Turbulent Combustion: Theory and Modelling
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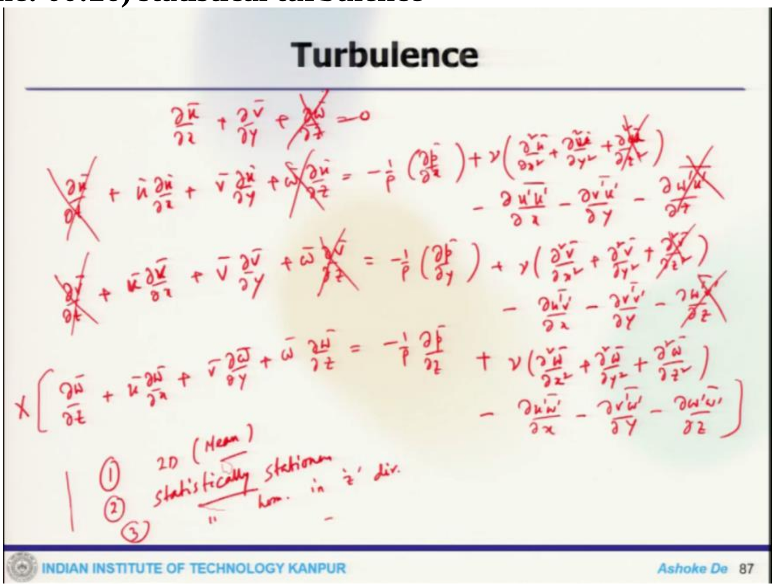
Lecture-34
Combustion in 2 Phase Flows (Contd...)

Statistical turbulence

Welcome back. So, let us continue the discussion of the statistical turbulence and we are looking at this simple jet flow and looking at the solution.

(Refer Slide Time: 00:26) statistical turbulence



We derive the governing equations for the with the assumption like mean 2-dimensional flow and statistically stationary system.

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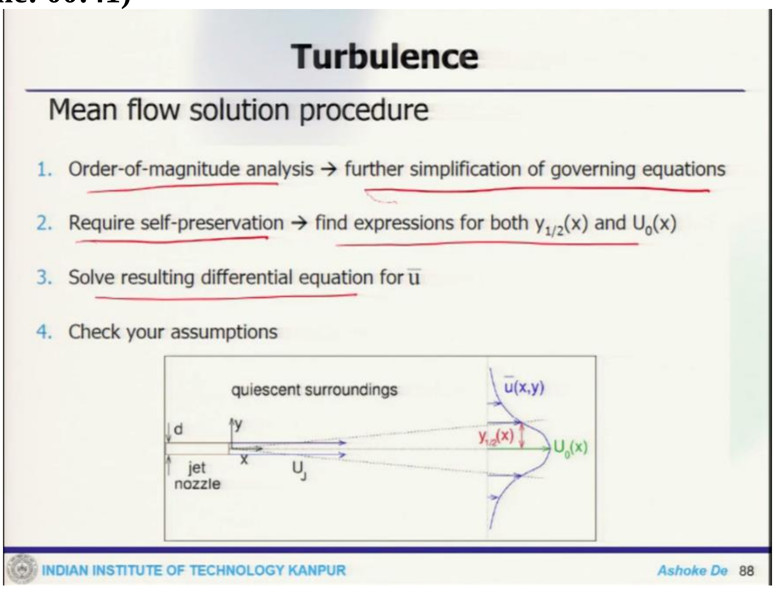

Now, we can find out the mean flow solution procedure. We will do by order-of-magnitude analysis. We can further simplify the governing equation. Then requires self-preservation to find the expression for both $y_{\frac{1}{2}}(x)$ and $U_0(x)$, then we can solve the resulting differential equations and we can see the assumptions that we had.

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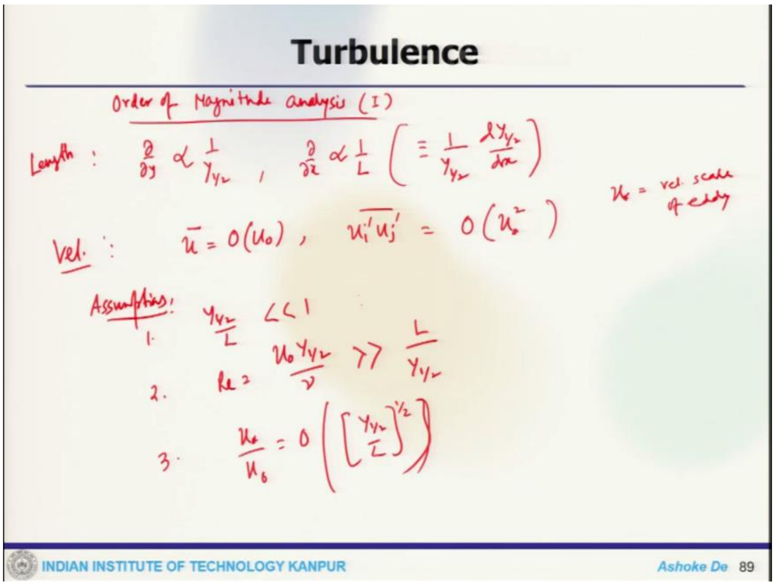
Magnitude analysis



Now, first thing is that order of magnitude analysis 1.

We can see the different length scales or variations in the flow.

$$\frac{\partial}{\partial y} = \frac{1}{y_{1/2}} \;, \qquad \frac{\partial}{\partial x} = \frac{1}{L}\left[\equiv \frac{1}{y_{1/2}}\frac{dy_{1/2}}{dx}\right]$$

Now we can look at the velocity scale

$$\overline{u} = O(u_o) \;, \qquad \overline{u'_i u'_j} = O(u_*{}^2)$$

u * can thought about as a typical velocity scale of eddy.

Now, we can have some assumptions

1. $\frac{y_{1/2}}{L} \ll 1$
2. $Re = \frac{u_o y_{1/2}}{\nu} \gg \frac{L}{y_{1/2}}$
3. $\frac{u_*}{u_o} = O\left(\left[\frac{y_{1/2}}{L}\right]^{1/2}\right)$

Point 1. means the streamwise evolution of the z is quite slow. Point 2. tells that at the large Re. So, the viscous effect is quite negligible on the flow. Point 3. means the turbulence adapts itself to changes in mean flow.

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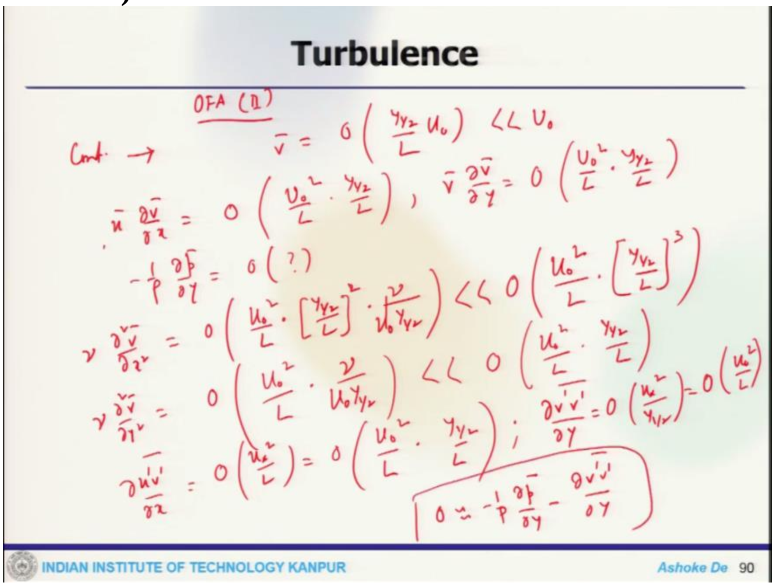

Now, we look at the order of analysis 2.

In that case from continuity we can write

$$\overline{v} = O\left(\frac{y_{1/2}}{L} u_0\right) \ll \; u_0$$

Now, we can estimate the terms in the y momentum equation.

$$\overline{u}\frac{\partial \overline{v}}{\partial x} = O\left(\frac{{u_0}^2}{L}.\frac{y_{1/2}}{L}\right), \quad \overline{v}\frac{\partial \overline{v}}{\partial y} = O\left(\frac{{u_0}^2}{L}.\frac{y_{1/2}}{L}\right)$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = O(?)\ \text{Not known}$$

Then the diffusion component

$$\nu\frac{\partial^2 \overline{v}}{\partial x^2} = O\left(\frac{{u_0}^2}{L}.\left[\frac{y_{1/2}}{L}\right]^2.\frac{\nu}{u_0 y_{1/2}}\right) \ll O\left(\frac{{u_0}^2}{L}.\left[\frac{y_{1/2}}{L}\right]^3\right)$$

$$\nu\frac{\partial^2 \overline{v}}{\partial y^2} = O\left(\frac{{u_0}^2}{L}.\frac{\nu}{u_0 y_{1/2}}\right) \ll \; O\left(\frac{{u_0}^2}{L}.\frac{y_{1/2}}{L}\right)$$

$$\frac{\partial \overline{u'v'}}{\partial x} = O\left(\frac{{u_*}^2}{L}\right) = \; O\left(\frac{{u_0}^2}{L}.\frac{y_{1/2}}{L}\right)$$

$$\frac{\partial \overline{v'v'}}{\partial y} = O\left(\frac{{u_*}^2}{y_{1/2}}\right) = \; O\left(\frac{{u_0}^2}{L}\right)$$

Now the required balance of leading term using assumption one would get us

$$O \approx \frac{-1}{\rho}\frac{\partial \overline{p}}{\partial y} - \frac{\partial \overline{v'v'}}{\partial y}$$

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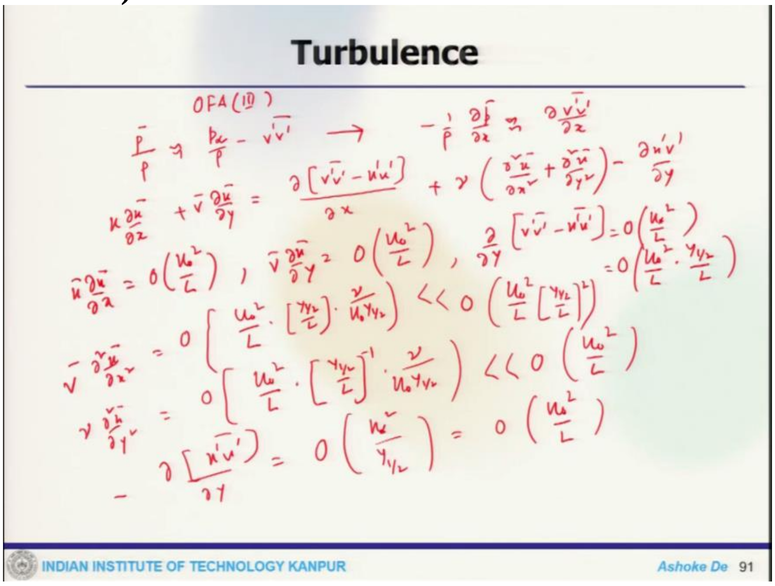

So, we can now move to at the third level. Now, we have a simplified y momentum equation, which is now

$$\frac{\overline{p}}{\rho} \approx \frac{\overline{p_\infty}}{\rho} - \overline{v'v'} \;\; \rightarrow \;\; \frac{-1}{\rho}\frac{\partial \overline{p}}{\partial x} \approx \frac{\partial \overline{v'v'}}{\partial x}$$

now if we substitute this in x momentum equation.

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = \frac{\partial[[\overline{v'v'} - \overline{u'u'}]}{\partial x} + \nu\left(\frac{\partial^2 \overline{u}}{\partial x^2} + \frac{\partial^2 \overline{u}}{\partial y^2}\right) - \frac{\partial \overline{v'u'}}{\partial y}$$

Now, if you do this estimate that terms in x momentum equation using assumption 2 and 3. We can get rest of the orders

$$\overline{u}\frac{\partial \overline{u}}{\partial x} = O\left(\frac{{U_0}^2}{L}\right)$$

$$\overline{v}\frac{\partial \overline{u}}{\partial y} = O\left(\frac{{U_0}^2}{L}\right)$$

$$\frac{\partial[[\overline{v'v'} - \overline{u'u'}]}{\partial x} = O\left(\frac{{U_*}^2}{L}\right) = O\left(\frac{{U_0}^2}{L}, \frac{y_{1/2}}{L}\right)$$

$$\nu\frac{\partial^2 \overline{u}}{\partial x^2} = O\left(\frac{{u_0}^2}{L}.\frac{y_{1/2}}{L}.\frac{\nu}{u_0 y_{1/2}}\right) \ll O\left(\frac{{u_0}^2}{L}.\left[\frac{y_{1/2}}{L}\right]^2\right)$$

$$\nu\frac{\partial^2 \overline{v}}{\partial x^2} = O\left(\frac{{u_0}^2}{L}.\left[\frac{y_{1/2}}{L}\right]^{-1}.\frac{\nu}{u_0 y_{1/2}}\right) \ll O\left(\frac{{u_0}^2}{L}\right)$$

$$\frac{\partial \overline{v'u'}}{\partial y} = O\left(\frac{{u_*}^2}{y_{1/2}}\right) = O\left(\frac{{u_0}^2}{L}\right)$$

Now the last part is that so now we required the balance of the leading term. So, the streamwise advection is

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{\partial \overline{v'u'}}{\partial y}$$

So, first term is streamwise advection, second term is lateral advection. RHS term is lateral turbulent diffusion. Now this can be solved along with the continuity equation which is

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$$

So these two equations should be solved then we get the self preservation.

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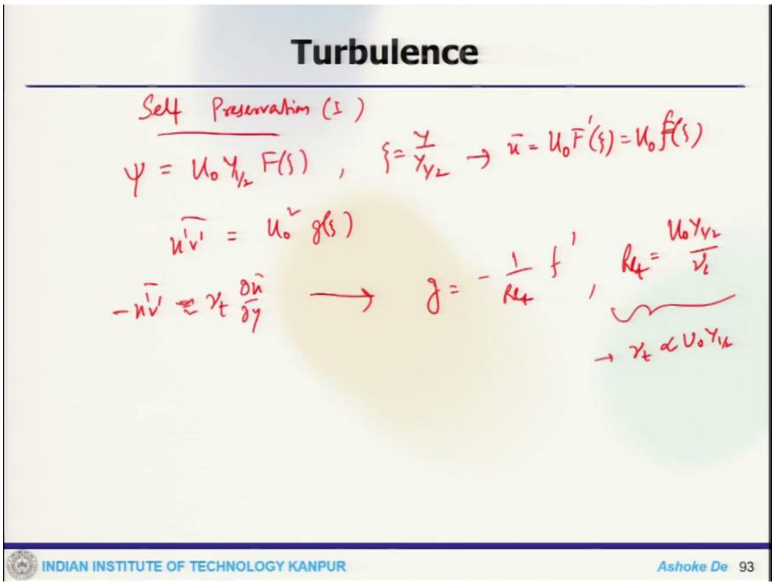
self preservation



We can introduce some stream function that automatically satisfy mass conservation where

$$\psi = u_0 y_{1/2} F(\xi) \quad \xi = \frac{y}{y_{1/2}}$$

That means I can have mean velocity profile as $u = u_0 F'^{(\xi)} = u_0 f(\xi)$.

Now similarity solution for turbulent stress which is from the again experimental observation which will give $\overline{u'v'} = {u_0}^2 g(\xi)$.

And if we establish the relation between f and g using the boussinesq hypothesis so, that is give us that

$$\overline{u'v'} = \nu_t \frac{\partial \bar{u}}{\partial y} \to g = \frac{1}{Re_t} f', \quad Re_t = \frac{u_0 y_{1/2}}{\nu_t}$$

Now, Re_t must be constant in x if flow is self preserving, which means ν_t should be equal to $u_0 y_{1/2}$ and also turbulent viscosity is assumed to be constant in y.

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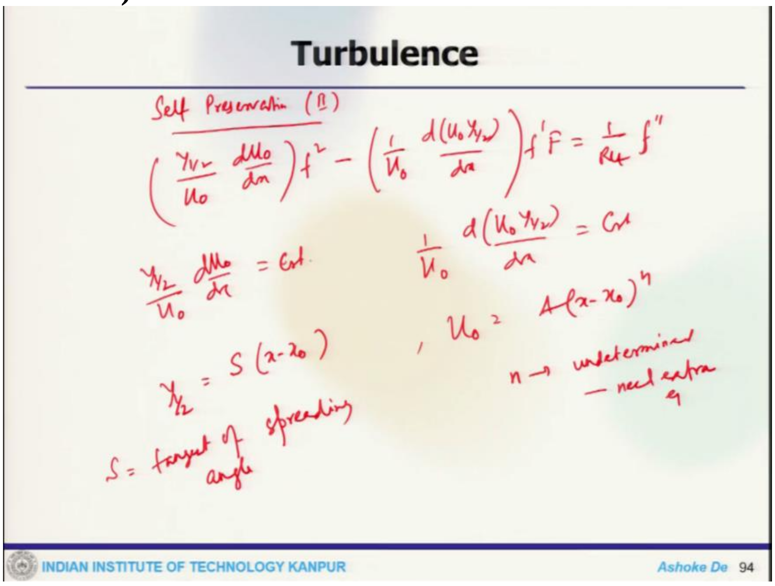

So, then my Self Preservation 2 would provide me substitute the self-similar solution into x momentum equation. We will get

$$\left(\frac{y_{1/2}}{u_0}\frac{du_0}{dx}\right)f^2 - \left(\frac{1}{u_0}\frac{d\left(u_0 y_{\frac{1}{2}}\right)}{dx}\right)f'F = \frac{1}{Re_t}f''$$

This will be self preserving if

$$\frac{y_{\frac{1}{2}}}{u_0}\frac{du_0}{dx} = constant$$

$$\frac{1}{u_0}\frac{d\left(u_0 y_{\frac{1}{2}}\right)}{dx} = constant$$

which follows that $y_{\frac{1}{2}} = S(x - x_0)$, $u_0 = A(x - x_0)^n$

S is tangent of spreading angle and n is the exponent which is undetermined. So, we need an extra equation.

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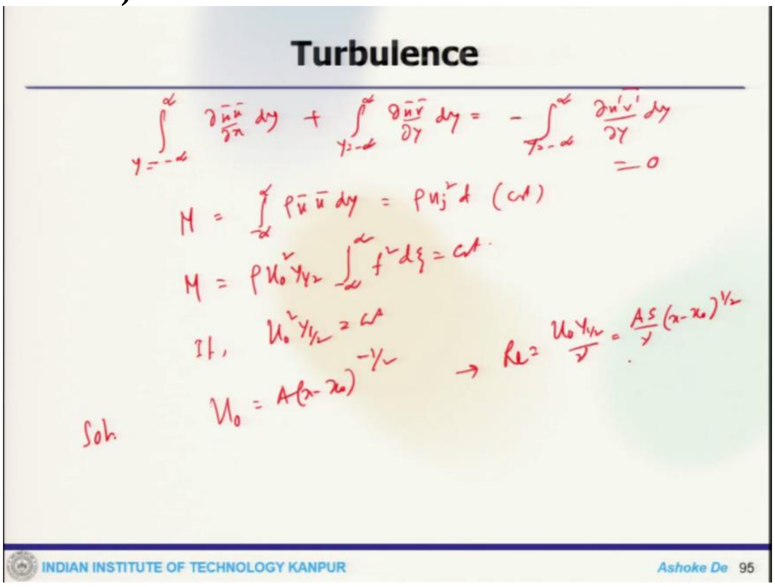

Now, we will get an extra equation for u_0 from

$$x_0 \int_{y=-\infty}^{\infty} \frac{\partial \overline{uu}}{\partial x} dy + \int_{y=-\infty}^{\infty} \frac{\partial \overline{uv}}{\partial y} dy = -\int_{y=-\infty}^{\infty} \frac{\partial u'\overline{v'}}{\partial y} dy = 0$$

Now the streamwise momentum is conserved.

we get

$$M = \int_{y=-\infty}^{\infty} \rho \overline{uu}\, dy = \rho {u_j}^2 d\ (constant)$$

$$M = \rho {u_o}^2 y_{\frac{1}{2}} \int_{-\infty}^{\infty} f^2 d\,\xi = constant$$

This will be also Self Preserving, if ${u_o}^2 y_{\frac{1}{2}}$ is constant. The solution which will get for the u_o

$$u_o = A(x - x_0)^{-1/2}$$

where Reynolds number is $Re = \frac{u_0 y_{1/2}}{\nu} = \frac{As}{\nu}(x - x_0)^{-1/2}$

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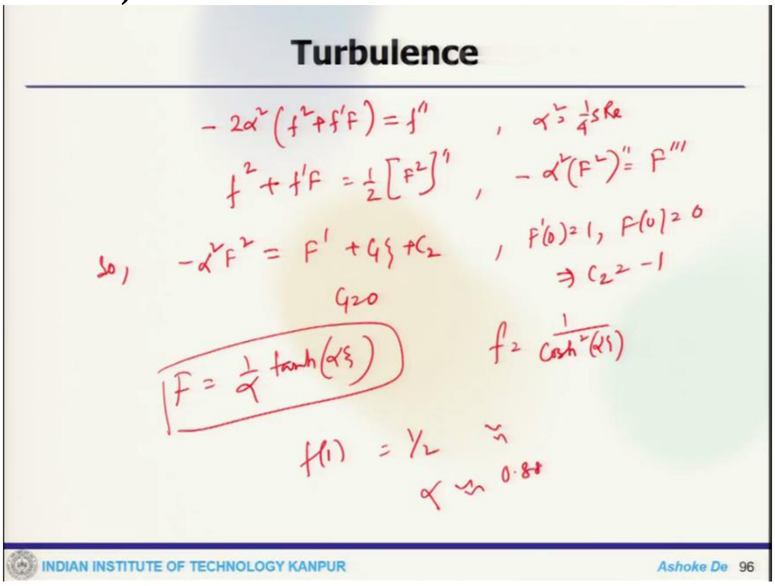

Now, the solution of the resulting equation, which will give

$$-2\alpha^2(f^2 + f'F) = f'' \qquad , \alpha^2 = \frac{1}{4} S\, Re$$

we can get

$$f^2 + f'F = \frac{1}{2}[F^2]'' \qquad , -\alpha^2 [F^2]'' = F'''$$

So, $-\alpha^2 F^2 = F' + C_1\xi + C_2$ Here C_1 becomes 0 because of symmetry.

$F'(0) = 1, F(0) = 0$ Which gets $C_2 = -1$. So, the solution for F and f will be

$F = \frac{1}{\alpha}\tanh(\alpha\xi)$ $\qquad f = \frac{1}{\cosh(\alpha\xi)^2}$

$f(1) = \frac{1}{2}$ $\qquad \alpha \approx 0.88$

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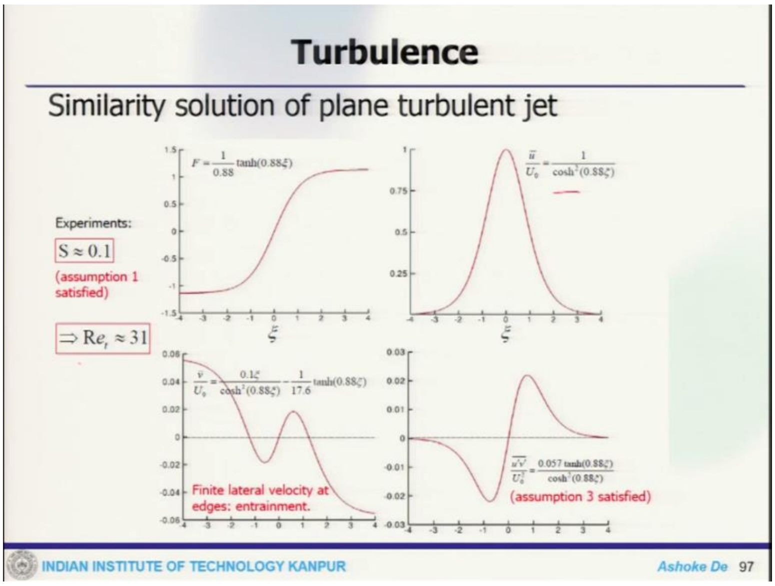
similarity solution for a plane turbulent jet



So, you can see for a similarity solution for a plane turbulent jet. We can see how it varies. This is the profile of u by u_0 this is for turbulent Reynolds number of 31.

So, this gives you an idea how things vary with.

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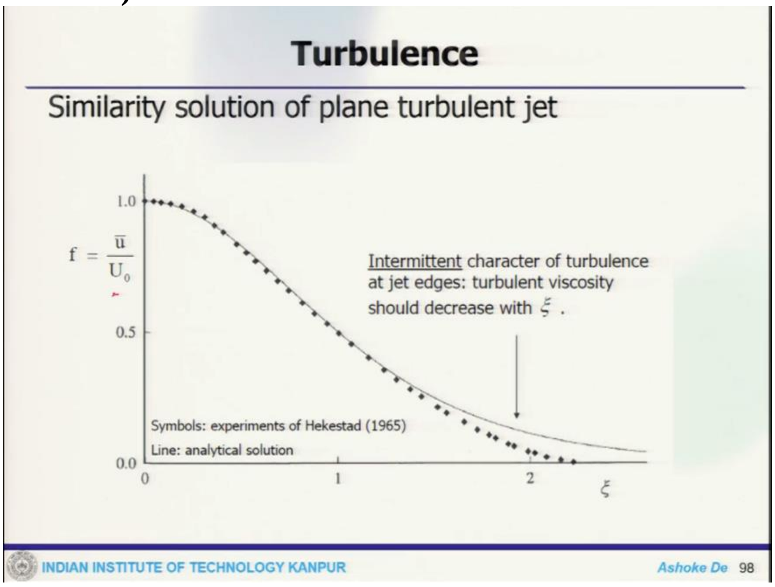

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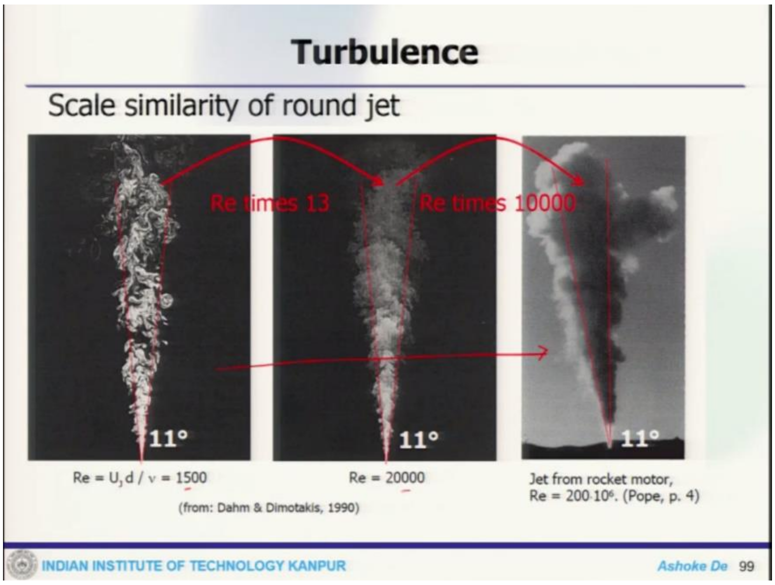
scale similarity of the round jet



Now, at the same time, these non-dimensional velocity This is how it varies and this will give you an idea about the scale similarity of the round jet. The similarity means there will be similarity of the large-scale structure. But, once we go from this first to last in above slide, the range of small scale structure actually increases.

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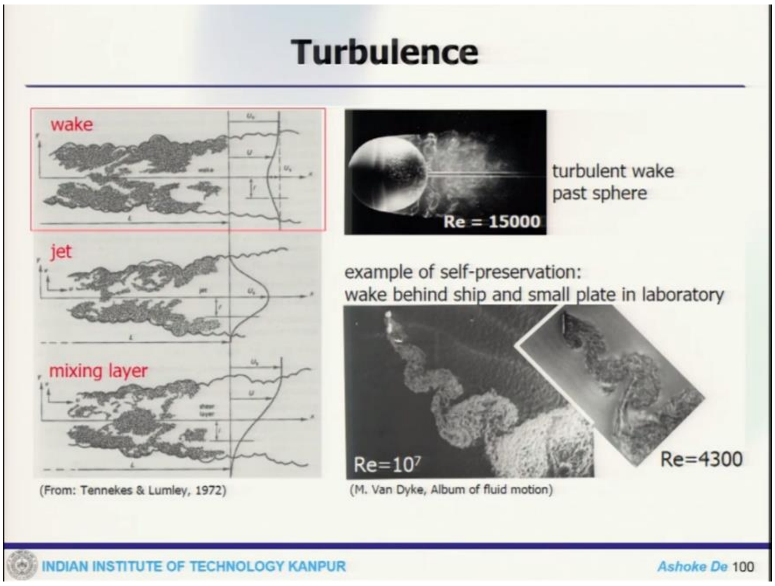

So, this is again some of the pictures like this is mixing layer jet, wake, this is the turbulent wake behind the sphere. So, this shows the self-preservation this is a high Reynolds number and a low Reynolds number. These are all example of self-preservation of the system.

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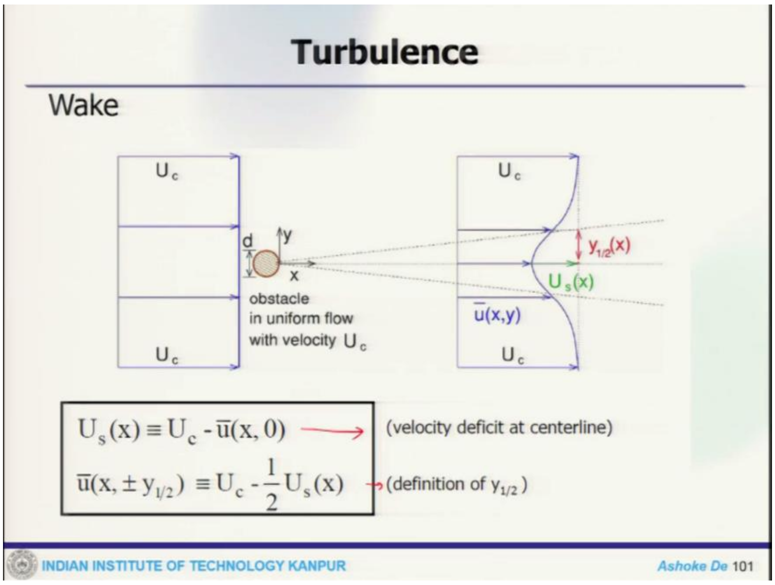
velocity deficit in wake



Now, we can see what happens. There is a circular cylinder or a circular obstacle here and is behind the width. The flow field takes on this kind of profile, but because of this pattern, there is a deficit. So, there is a velocity deficit at the center line in the wake and the half width is shown.

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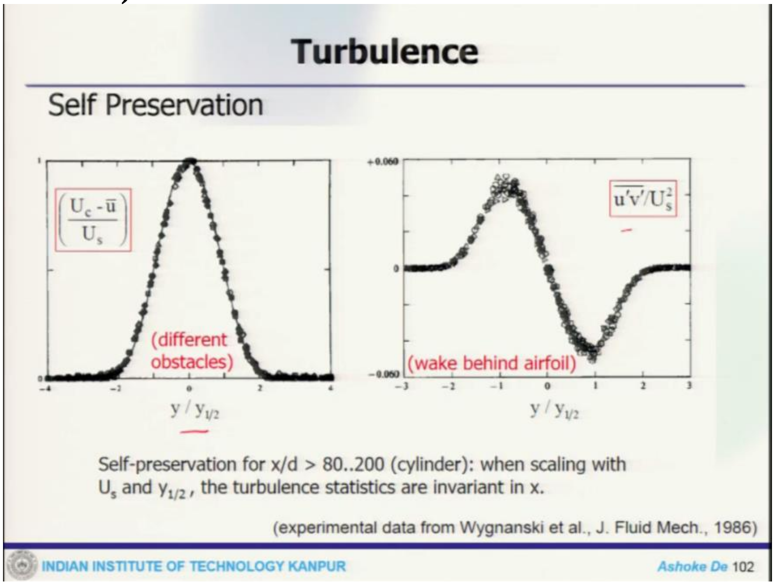

Now, when you talk about the self-preservation and this is what $\frac{y}{y_{1/2}}$ plot. shows nice plots and this is how the wake behind the fluctuating component behaves. So, this gives you an idea and there are flows which essentially shows this kind of behaviour.

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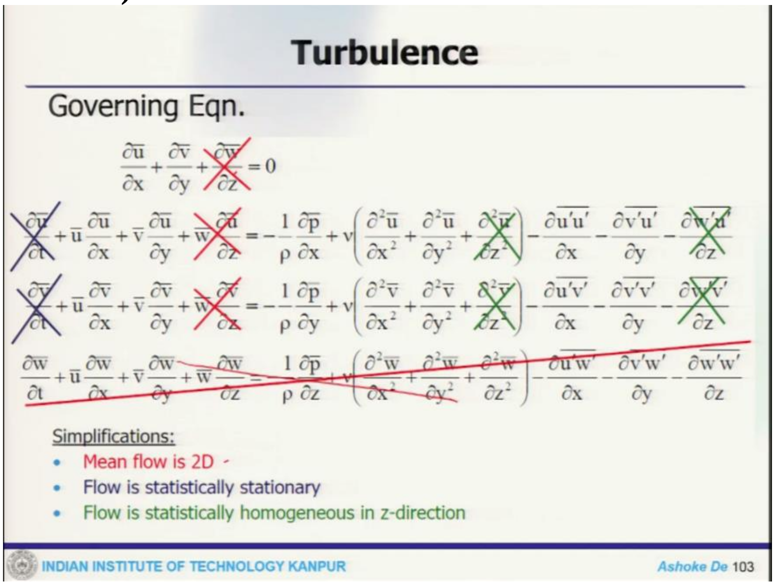

Now, coming back to that, we had mean flow in 2D so, we got rid of the terms as above, because of the flow statistically stationery and flow is statistically homogeneous in z-direction. So, this is the equation we started off.

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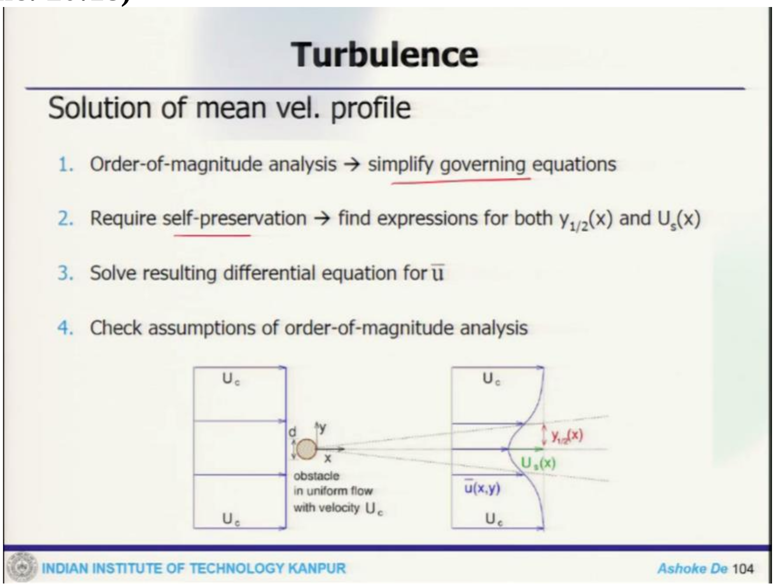

Now we find out the solution of mean velocity profile in the wake. We will again start with order of magnitude analysis and then simplify the governing equations. We find out the self-preservation and solve the required equations. And once you do that, we can get the solution for the mean velocity profile at the wake. So, this is how we proceed further. One can go through this in standard textbook and all these.

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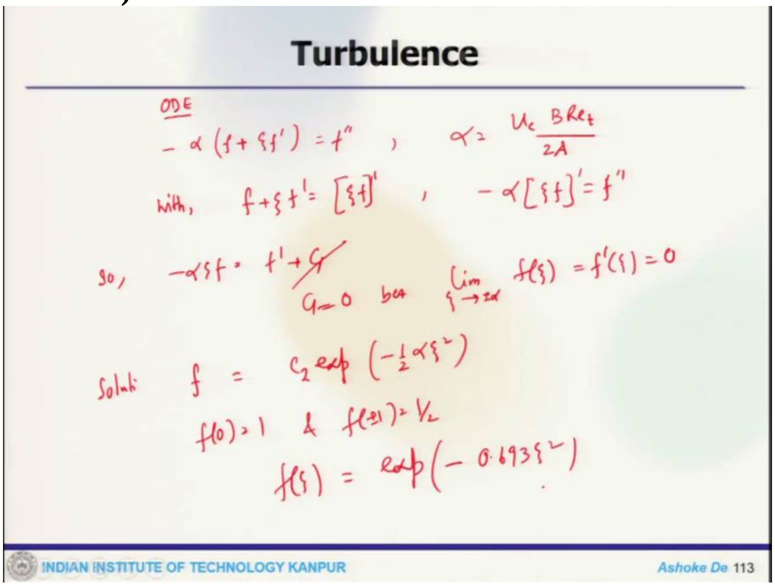

We can come down to the ordinary differential equation that you get. So this will give you an idea about the solution, which would be

$$-\alpha(f + \xi f') = f'' \qquad , \qquad \alpha = \frac{u_c B Re_t}{2A}$$

With,

$$f + \xi f' = [\xi f]' \qquad , \qquad -\alpha[\xi f]' = f''$$

we get

$$-\alpha\xi f = f' + C_1 \quad , \quad C_1 = 0 \; because \; \lim_{\xi \to \pm\infty} f(\xi) = f'^{(\xi)} = 0$$

And solution of f

$$f = C_2 \exp\left(\frac{-1}{2}\alpha\xi^2\right)$$

$$f(0) = 1, \quad f(\pm 1) = 1/2$$

Solution will be

$$f = C_2 \exp(-0.693\xi^2)$$

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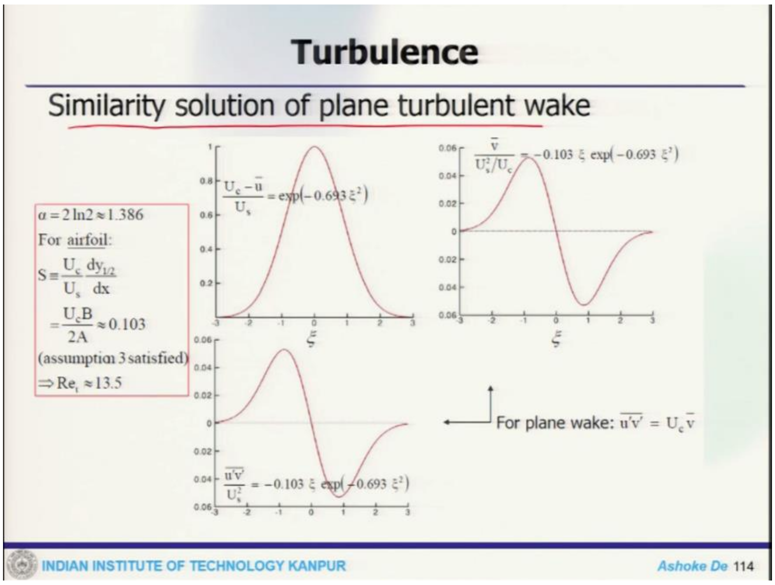
Similarity solution of plane turbulent wake



So, if you again, plot, so this is a similarity solution of a plane turbulent wake. if we plot that, then you get this kind of profile. So, this standard calculation one can continue, and basically derived that. The is a solution of the f below.

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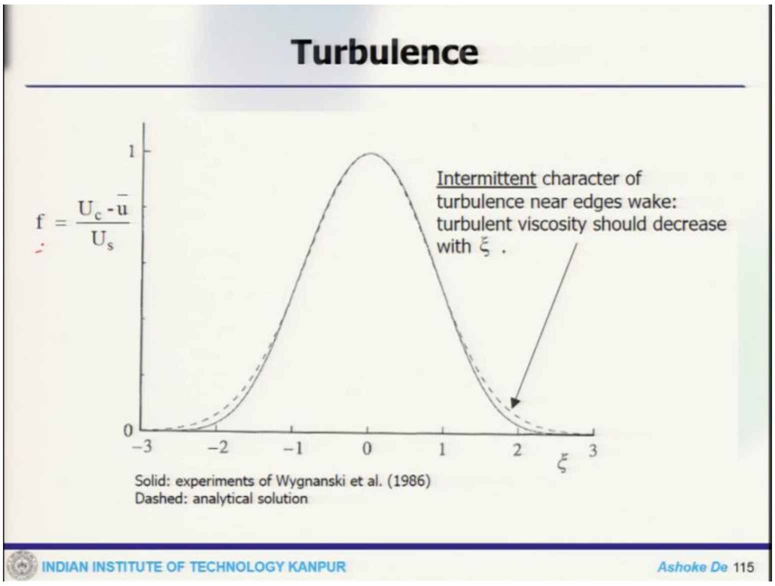

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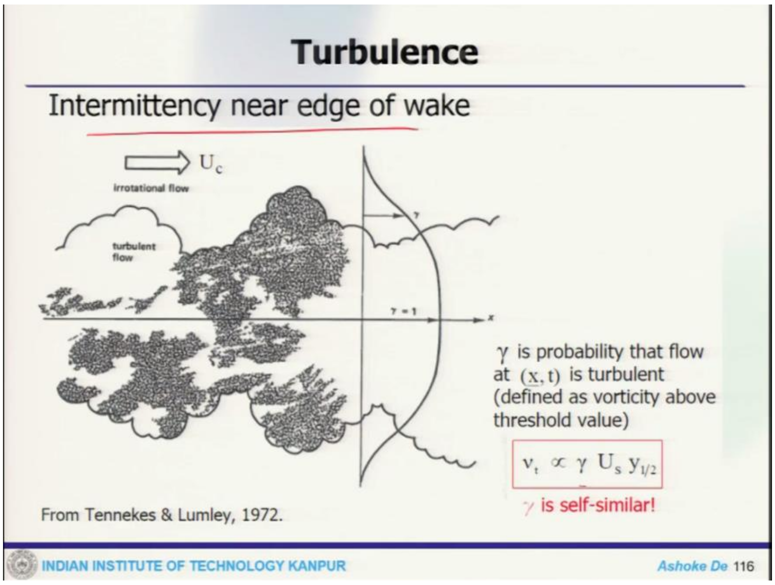
Intermittency near edge of wake



But if you look at this particular picture, here is the U_c coming in and you can see the intermittency near each of the wake. So γ is the probability that flow at x, t is turbulent,

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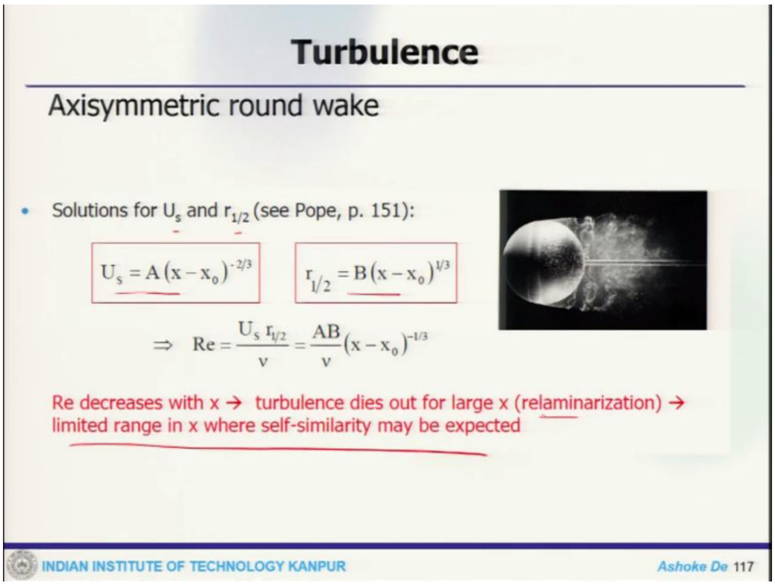
Axisymmetric round wake



Now, if you have an axisymmetric round wake, then the solution of U_s and $r_{1/2}$, you can find out the solution of this pattern. Reynolds number decreases with x, turbulence dies out of large x, so it is a relaminarization. So limited range in x where self-similarity may be expected.

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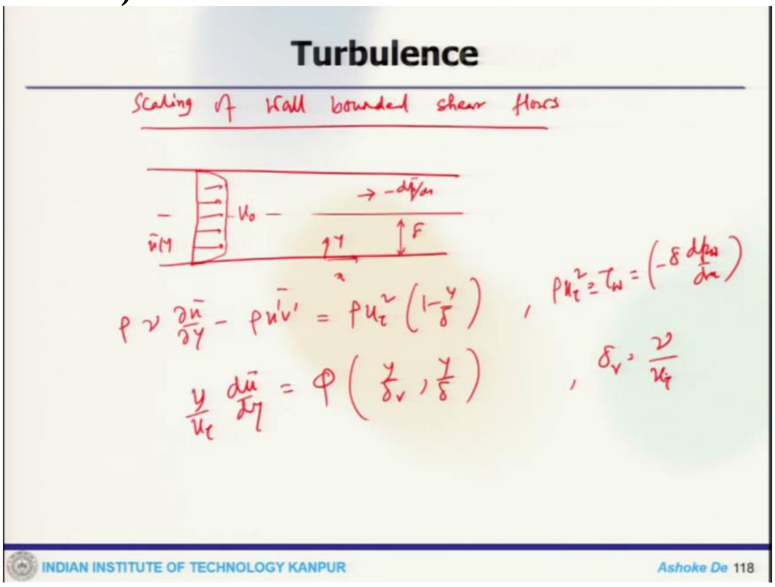

Now we can look at the scaling of wall bounded shear flow. So, you can take a center line, there could be a nice velocity profile. So profile for total shear stress which will give me

$$\rho \upsilon \frac{\partial \overline{u}}{\partial y} - \rho \overline{u'v'} = \rho {u_\tau}^2 \left(1 - \frac{y}{\delta}\right)$$

Where

$$\rho {u_\tau}^2 = \tau_w = \left(-\delta \frac{dp_w}{dx}\right)$$

So scaling on mean velocity gradient in based on dimensional analysis one can get

$$\frac{y}{u_\tau}\frac{\partial \overline{u}}{\partial y} = \phi\left(\frac{y}{\delta_\upsilon}, \frac{y}{\delta}\right) \qquad , \delta_\upsilon = \frac{\upsilon}{u_\tau}$$

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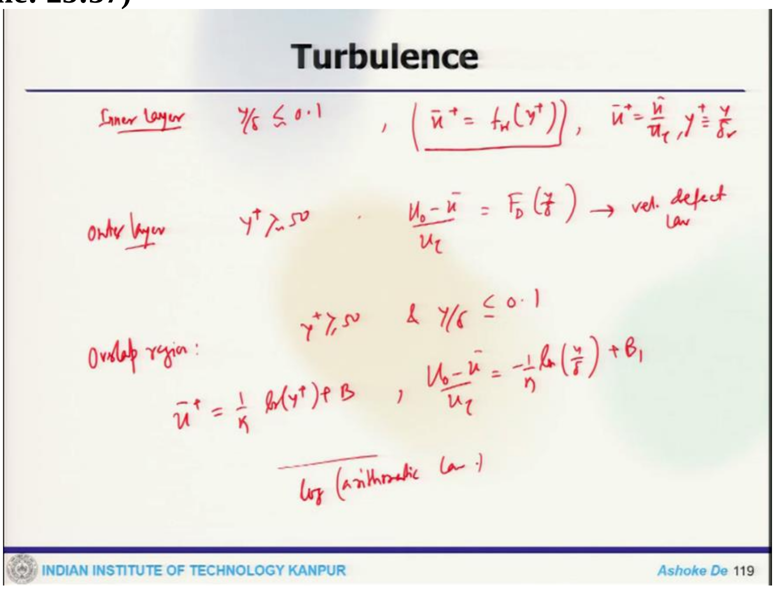

So, we have 3 different layer one is the inner layer where $\frac{y}{\delta} \leq 0.1$, this is close to the wall and length scale delta is not that important. So, what we get

Inner layer

$\overline{u}^+ = f_w(y^+), \;\; \overline{u}^+ = \frac{\overline{u}}{u_\tau} \;, \; y^+ = \frac{y}{\delta_v}$

this is called law of the wall.

Outer layer where $y^+ > 50$, this is far from the wall then we get

$$\frac{u_0 - \overline{u}}{u_\tau} = F_D(\frac{y}{\delta})$$

This is known as velocity defect law.

And there could be overlap region which we can identify which is $y^+ > 50$ and $\frac{y}{\delta} \leq 0.1$ we have

$$\overline{u}^+ = \frac{1}{\kappa}\ln(y^+) + B \qquad , \qquad \frac{u_0 - \overline{u}}{u_\tau} = -\frac{1}{\kappa}\ln\left(\frac{y}{\delta}\right) + B_1$$

this is one can think about log arithmetic law.

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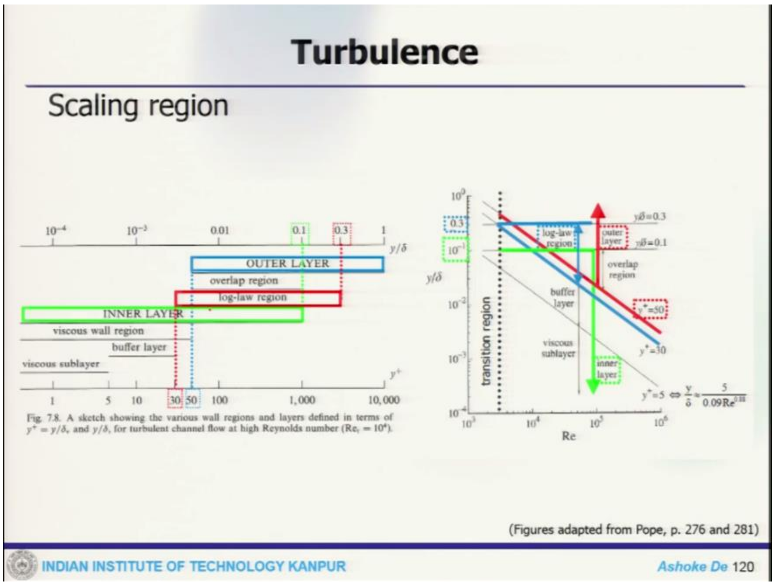
Scaling region



So once you put that thing, so there would be inner layer, log layer, wall region and outer layer. these are the scaling region. This is adapted from Pope; one can go through the details there. And this is variation with the Reynolds number and $\frac{y}{\delta}$. So, that is what you get when you look at these different scaling at the different region, in a wall bounded shear flow. We will stop today and continue the discussion in the next lecture. Thank you.