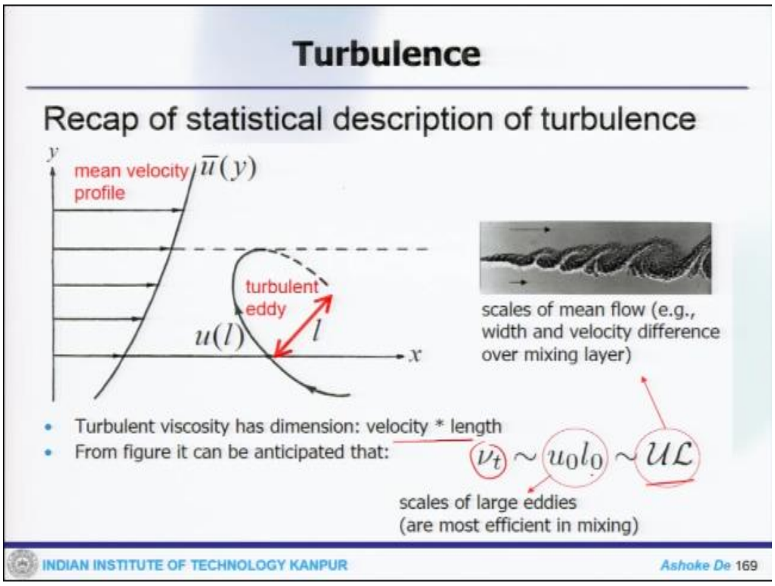

And this is if you scales of the mean flow is should be of U and L. So, it is actually the eddies viscosity scales of the order of this velocity and length scale. Now, we can come to the sum of the hypothesis. So, in analogy with molecules stress allowed a closer look at the molecular using kinetic gas theory. So, if you consider a 1 dimensional velocity profile so, this is at the macroscopic level and this is what happens in the microscopic level the molecules. So macroscopic momentum flux across this plane is σ_{xy}, which would be proportional to c n m.

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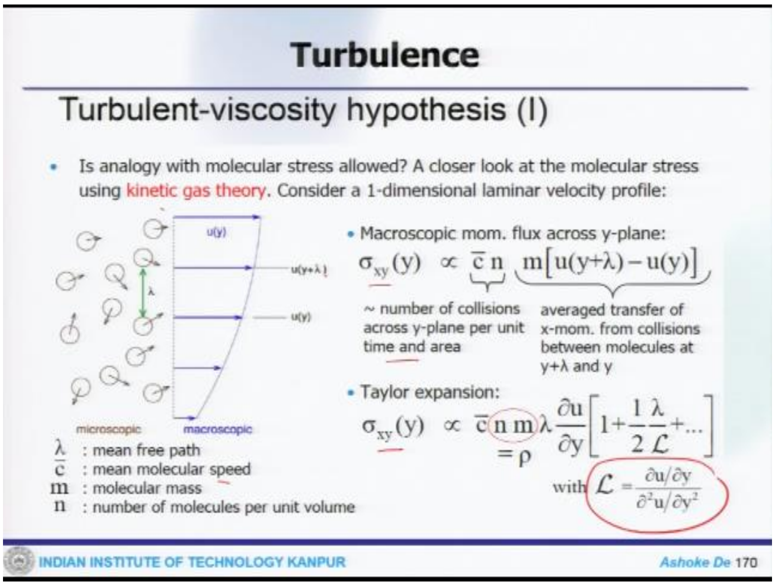

So, this is the number of collisions across y plan per unit time and area and these component contribute to averaged transfer of x momentum from collisions between molecule at y + λ and y between this. So, using the taylor expansion one can write this sigma x y in terms of this and then lambda del u by del y where L is represented like this, here lambda is the mean free path, c is the mean moleculer speed, m is the molecular mass and n represents the Number of molecules per unit volume.

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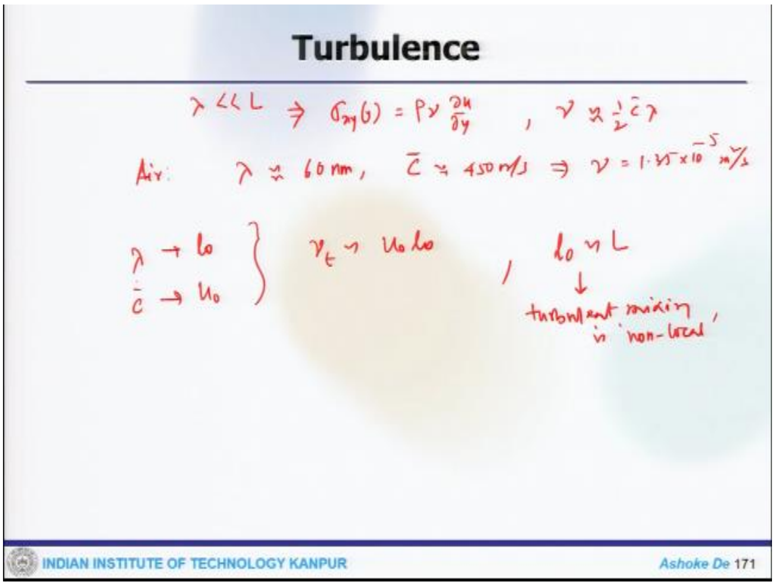

Now, in practical application typically lambda is quite small than L which signifies that σ_{xy} would be

$$\sigma_{xy} = \rho\nu\frac{\partial u}{\partial y}$$

where ν is

$$\nu \approx \frac{1}{2}\bar{c}\lambda$$

now, for here you can have for example, here this λ is approximate 60 nanometer, $\bar{c}$ is roughly 450 meter per second which gets you ν as $1.35\text{x}10^{-5}$ meter square per second. Now, then we get the model for turbulent viscosity where lambda is scales $\bar{c}$ goes to u_o where ν_t is scale like $u_o\, l_o$.

In general this l_o is of the order of l for both boundary free or wall bounded flows. So that means, the turbulent mixing is non local. So this turbulent viscosity a hypothesis still it is provides an reasonable result in particular when production and viscous dissipation rate optic in local balance.

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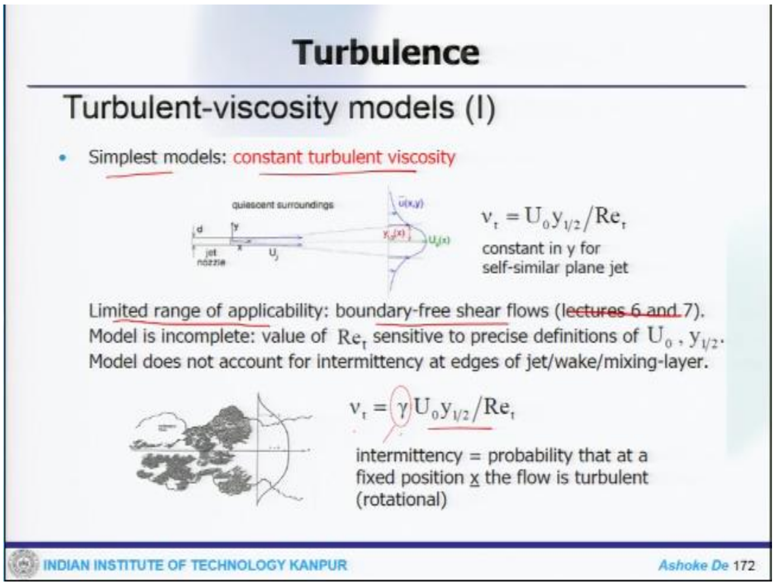

Now, that will actually paves the way for building up the turbulent eddy viscosity or turbulent viscosity models. So simplest one can think about it is a constant turbulent viscosity. So, here simple jet like this, where

$$\nu_t = U_o y_{1/2}/Re_t$$

So, this simplest one has a limited range of applicability, only boundary free shear flows. Now, model is incomplete value of Re_t sensitive to define U_o and Y_o and also it has does not account for the intermittency.

So, that is why the ν_t if you taken factor for intermittency ahead of this which will look like this. So, this intermittency will give you the probability that at a fixed position x the flow is turbulent.

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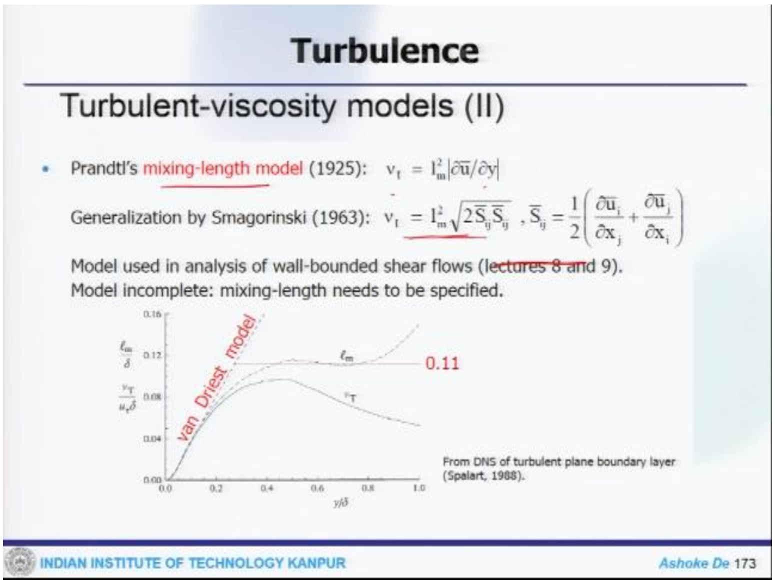

So, some sort of an now, in 1925 way back the Prandtl's proposed the mixing length model, which is like

$$\nu_t = l_m^2 \left|\frac{\partial \bar{u}}{\partial y}\right|$$

and then later on in 1963 Smagorinsky generalize that ν_t would be this

$$\nu_t = l_m^2 \sqrt{2\overline{S_{ij}}\,\overline{S_{ij}}}$$

is where S_{ij} is the standard tensor and, you can look at this where this model can be and later on also Smagorinsky, use these hypotheses to derive the some other turbulence model.

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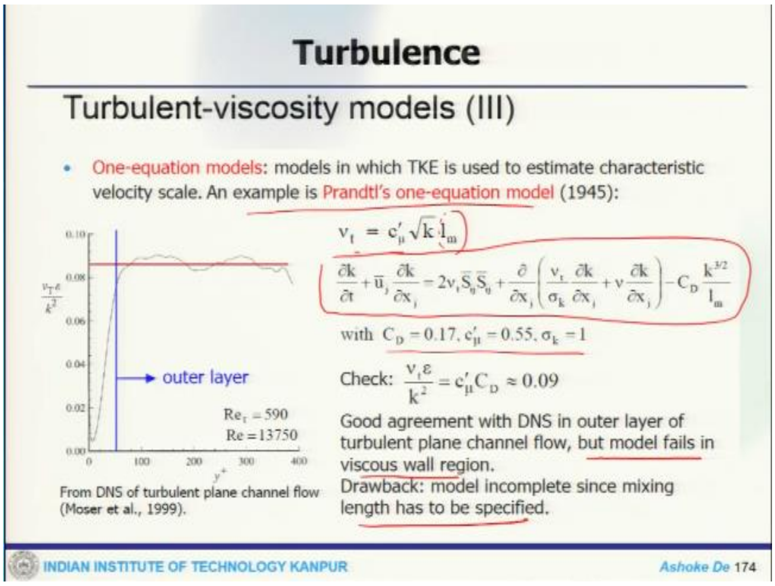

Now, then the development move to an one equation model, where the model in which turbulent kinetic energy use to estimate characteristics velocity scale. So, this is the example of prandtl one equation model 1945 where ν_t is approximated like this constant root of kinetic energy l m and this is the kinetic energy equation that used to be solved, where these are the model coefficients. Now, this model has not good agreement with the DNS in outer layer of the turbulent plane channel flow, but it fails in viscous all region.

So, it is incomplete since mixing length has to be specified that means this particular term l m which needs to be specified before and when you start your calculation, so, you need to specify that and which creates the problem here.

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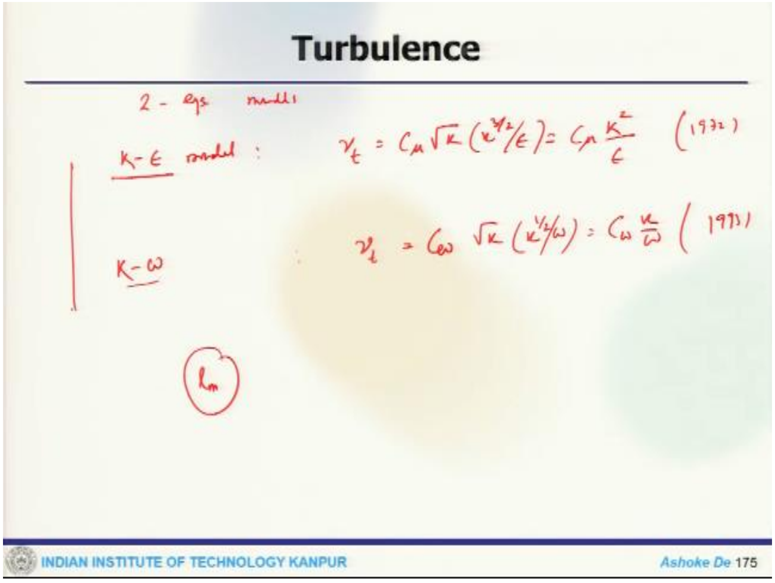

Now, later on, there is the development moved at and we have got two equations model where simple two equation model you solve k and epsilon model where kinetic energy and along with the dissipation where

$$\nu_t = C_\mu \sqrt{k}\left(\frac{k^{\frac{3}{2}}}{\varepsilon}\right) = C_\mu \frac{k^2}{\varepsilon}$$

So, this was proposed by Johnson launder in 1972 then different variant of like k Omega where do you solve for kinetic energy and the turbulent frequency.

In that case,

$$\nu_t = C_\omega \sqrt{k}\left(\frac{k^{\frac{3}{2}}}{\omega}\right) = C_\omega \frac{k}{\omega}$$

This is proposed by Wilcox in 1993. See, this two equation models are complete in the sense that no flow dependent specification like such as l_m is required. So, that way they look complete here do you solve for the kinetic energy or epsilon or the turbulent frequency. So, typically, these days event in the industrial application these models are quite heavily used.

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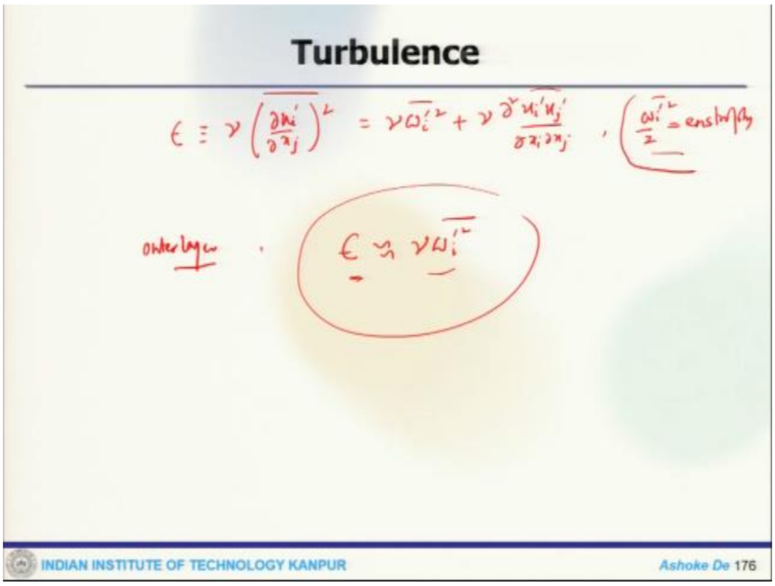

Now, how we actually look at the relationship between discuss dissipation rate with the end stopping. So, viscous dissipation is given as

$$\epsilon \equiv \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} = \nu\overline{\omega_i'^2} + \nu\frac{\partial^2 u_i'\overline{u_j'}}{\partial x_i \partial x_j}$$

where $\overline{\omega_i'^2}/2$ is the enstrophy So, the enstrphy is a measure of the. So, this actually the measure of the intensity of vorticity fluctuation. So,now one can see in the small scale you get to see the largest amount of this contribution of enstrophy.

And contribution of the turbulent kinetic energy at the large scale. Now, for wall bounded flows shear's flows wall bounded shear's flows outer layer you get this dissipation is some how connected with enstrophy like that. So, that means, that this equation can be approximated as equals for enstrophy there is a paradox from kolmogorov in 1941 that epsilon is actually determined by the energy transfer rate at large scale and in that it should be independent of viscosity , but when you look at these they are sort of correlated.

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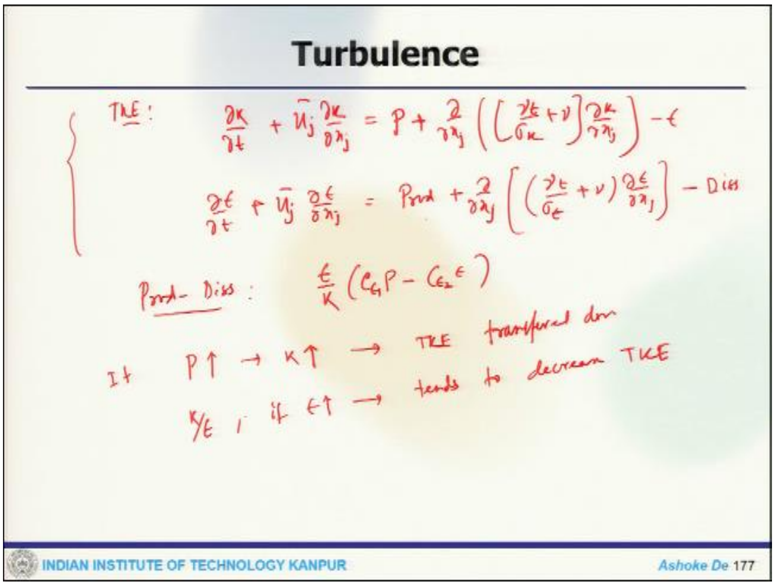

So, then we get the transport equation for epsilon where we got turbulent kinetic energy which is

$$\frac{\partial k}{\partial t}+\overline{u_J}\frac{\partial k}{\partial x_j}=P+\frac{\partial}{\partial x_j}\left(\left(\frac{\nu_t}{\sigma_x}+\nu\right)\frac{\partial k}{\partial x_j}\right)-\epsilon$$

similarly, we get the equation for epsilon which is

$$\frac{\partial \epsilon}{\partial t}+\overline{u_J}\frac{\partial \epsilon}{\partial x_j}=Prod+\frac{\partial}{\partial x_j}\left(\left(\frac{\nu_t}{\sigma_x}+\nu\right)\frac{\partial \epsilon}{\partial x_j}\right)-Diss$$

So, there is a production time there is a dissipation so, there are some empirical relation for that this defense of these 2 is approximated

$$\frac{\epsilon}{k}\left(C_{\epsilon_1}P-C_{\epsilon_2}\epsilon\right)$$

now, if p increases kinetic energy increases. So, turbulent kinetic energy transport down now energy cascade at timescale k by epsilon which will if your epsilon increases, so, which will tends to decrease turbulent kinetic energy. So, now, that a two equation model where you have k and epsilon.

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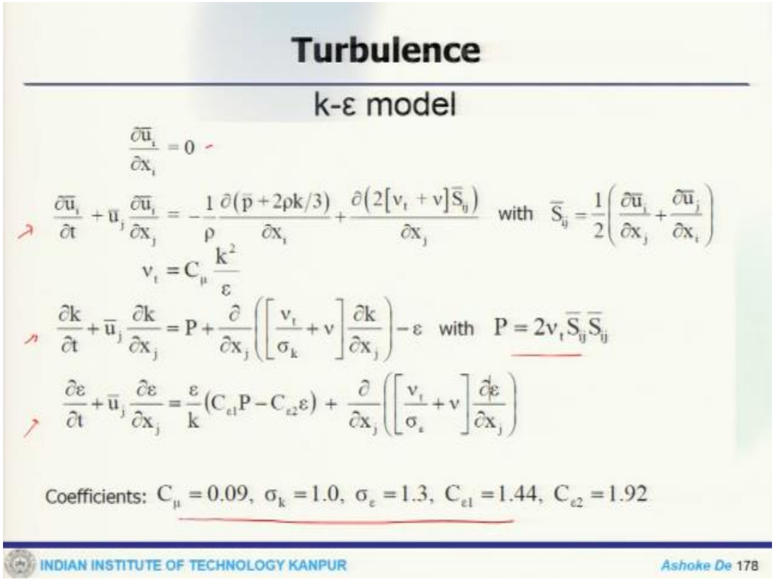

And if you put everything together this is what you need to solve in a RANS context where the first equation comes from the continuity second equation is your momentum this is for turbulent kinetic energy there P is the production of kinetic energy this is epsilon are these are some of the modern constants. So, that is a complete model. And now, this is quite I mean readily available in any commercial CFD package or code or also most of the code which are used by the academicians also simple to use, easy to implement and can be used for a wide range of flows with their limitations. So, one has to know the limitations.

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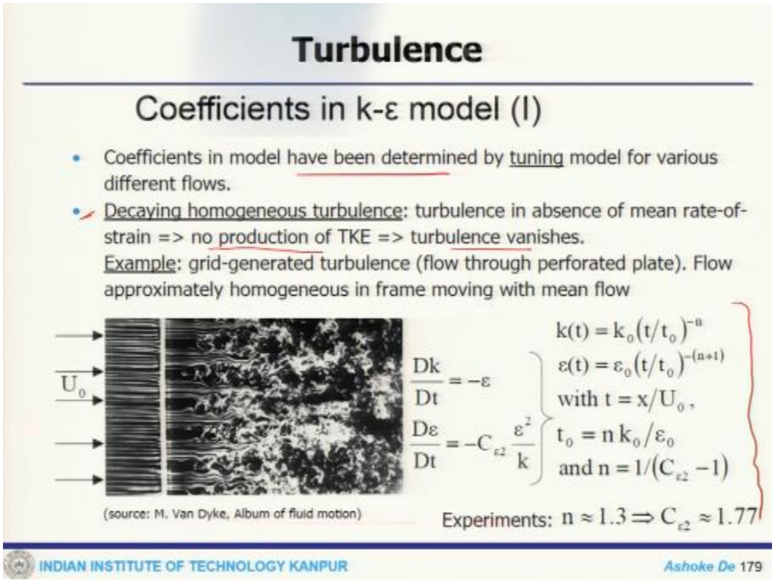

Now, if you look at those coefficients, which are present they are in the k-epsilon and model. So, they are been determined by tuning the model for different flows. Now, one important flow is the decaying homogeneous turbulence, so, turbulence is absence of mean rate of strain, so, there is no production turbulent kinetics energy turbulence vanishes. So, that is where these are the coefficients are which are kind of looking at a different flows, these coefficients are tuned over a period of time and they are.

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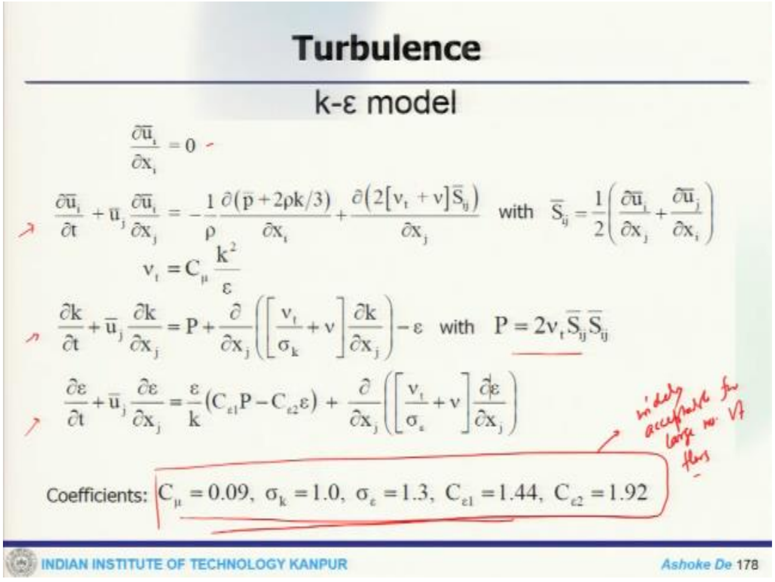

Now these coefficients are some how one can say widely acceptable for large number of flows. Having said that, one has to make a note here that any specific application this requ ires some tuning.

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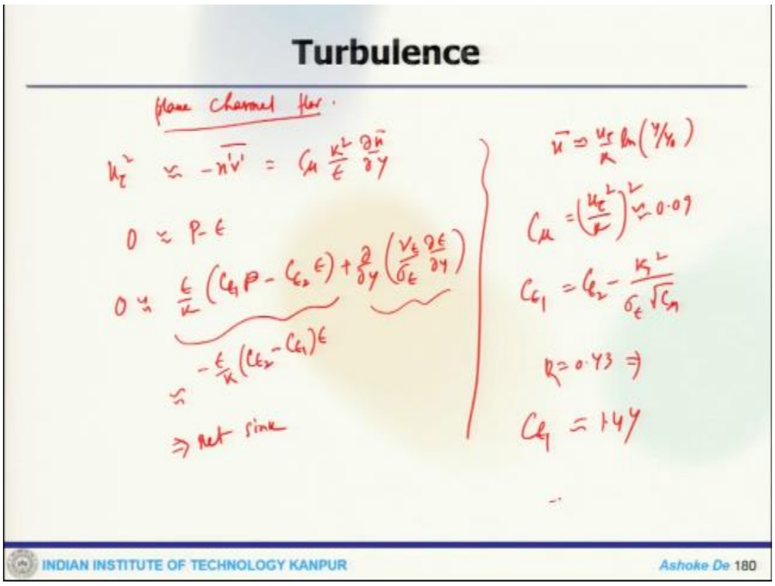

Now, behavior of this model in logarithmic layer for let us say plane channel flow. So, if you look at the behavior, so, u_τ^2 will be

$$u_\tau^2 = C_\mu \frac{k^2}{\epsilon} \frac{\partial \bar{u}}{\partial y}$$

this would be P – epsilon and

$$0 \approx \frac{\in}{k}\left(C_{\epsilon_1} P - C_{\epsilon_2}\epsilon\right) + \frac{\partial}{\partial y}\left(\frac{\nu_t}{\sigma_t}\frac{\partial \varepsilon}{\partial y}\right)$$

So, this term which will get back as

$$\approx \left(C_{\epsilon_1} - C_{\epsilon_2}\right)\epsilon$$

this is the turbulent diffusion term, diffusion transport Epsilon from one region towards a log region.

So, this would be Net sink term. So, using $\bar{u}$ equals to

$$\bar{u} = \frac{u_\tau}{k} lm\left(\frac{y}{y_o}\right)$$

$$C_\mu = \left(\frac{u_\tau^2}{k}\right)^2 \approx 0.09$$

$$C_{\epsilon_1} = C_{\epsilon_2} - \frac{k_1^2}{\sigma_t \sqrt{C_\mu}}$$

where for kappa equal to point 0.43 that gets you C_{ϵ_1} is 1.44 as I said these coefficients are widely acceptable.

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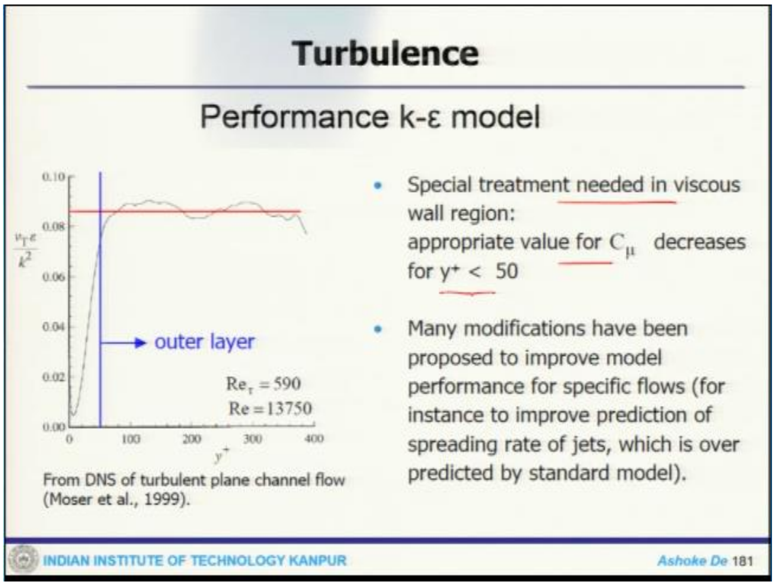

Now, if you look at this performance of this model, so, you requ ire special treatment in the viscous wall there are approximately value of similar decreases for y+ less than 50 that plane is go to outer layer it is fine within that. So, lot of modification have been proposed people have looked at the experiments and huge set of DNS database of channel flow to look at what kind of modification to be made for this model to be accurately used in the that near all region.

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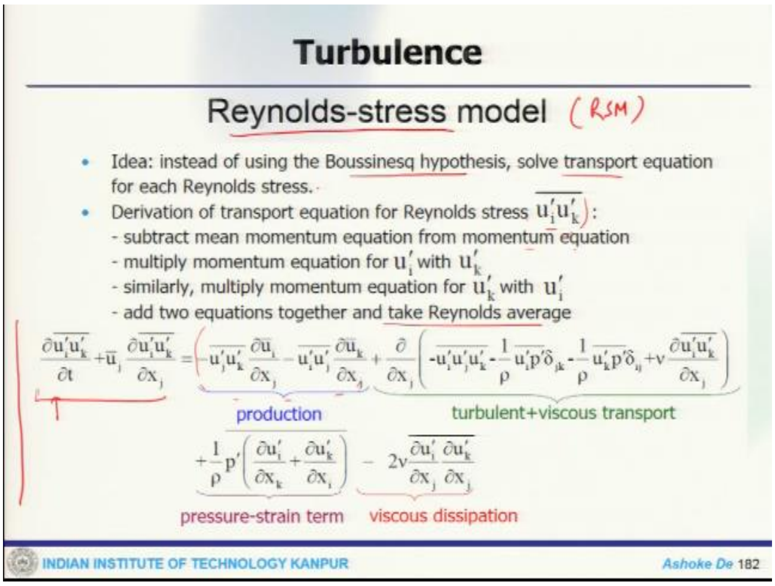

Now, then, after that, there is another set of model which is called Reynolds stress model, which is slightly more involved. Here the whole idea is that instead of using Boussinesq hypotheses, we solve for the transport equation for each Reynolds stress. And now, each Reynolds stress means we get so, many equations for the 6 equation for these renolyds stress and how do you opt in that equation, you subtract the mean momentum equation from the momentum or equation multiply momentum equation for u i prime with u k prime.

Similarly multiply moment equation for u k prime and two equation together and once you take the reynolds average this is what you get. So, this is your each Reynolds stress component and set it and so, this side you can think about the material derivative of that these 2 term all together contribute to the productionand these can contribute to the turbulent viscous transport see the pressure term this is a pressure strain term and this is viscous distribution.

So, essentially your left hand side this is the material derivative of the Reynolds stress term or the stress component where right hand side you have production turbulence + viscous transport. So, if you look at compared to two equation models, this equation set of equations are qu ite exhaustive. So, they are not that straight forward to be implemented, but, it has certain advantages over the two equation models.

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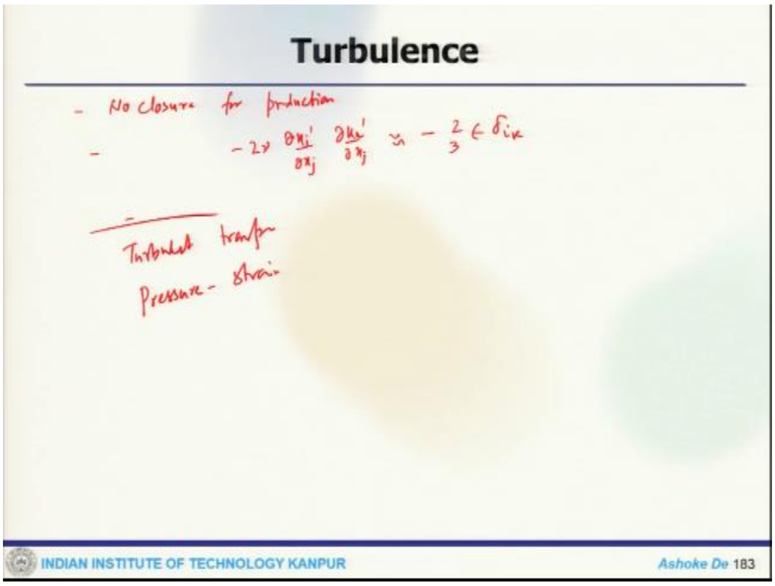

So, like one can think about there is no closure which is requ ired for production term, that is one of the important aspect of it, then, one can think about isotropic of the microstructure except close to the wall which can be approximated as

$$-2\nu\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}\approx-\frac{2}{3}\in\delta_{ix}$$

then there is a negligible contribution of microstructure to Reynolds shear stress. So, viscous dissipation rate of the Reynolds shear stress can be neglected.

Now, there are other cloure problem. So, like turbulence transport that is one issue pressure strain term that is another problem because it is responsible for return to isotropic and destruction Reynolds stress term and also it is computationally expensive. So, we will see how I mean other terms that effect this Reynolds stress equation, stop there today and continue in the next lecture.