Turbulent Combustion: Theory and Modelling
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Lecture-36
Turbulence – (contd..)

Welcome back. So we are in the middle of the discussion of this scaling of the turbulent flow and we are looking at wall bounded flows and shear flows. So we looked at the effect of the roughness also, now we will continue from there.

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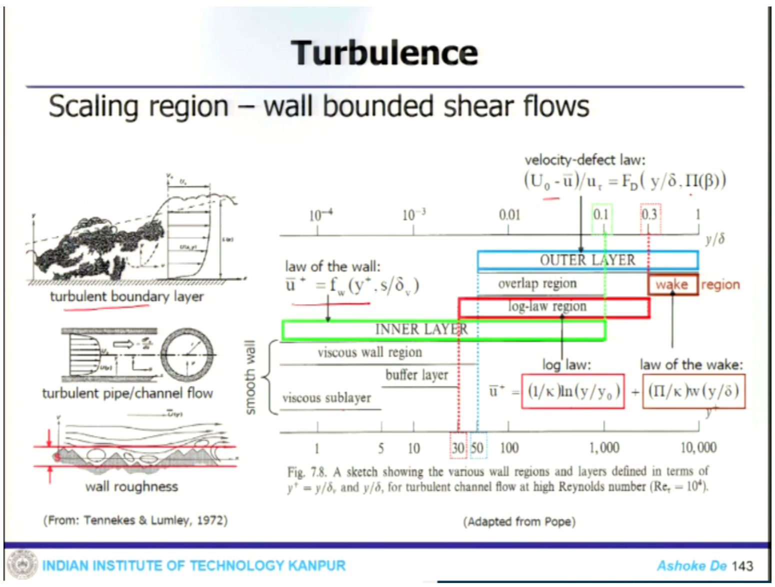

And what happens in the boundary layer and you have turbulent boundary layer over flat plate. You have turbulent pipe or channel flow. You have wall roughness. So if you sort of combine that together, they will get into 3 different layers one is the inner layer there would be log-law their overlap region and the outer region. So this is how the complete velocity profile in the boundary layer, they and obviously you have a in a smooth wall. You have some viscous sub layer buffer layer and viscous wall region. So in the inner layer, this is the law of the wall.

And when you come to the log region, there will be log law plus with law of the wake. And then you come to the outer layer where you have a velocity defect law and this is what it combines everything together.

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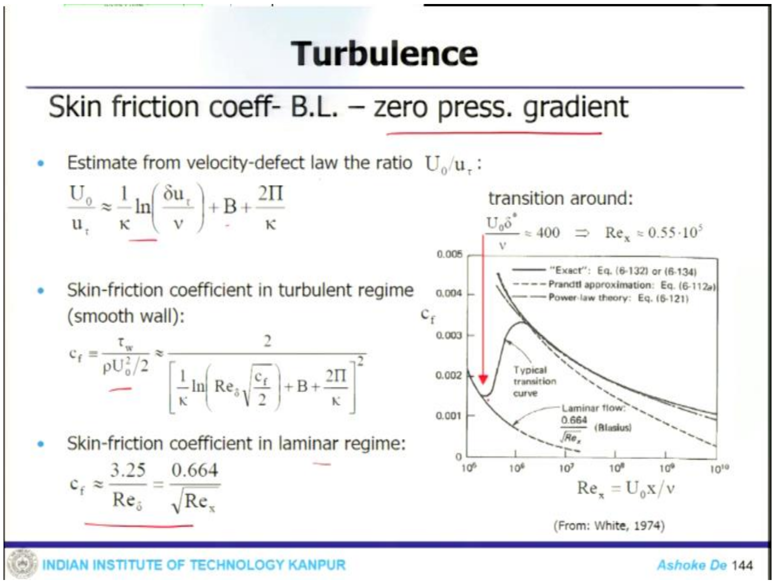

Now one important thing is that we can find out the skin friction coefficient of the boundary layer for zero pressure gradient case. Now, we have the velocity defect law, which is

$$\frac{U_o}{u_\tau} \approx \frac{1}{\kappa} ln\left(\frac{\delta u_\tau}{\upsilon}\right) + B + \frac{2\Pi}{\kappa}$$

and skin friction coefficient in the turbulent region for the smooth wall is defined as

$$c_f = \frac{\tau_w}{\rho U_o^2/2} \approx \frac{2}{\left[\frac{1}{\kappa}\ln\left(Re_\delta\sqrt{\frac{c_f}{2}}\right) + B + \frac{2\Pi}{\kappa}\right]}$$

If you plot c_f versus Re_x this is how it moves and when it becomes turbulent, this is how now in the laminar region.

It would be nicely one can find out it would be order of points Re_x that is the curve which shows that in the laminar but this is the zone where retrieves and then it becomes turbulent. Then it follows like that.

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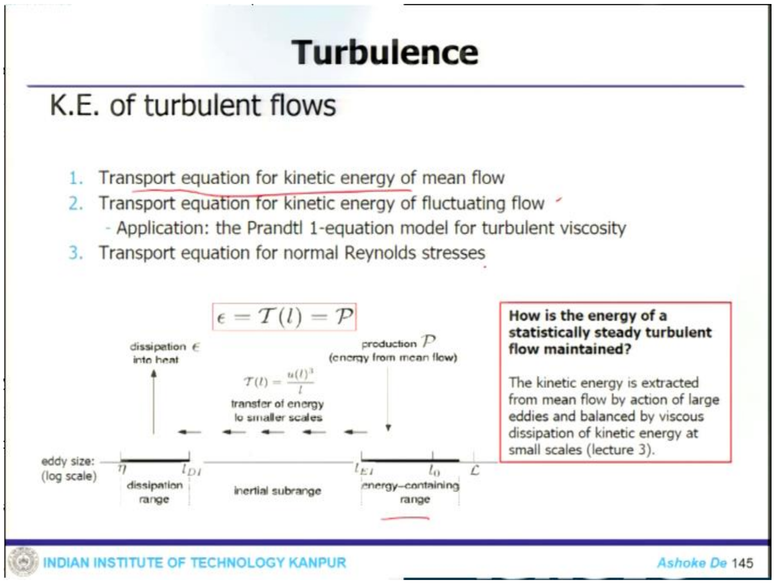

Now moving ahead we can look at the kinetic energy of turbulent flow what we can we have already seen that there are large eddies which are actually at the energy containing eddies and the energy transfer from the large eddy to the smaller eddy then from smaller eddy through dissipation it dissipates. So this is how the energy cascading takes place so we can find out or estimate the kinetic energy so transport equation for kinetic energy of the mean flow.

We can same transport equation for fluctuating flow and normal Reynolds stresses.

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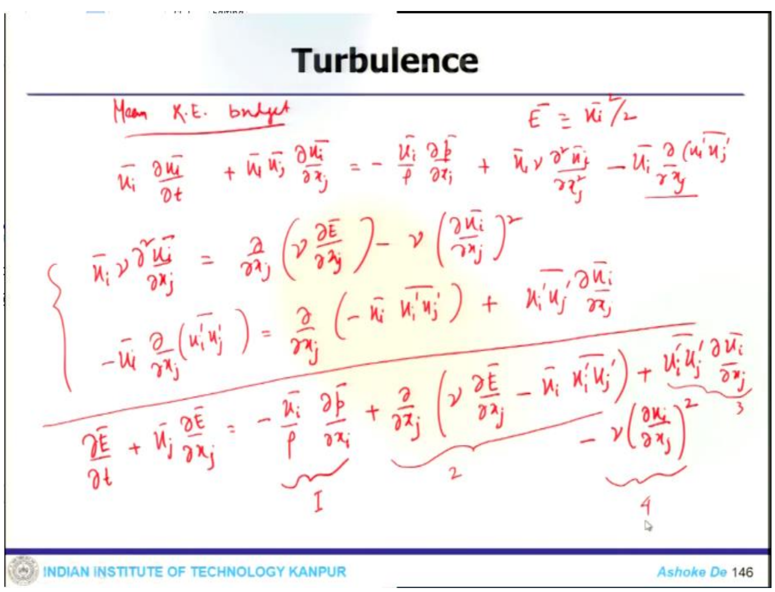

So these things we can actually now we first look at mean kinetic energy budget. So mean kinetic energy budget what we look at so this is we can derive the transport equation of mean kinetic energy. Let us say E which is

$$\bar{E} = \frac{\bar{u}_i^{\,2}}{2}$$

$$\bar{u}_i \frac{\partial \bar{u}_i}{\partial t} + \bar{u}_i \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{\bar{u}_i}{\rho} \frac{\partial \bar{p}}{\partial x_j} + \bar{u}_i \nu \frac{\partial^2 \bar{u}_i}{\partial x_j^2} - \bar{u}_i \frac{\partial (\overline{u_i' u_j'})}{\partial x_j}$$

Now the diffusion term we can rewrite like

$$\bar{u}_i \nu \frac{\partial^2 \bar{u}_i}{\partial x_j^2} = \frac{\partial}{\partial x_j}\left(\nu \frac{\partial \bar{E}}{\partial x_j}\right) - \nu \left(\frac{\partial \bar{u}_i}{\partial x_j}\right)^2$$

and the Reynolds term also we can rewrite which is

$$-\bar{u}_i \frac{\partial (\overline{u_i' u_j'})}{\partial x_j} = \frac{\partial (-\bar{u}_i \ \overline{u_i' u_j'})}{\partial x_j} + \overline{u_i' u_j'} \ \frac{\partial \bar{u}_i}{\partial x_j}$$

Now if I put these things into the previous equation that will get me

$$\frac{\partial \bar{E}}{\partial t} + \bar{u}_j \frac{\partial \bar{E}}{\partial x_j} = -\frac{\bar{u}_i}{\rho} \frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial \bar{E}}{\partial x_j} - \bar{u}_i \ \overline{u_i' u_j'}\right) + \overline{u_i' u_j'} \frac{\partial \bar{u}_i}{\partial x_j} - \nu \left(\frac{\partial \bar{u}_i}{\partial x_j}\right)^2$$

So that is what we get for mean kinetic energy. Now here we can say let us in the right hand side this would be the term one. This whole thing, we can say it is 2, this guy is 3 and this guy is 4.

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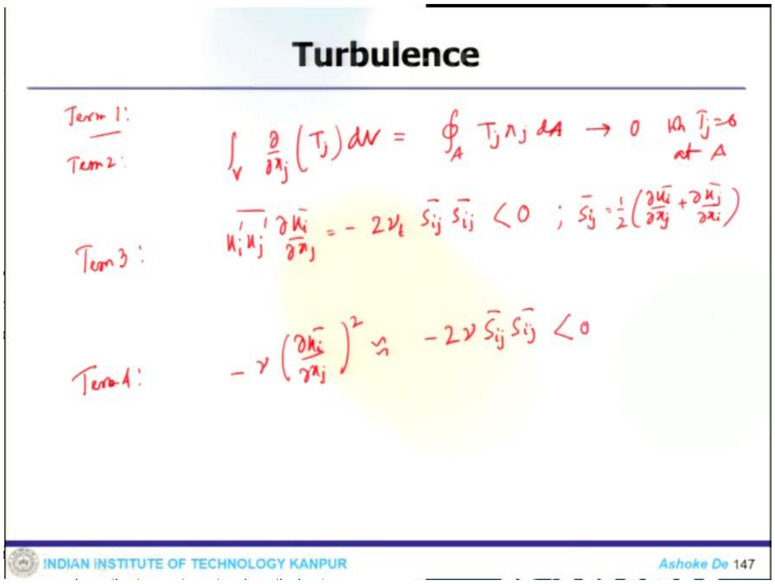

Now term 1 is energy production by pressure gradient then term 2 is the spatial distribution of the energy by viscous and turbulent diffusion, which means if I integrate over volume by T_j dv which will get me the surface integral of T_j n_j dA which is 0 when T_j equals to zero at A. Now term 3 which is this term which is a loss of energy by Reynolds stress. So using the Boussinesq hypothesis one can write that,

$$u_i'\overline{u_j'}\frac{\partial \overline{u_\iota}}{\partial x_j} = -2\nu_i \bar{S}_{ij}\bar{S}_{ij} < 0$$

where S_{ij} is

$$\bar{S}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u_\iota}}{\partial x_j} + \frac{\partial \overline{u_J}}{\partial x_i}\right)$$

which is the rate of strain tensor of the mean flow.

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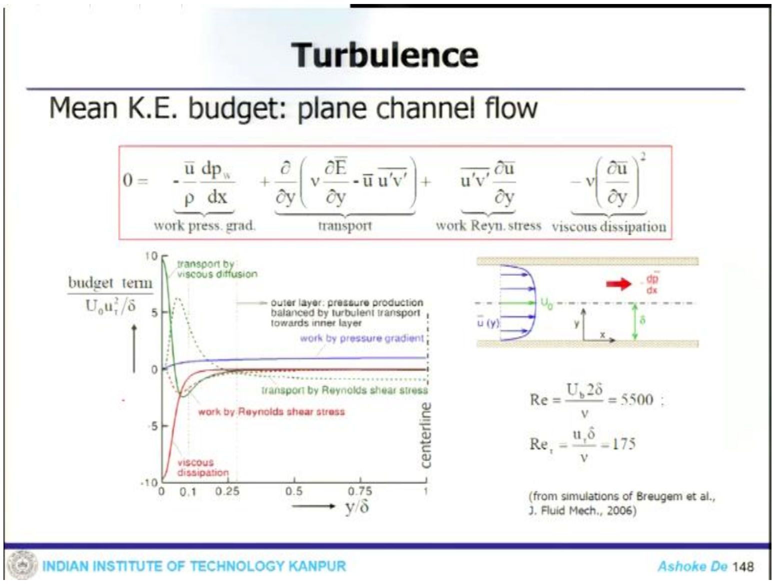

And term 4 which is again loss of energy by viscous dissipation. So these are the different terms in that kinetic energy equation. Now if we look at this mean kinetic energy equation for plane channel flow, so that gives me this is a pressure gradient term, transport term, Reynolds stress term, viscous dissipation term and if you plot this budget term that means this is the individual term, so you can see the contribution from each of these terms. I mean due to pressure gradient or the transport and all these things.

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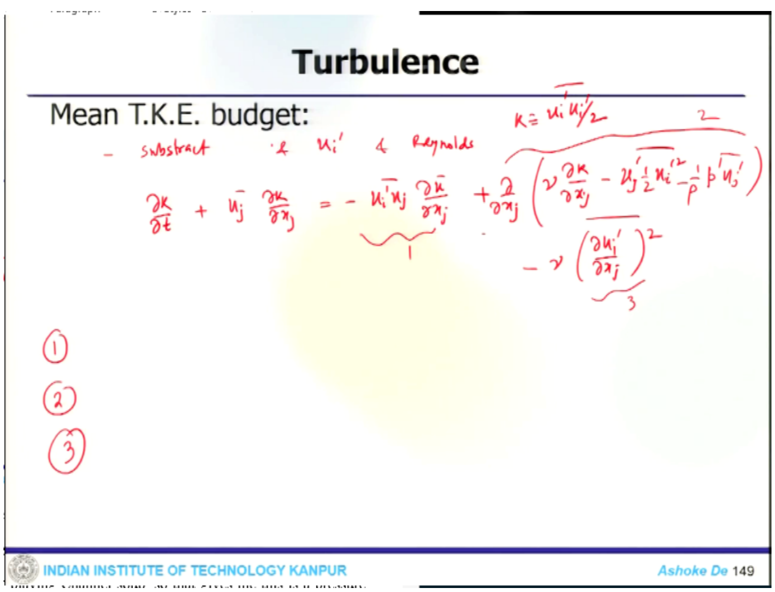

Now we can look at mean turbo kinetic energy budget so we can derive this transport equation from the kinetic energy which we have defined by 2. So this is mean turbulent kinetic energy budget. So turbulent kinetic energy would be this. Now we can subtract the mean momentum equation from the momentum equation and multiply with the u i prime and take Reynolds averaging. So we first subtract mean momentum equation from the momentum equation and then multiply by u_i prime and take Reynolds averaging.

$$\frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = -u_i' \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\frac{\nu \partial k}{\partial x_j} - u_j' \frac{1}{2} \overline{x_i'^2} - \frac{1}{\rho}\, p' \overline{u_j'} \right) - \nu \overline{\left(\frac{\partial u_i'}{\partial x_j} \right)^2}$$

now again we can have this is the term 1, this term is 2, and this is term 3. So term 1 is the production of time where kinetic energy by Reynolds stress on mean flow. So turbulent flow is essentially unstable. So in stability process transfer of energy from the mean flow to large eddies.

Now term 2, which is a spatial distribution of turbulent kinetic energy by viscous diffusion and fluctuating velocity plus pressure? And term 3 the loss of turbulent kinetic energy due to viscous dissipation. Now for a plane channel flow, if you look at the mean turbulent kinetic energy. So this is the Reynolds stress component, this is transport component, dissipation component and you can look at the effect of different budget terms for plane channel flow. So once we get this.

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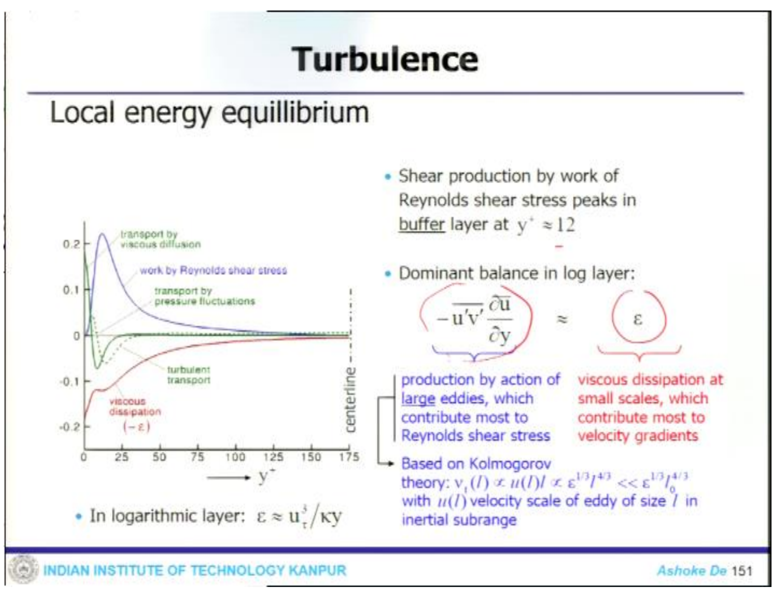

Now also you estimate the local energy equilibrium. So where is the shear production by the Reynolds stress at certain y^+ and you have dominant balance in the log layer where this is the production of by action of large eddies which contributes to maximum of shear stress and this is the dissipation at small scale. So this production term is based on Kolmogorov theory where

$$\nu(t) \propto \varepsilon^{1/3} l^{4/3}$$

where so that means in log layer there could be a balance. Whatever is the energy produced by the large eddies. This should be dissipated by the small eddies.

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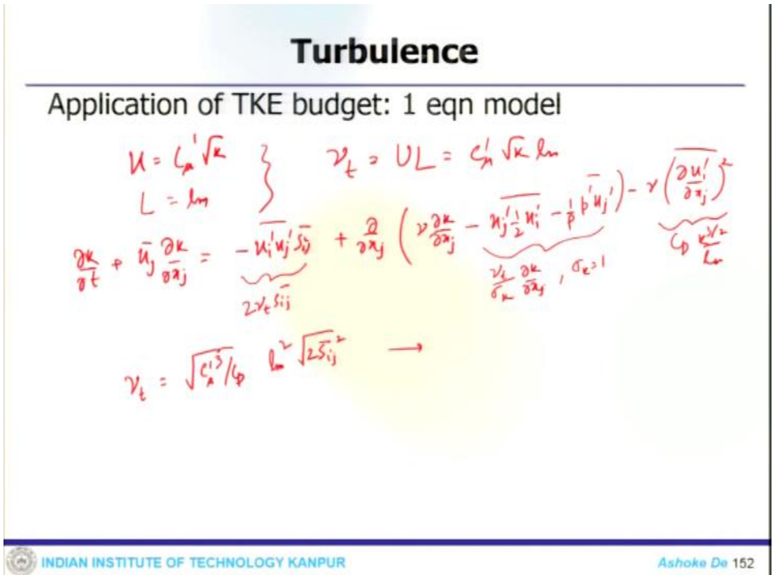

So we can apply this turbulent kinetic energy budget for 1 equation model. So the macroscopic velocity scale is

$$U = C_\mu' \sqrt{k}$$

and macroscopic length scale is

$$L = l_m$$

which will get us

$$\nu_t = UL = C_\mu' \sqrt{k}\, l_m$$

Now we can have a turbulent kinetic energy transport equation, which is in terms of

$$\frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = -u_i' \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\frac{\nu \partial k}{\partial x_j} - u_j' \frac{1}{2} \overline{x_i'^2} - \frac{1}{\rho}\, p' \overline{u_j'} \right) - \nu \overline{\left(\frac{\partial u_i'}{\partial x_j} \right)}^2$$

So this is estimated as

Now local energy equilibrium between production and dissipation in that concept we can write

$$\nu_t = \sqrt{\frac{c_\mu'^3}{C_D}}\, l_m^2 \sqrt{2S_{ij}^2}$$

which is Prandtl mixing length hypothesis. So advantage of this one equation model is that it accounts for non-equilibrium effect. So that is in one of the advantage but the biggest disadvantage is that mixing length must still be specified.

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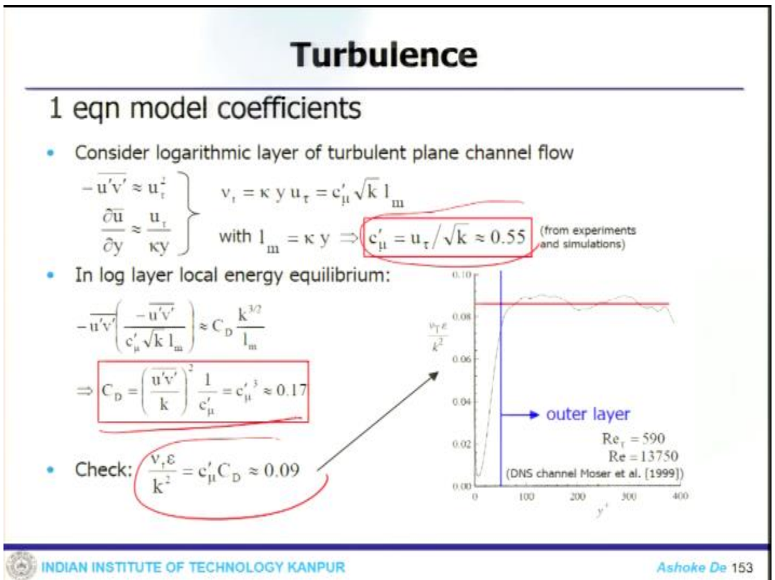

Now we can have these one equation coefficients that we can consider this logarithmic layer of turbulent plane channel flow where

$$-\overline{u'v'} \approx u_\tau^2$$

$$\frac{\partial \bar{u}}{\partial x} \approx \frac{u_\tau}{\kappa y}$$

So this is the coefficient for the c_μ'

$$c_\mu' = \frac{u_\tau}{\sqrt{k}} \approx 0.55$$

Now in log layer due to local energy equilibrium you get

$$C_D = \left(\frac{\overline{u'v'}}{k}\right)^2 \frac{1}{c_\mu'} = c_\mu'^3 \approx 0.17$$

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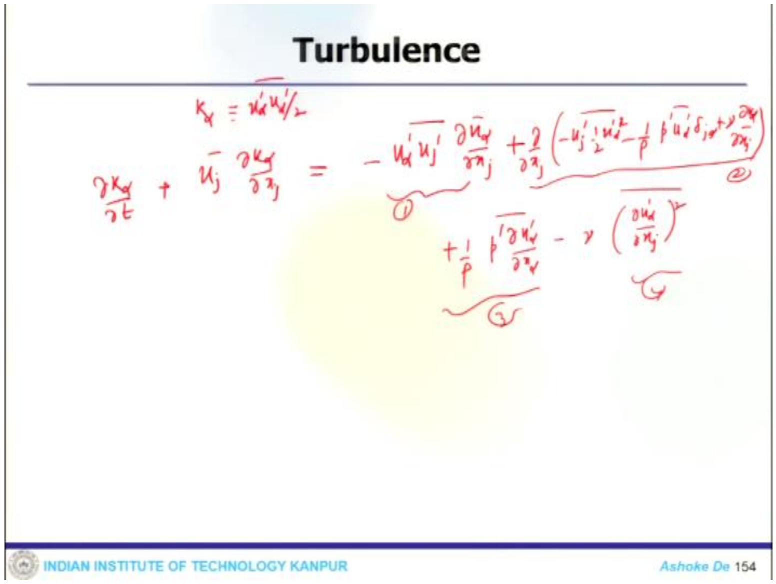

Now we can find out the eddy viscosity for velocity component. Now for individual velocity component kinetic energy would be

$$k_\alpha = \frac{u'_\alpha \overline{u'_\alpha}}{2}$$

Now you can subtract the mean momentum equation from the momentum equation and multiply by u_α, so we get this equation. So which is in terms of which we have got

$$\frac{\partial k_\alpha}{\partial t} + \bar{u}_j \frac{\partial k_\alpha}{\partial x_j} = -u'_\alpha \bar{u}_j \frac{\partial \bar{u}_\alpha}{\partial x_j} + \frac{\partial}{\partial x_j}\left(-u'_j \frac{1}{2}\overline{u'^2_\alpha} - \frac{1}{\rho}\, p'\overline{u'_\alpha}\delta_{j\alpha} + \frac{\nu \partial k_\alpha}{\partial x_j}\right) + \frac{1}{\rho}\, p' \overline{\frac{\partial u'_\alpha}{\partial x_\alpha}} - \nu \overline{\left(\frac{\partial u'_\alpha}{\partial x_j}\right)^2}$$

So right hand side you got term 1 term 2, 3 and 4. So you can find out the energy power fluctuating component like this.

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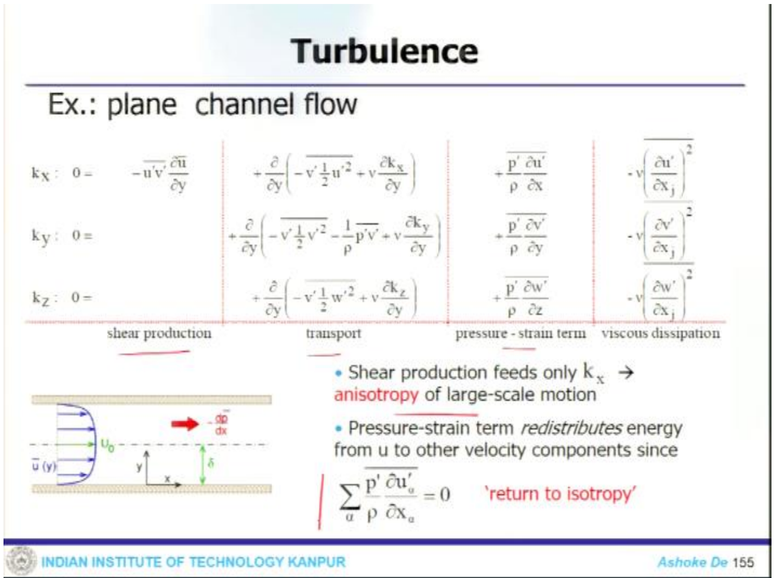

For a plane channel flow if k_x is 0 then this would be 0, so you can put it this is the shear production component, these are the transport component, pressure strain term and viscous dissipation. So the shear production feeds to k_x which is anisotropy of large-scale motion, so that essentially tells you how much anisotropy is there and pressure stain redistributes the energy of u to other velocity components through this.

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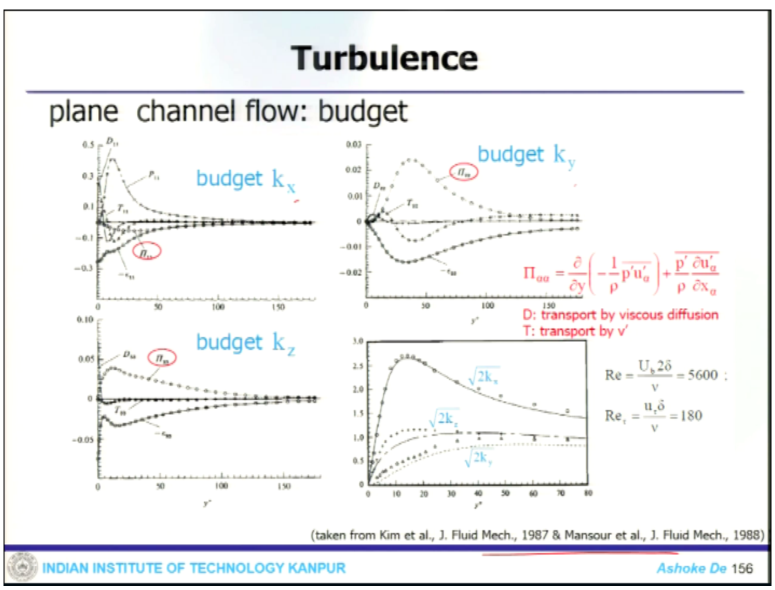

Now we can look at the different budget like budget of k_x k_y k_z and all these details. So this is taken from this multiple literature where people have looked at for channel flow in details and they computed individual budget terms and all these things.

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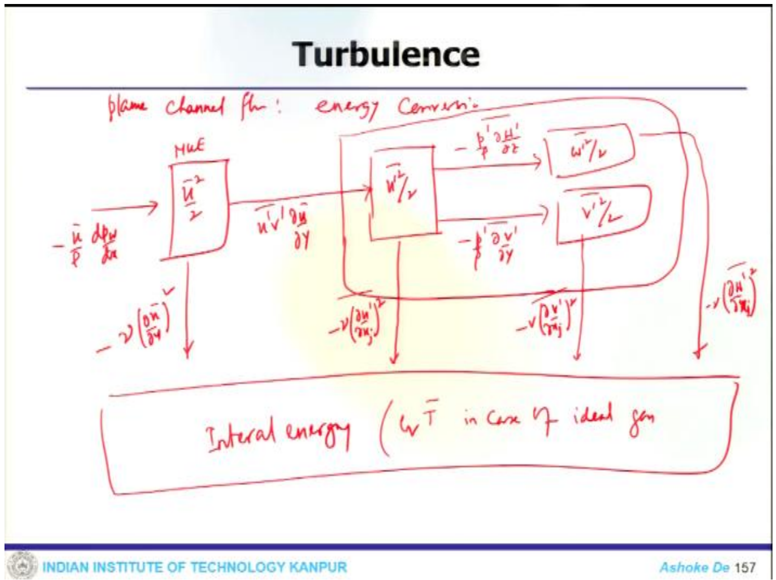

Now if we look at for plane channel flow and energy conversion so we can say that say this is my $\frac{\bar{u}^2}{2}$. This is mean kinetic energy which is -$\frac{\bar{u}}{\rho}\frac{dp_w}{dx}$, then it goes to a term which is essentially $\frac{\bar{u}'^2}{2}$ and that -$\frac{p\prime}{\rho}\frac{dw\prime}{dz}$, that get you by -$\frac{p\prime}{\rho}\frac{dv\prime}{dy}$, you the $\frac{\bar{v}'^2}{2}$. So that is another block where u'v'$\frac{du}{dy}$.

Now here this is nu del u prime by del y square then this - nu del u prime by del x j square, then this del v prime by del x j square and this one also - nu del w prime by del x j. So all this will contribute to the one single component is the internal energy which is C v T bar in case of ideal gas. So this is how the complete picture of the energy conversion look like for a plane channel flow.

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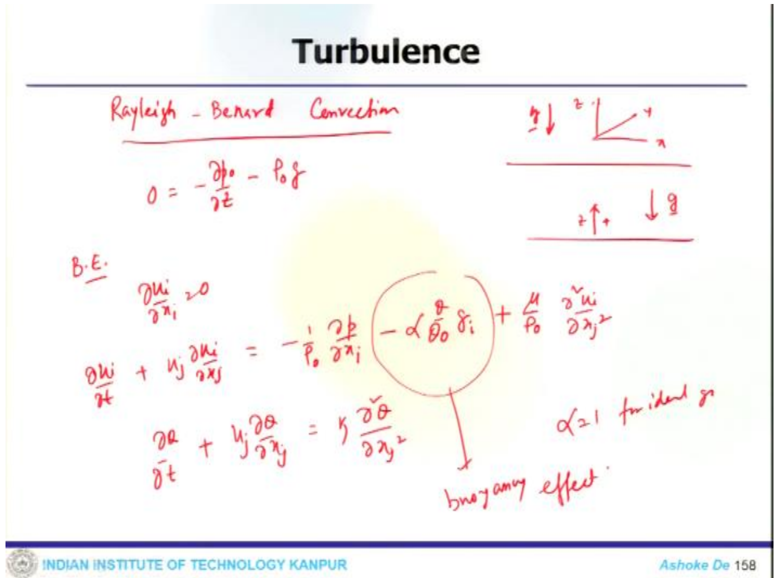

Now, we move to the another topic where you have the density which varies due to temperature. That means the between 2 parallel plates will look at Rayleigh Benard Convection. So we have 2 parallel plates plus g this is g direction. Axis x y z and this is how my g is going to act. So these kinds of things one can observe in atmospheric boundary layer where heating up our surface by shortwave radiation from sun during the day cooling by long-wave radiation during the night.

Now here Boussinesq approximation would be required. So the temperature variation is connected with the pressure and hydrostatic variation, which is like this and the Boussinesq equations we write

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_o}\frac{\partial p}{\partial x_i} - \frac{\theta}{\theta_o} g_i + \frac{\mu}{\rho_o}\frac{\partial^2 u_i}{\partial {x_j}^2}$$

So alpha equals to 1 for ideal gas and this is the term which is known as buoyancy effect, so we have already looked at this equation.

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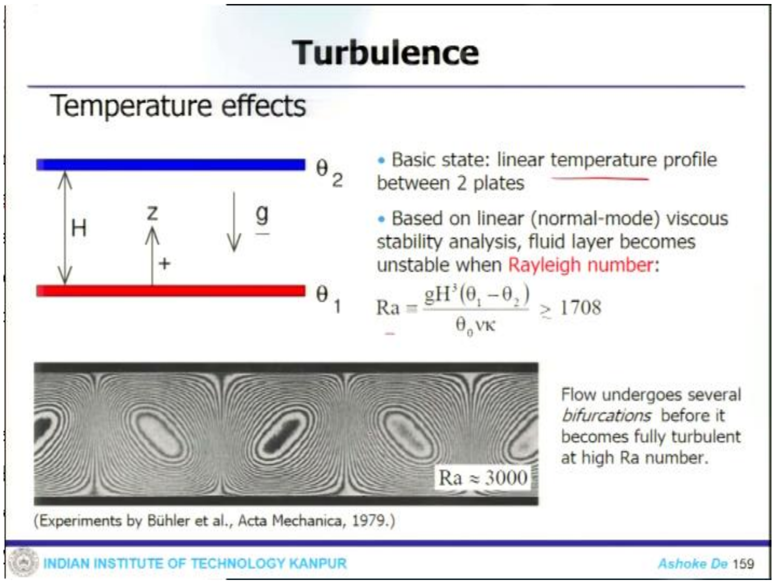

Now how temperature affects this is a parallel plate and these are the 2 different temperature it has and you can see how basic state one can assume the linear temperature profile and based on linear normal mode stability analysis one can do and find out the rally number, and this is how the experimentally observed flow pattern in these things. So it goes through multiple bifurcation before it becomes fully turbulent.

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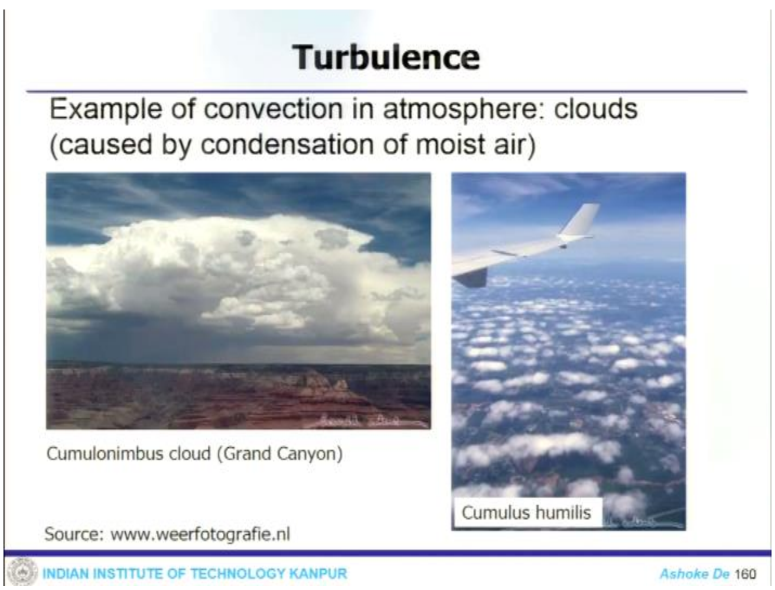

So this is another area where and you can see these are the some of the example of this relevant at conduction in atmospheric cloud where you can see these things.

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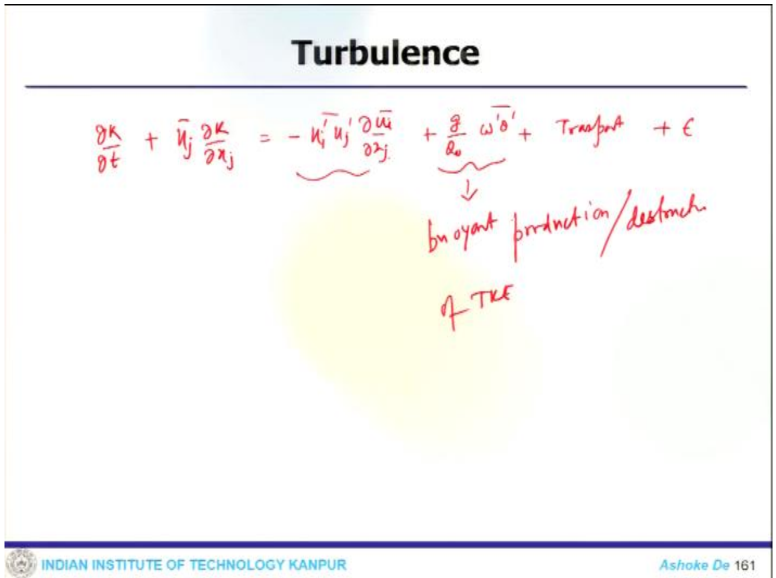

Now we can estimate the turbulent kinetic energy budget including buoyancy effect. So the derivation would be similar and we have already got the equation, so but will rewrite it, but there will be effect from the temperature gradient.

$$\frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = -u_i' \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} + \frac{g}{\theta_o} \omega' \overline{\theta}' \ + transport + \epsilon$$

So this again the term one which is a production by the general shear stress. This is the buoyant production or destruction of turbulent kinetic energy by gravitational forces.

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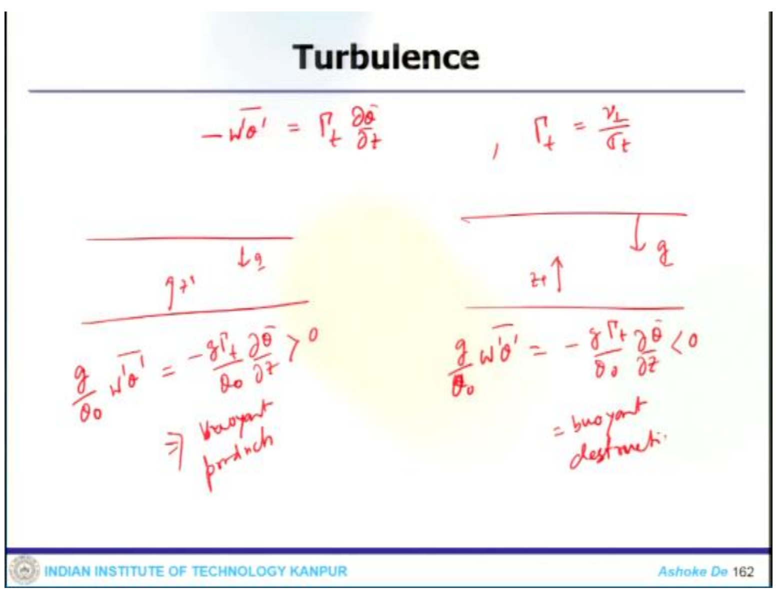

Now what is buoyant production or destruction, so the Boussinesq hypothesis says that

$$-w'\overline{\theta'} = \Pi_t \frac{\partial\bar{\theta}}{\partial t}$$

So this is a typical gradient diffusion hypothesis in turbulent heat flux were Π_t is approximated as

$$\Pi_t = \frac{\nu_t}{\sigma_t}$$

where σ_t is the turbulent Prandtl number ν_t is the turbulent diffusivity, so there could be 2 different cases, where one case this is an convective turbulence.

$$\frac{g}{\theta_o}\omega'\bar{\theta}' = \frac{g\Pi_t}{\theta_o}\frac{\partial\bar{\theta}}{\partial t} > 0$$

So this will be buoyant production situation or you can have a situation where

$$\frac{g}{\theta_o}\omega'\bar{\theta}' = \frac{g\Pi_t}{\theta_o}\frac{\partial\bar{\theta}}{\partial t} < 0$$

which is buoyant destruction. So we can look at this production and destruction term the buoyancy effect. Now the other like buoyancy versus shear and all these things that we will discuss in the next lecture will stop here today.