Turbulent Combustion: Theory and Modelling
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Lecture-33
Turbulence (Contd...)

Welcome back. So, let us continue the discussion on turbulence and the scaling of turbulence. So, we are looking at the different scale of turbulent flow field. So, like macro structure and then the micro structure and then also we have looked at the energy cascading phenomena. So, now, we are looking at the different statistical description of the turbulence and we have so far looked at the averaging methods.

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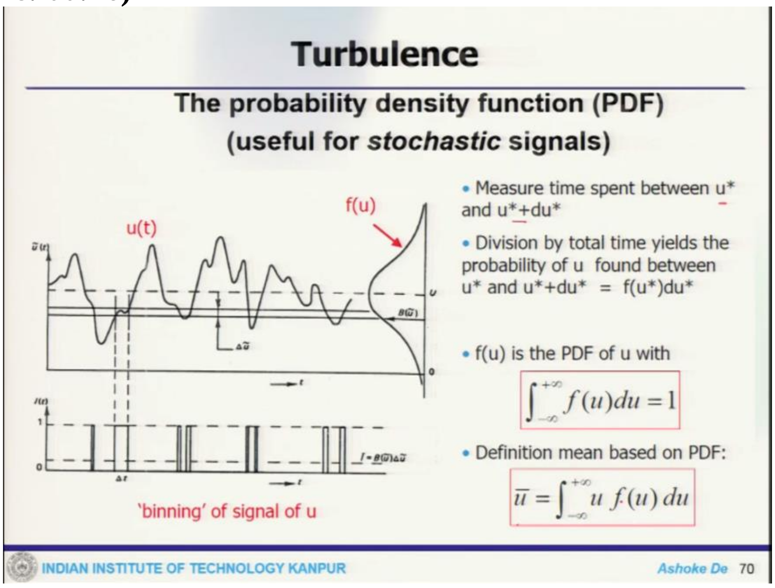

Probability density function (PDF) of Turbulent signal

Now, we will look at some of these other features like probability density function (PDF), which is quite useful when we look at them in a statistical signal like turbulent signal. Above slide provides a velocity profile and its time history of the velocity profile. This is instantaneous velocity profile and if we look at the PDF of it, we have measured time spent between u^* and $u^* + du^*$ velocity like one particular instant and the delta change of that.

So, using the f(u) PDF definition of

$$\int_{-\infty}^{\infty} f(u)\, du = 1$$

When you look at the definition of mean velocity based on PDF, we can get

$$\bar{u} = \int_{-\infty}^{\infty} u\, f(u)\, du$$

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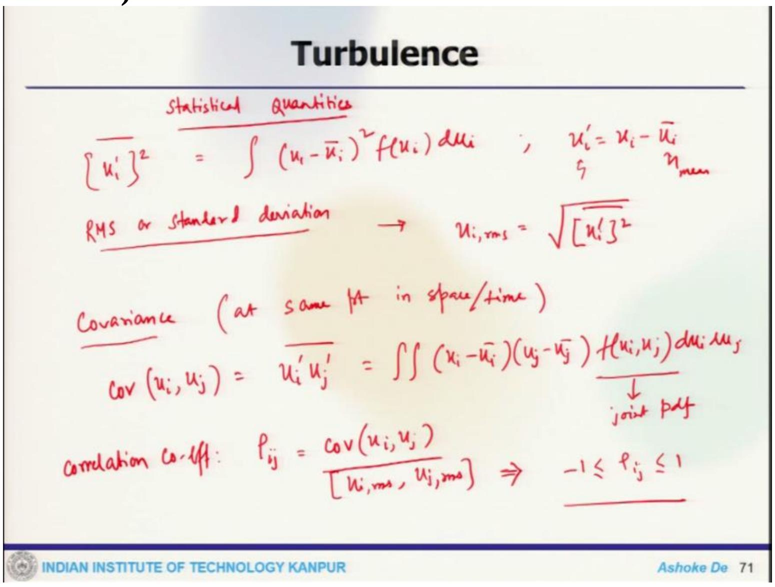

Statistical quantities of Turbulent signal

Now, we can look at some of these statistical quantities. We will look at the variance of a velocity component.

$$\overline{[u_i]^2} = \int (u_i - \overline{u_i})^2 \, f(u_i) \, du_i$$

Where

$u_i' = u_i - \overline{u_i}$

$\overline{u_i}$ is the mean quantity.

$u_i{}'$ is turbulent fluctuation.

Similarly, one can find out the RMS or standard deviation. Which give us

$$u_{i\,rms} = \sqrt{\overline{[u'_i]^2}}$$

now we can find out the covariance between two velocity components.

Covariance between two velocity components at same point in space and time.

$$Cov(u_i, u_j) = \overline{u'_i \, u'_j} = \iint (u_i - \overline{u_i}) \left(u_j - \overline{u_j}\right) f\left(u_i \, u_j\right) du_i \, du_j$$

$f\left(u_i \, u_j\right)$ function is called the joint PDF. Now, this concept of this probability density function, this should be quite handy when we look at some PDF based combustion model. Now, the other one is the correlation coefficient which is

$$\rho_{ij} = \frac{Cov(u_i, u_j)}{\left[u_{i\,rms}\, , u_{j\,rms}\right]}$$

So, that gives

$$-1 \leq \rho_{ij} \leq 1$$

this correlation coefficient is sort of bounded.

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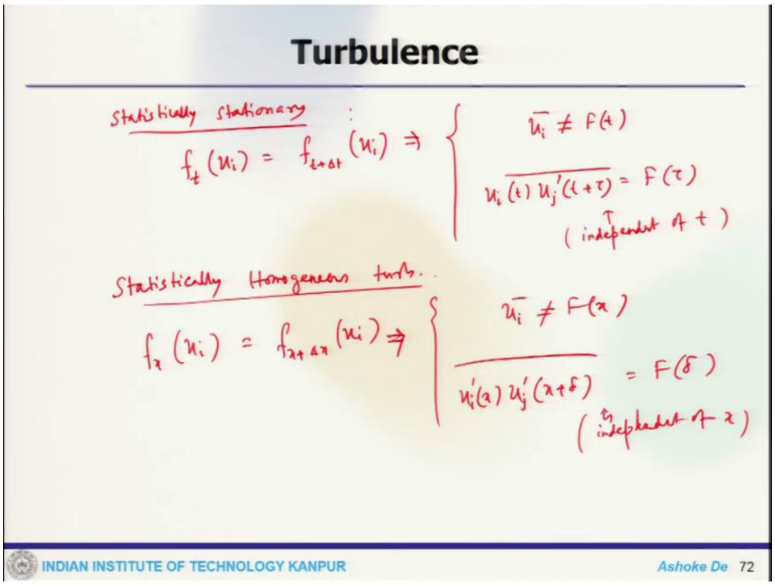

Now, there could be some special cases, let us say if we have 1. statistically stationary case or steady turbulence, the PDF becomes independent of time. Then we can write

$$f_t(u_i) = f_{t+\Delta t}(u_i) \Rightarrow \begin{cases} \overline{u_i} \neq f(t) \\ \overline{u'_i(t)u'_j(t+\tau)} = F(\tau) \neq f(t) \end{cases}$$

Now, item 2. statistically homogeneous turbulence. For statistically homogeneous turbulent we can write

$$f_t(u_i) = f_{t+\Delta t}(u_i) \Rightarrow \begin{cases} \overline{u_i} \neq f(x) \\ \overline{u'_i(x)u'_j(x+\delta)} = F(\delta) \neq f(x) \end{cases}$$

So, one case independent of time, other case we have independent of x.

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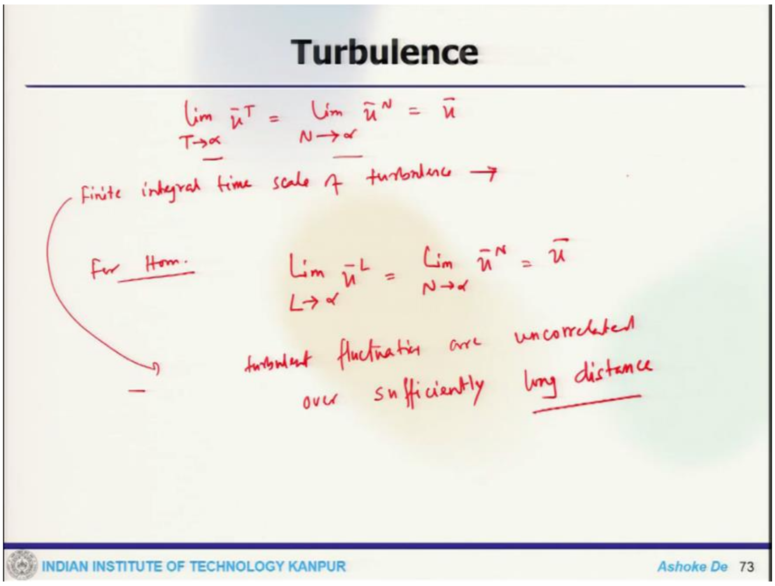

Now, for a stationary process, we can write that

$$\lim_{T\to\infty} \overline{u}^{T} = \lim_{N\to\infty} \overline{u}^{N} = \overline{u}$$

So that means that time averaging and ensemble averaging lead to the same thing. One can think about finite integral timescale of turbulence which means that turbulence fluctuations are uncorrelated over sufficiently long time.

Now, for homogeneous process, we can have this limit

$$\lim_{L\to\infty} \overline{u}^{L} = \lim_{N\to\infty} \overline{u}^{N} = \overline{u}$$

Line averaging become similar to and ensemble averaging. So, here the finite integral timescale of turbulence is that turbulent fluctuations are uncorrelated over sufficiently long distance. So, one case 1. the fluctuations are uncorrelated over a sufficiently long time and in this case for homogeneous process the turbulent fluctuations are uncorrelated over sufficiently long distance. So, these are the two hypotheses one can have.

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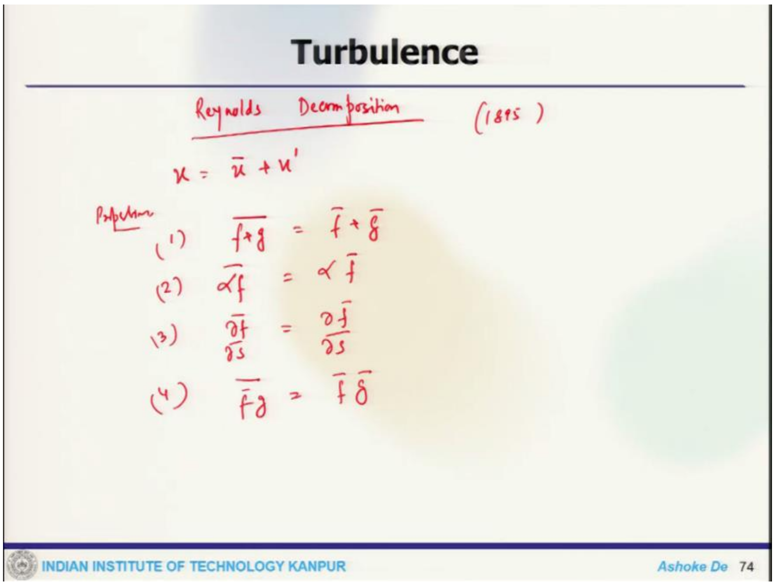

Properties of averaging

Now, that brings to the Reynolds decomposition. Reynolds who proposed in 1895 that if I have instantaneous flow field, I can have mean plus fluctuating component.

$$u = \overline{u} + u'$$

After Reynolds averaging the properties that one can have these are quite important.

1. $\overline{f + g} = \overline{f} + \overline{g}$
2. $\overline{\alpha f} = \alpha \overline{f}$
3. $\overline{\frac{\partial f}{\partial s}} = \frac{\partial \overline{f}}{\partial s}$
4. $\overline{\overline{f} g} = \overline{f}\, \overline{g}$

So, these are the some of the desired properties of averaging.

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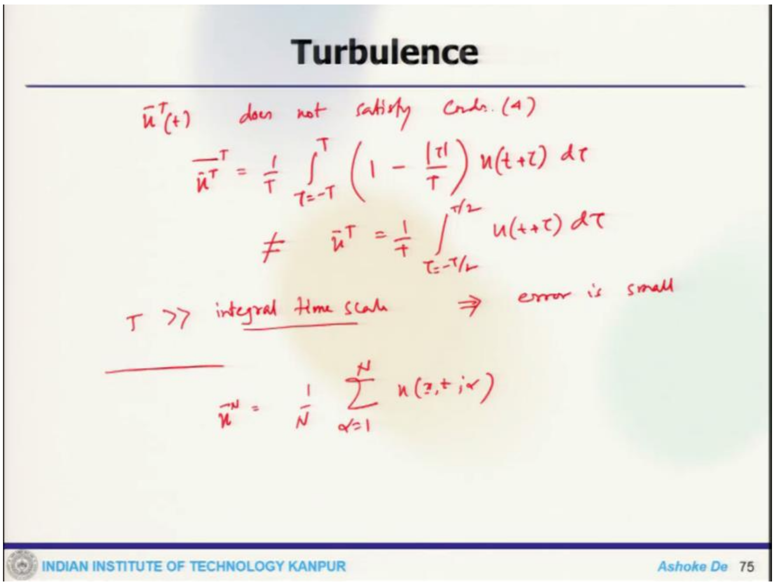

If you look at the time averaging that is $\overline{u}^{T}(t)$, time averaging actually it does not satisfy condition number 4, because of the doubling of the integral interval like

$$\overline{\overline{u}^{T}}^{T} = \frac{1}{T}\int_{-T}^{T}\left(1-\frac{|\tau|}{T}\right) u(t+\tau)\, d\tau$$

$$\neq \overline{u}^{T} = \frac{1}{T}\int_{-T/2}^{T/2} u(t+\tau)\, d\tau$$

If T>> integral timescale, which is the timescale of larger eddy essentially, integral timescale is the timescale of larger eddies. If it is much larger than that, in that case this error is small. Now, at the same time, if you look at the ensemble averaging it does satisfy all these properties. So, this is what we write from ensemble averaging is that

$$\overline{u}^{N} = \frac{1}{N}\sum_{\alpha=1}^{N} u(x,t,\alpha)$$

So, that satisfy all the condition.

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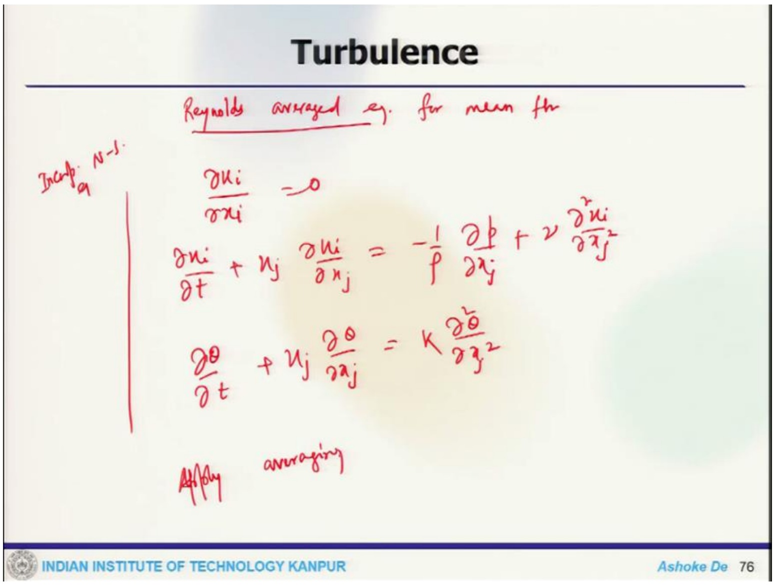

Reynolds averaging

Now we can get the Reynolds averaging equation for mean flow. So, we start with an incompressible Navier stokes equation

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_j} + v\frac{\partial^2 u_i}{\partial {x_j}^2}$$

$$\frac{\partial \theta}{\partial t} + u_j \frac{\partial \theta}{\partial x_j} = \kappa \frac{\partial^2 \theta}{\partial {x_j}^2}$$

So, this is our starting point that means, all incompressible NS equation now we apply the average over here. So, once we apply, averaging, Reynolds averaging to these below equations,

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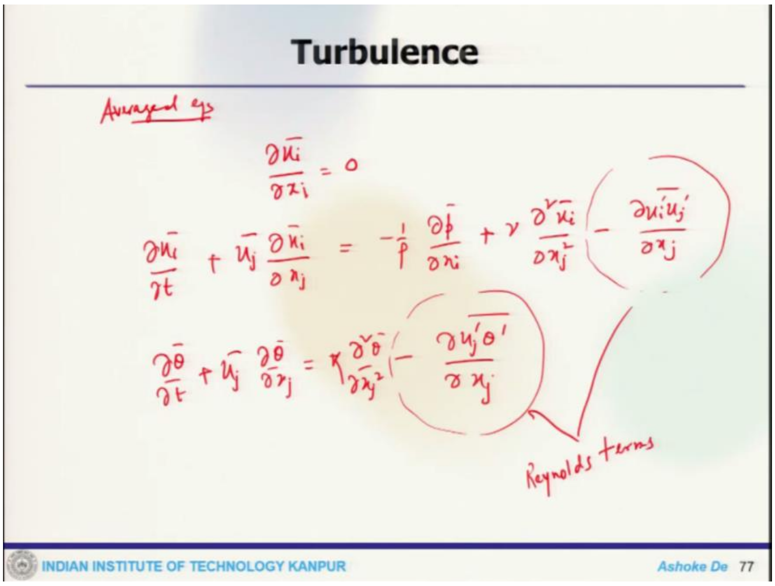

So, we can get the average equation

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0$$

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_j} + \nu \frac{\partial^2 \bar{u}_i}{\partial {x_j}^2} - \frac{\partial \overline{u'_i u'_j}}{\partial x_j}$$

$$\frac{\partial \bar{\theta}}{\partial t} + \bar{u}_j \frac{\partial \bar{\theta}}{\partial x_j} = \kappa \frac{\partial^2 \bar{\theta}}{\partial {x_j}^2} - \frac{\partial \overline{u'_j \theta'}}{\partial x_j}$$

So, you can see two additional term which are added compared to the starting equation or governing equations. Due to the averaging we get these additional terms and these terms are known as Reynolds terms.

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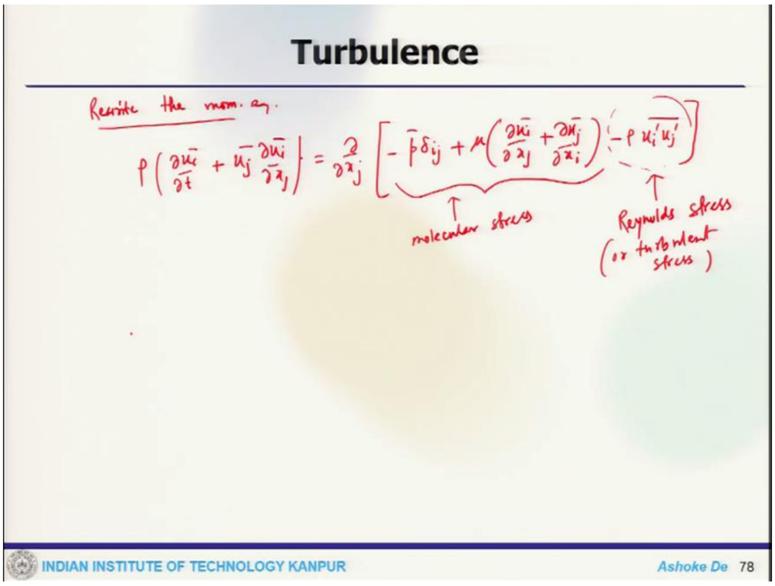

Now, we can rewrite the momentum equation

$$\rho\left[\frac{\partial \bar{u}_i}{\partial t}+\bar{u}_j\frac{\partial \bar{u}_i}{\partial x_j}\right]=\frac{\partial}{\partial x_j}\left[\underbrace{-\bar{p}\,\delta_{ij}+\mu\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)}_{\text{Molecular stress}}-\underbrace{\rho\overline{u'_i\,u'_j}}_{\substack{\text{Reynolds stress}\\ \text{Or}\\ \text{Turbulent stress}}}\right]$$

The extra term is arrived because of Reynolds averaging over the equation system. One can interpret is that, the turbulent fluctuations act on mean flow as if they induce an additional stress which is a direct result from the averaging and this additional stress originates from the transport of mean momentum by turbulent fluctuation.

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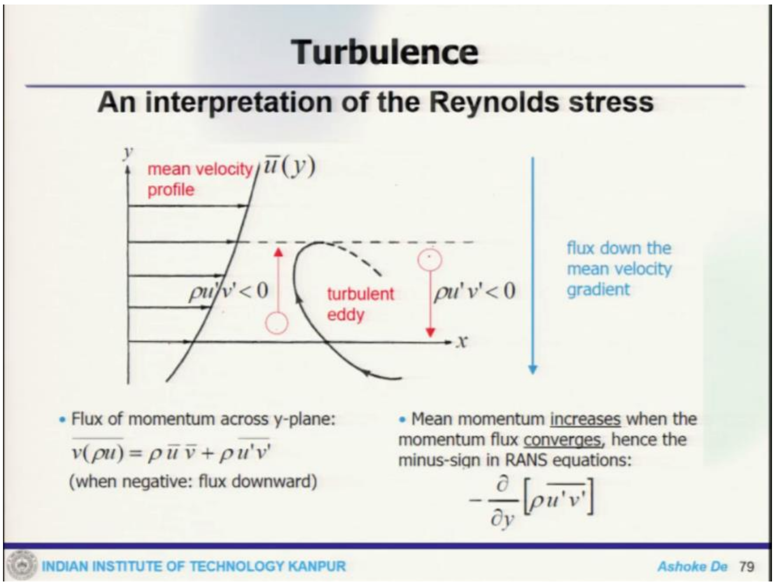

Interpretation of Reynolds stress

We can have a velocity profile like above and fluid particle may go in increasing y direction. The mean velocity increases, mean momentum also increases but between the points to maintain that momentum flux the condition $\rho u'v' < 0$ has to satisfy. Similarly, when moving own along y mean momentum decreases. But to maintain the momentum, this has to be increased. So, one can look at it in that way.

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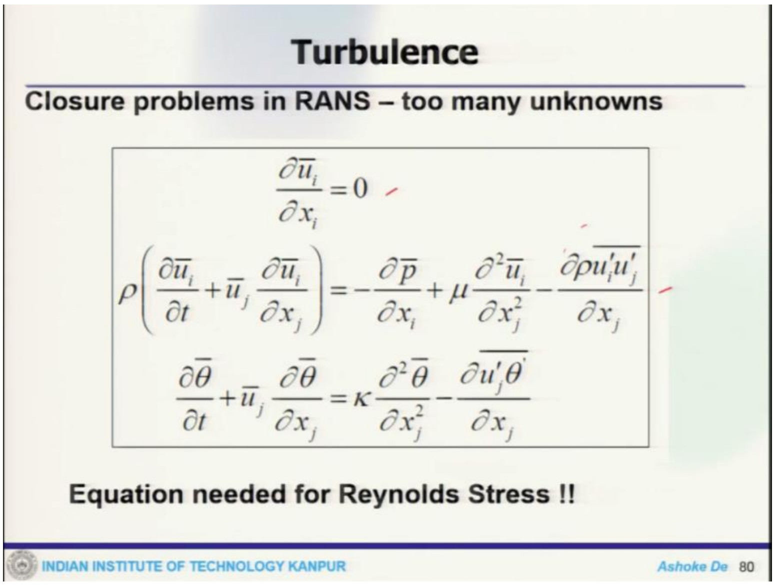

If we look at those equations completely, we need an equation for the Reynolds stress.

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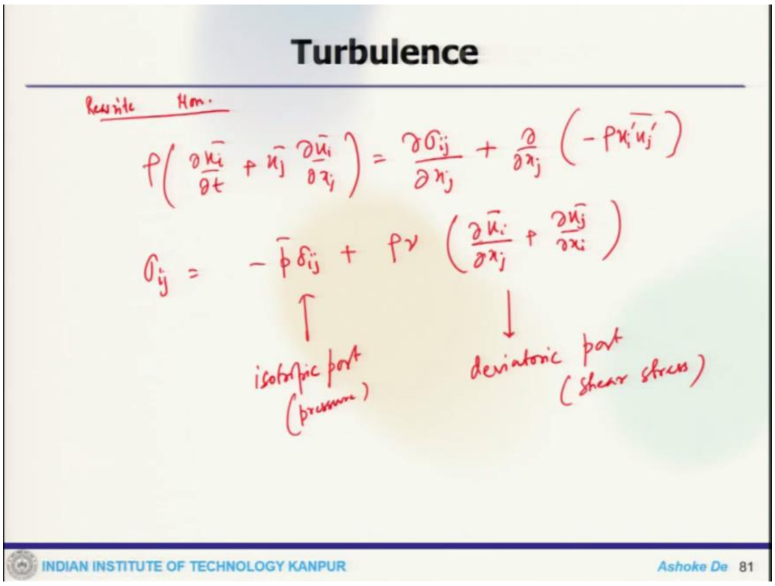

Closure problems in RANS

Now, closure for this stress what we can write, we have this momentum equation we can rewrite again in a slightly different form

$$\rho\left[\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j}\right] = \frac{\partial \sigma_{ij}}{\partial x_j} + \frac{\partial}{\partial x_j}\left[-\rho \overline{u'_i u'_j}\right]$$

$$\sigma_{ij} = \underbrace{-\bar{p}\,\delta_{ij}}_{\text{Isotropic part (Pressure)}} + \underbrace{\rho \nu \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)}_{\text{Deviatoric part (Shear stress)}}$$

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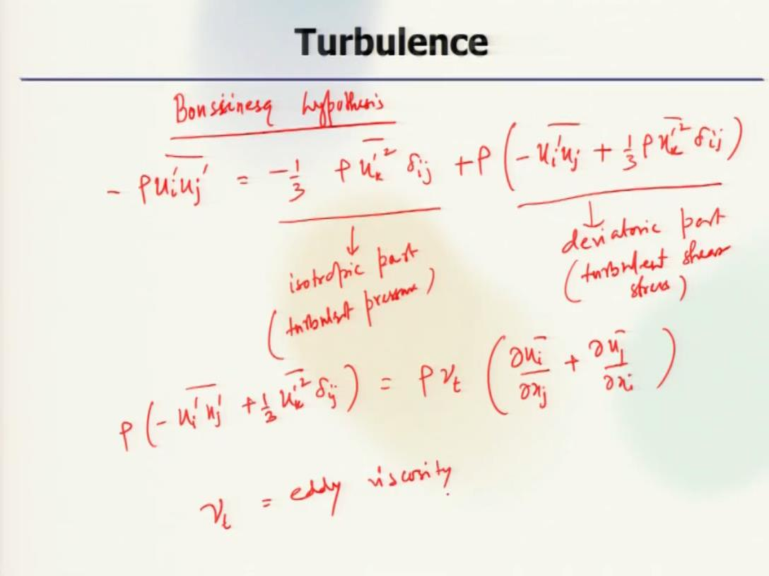

To close that Reynolds stress term, we bring in the boussinesq hypothesis. so that we can write these Reynolds stress term in a slightly different way.

$$-\rho\overline{u'_i\,u'_j} = \underbrace{-\frac{1}{3}\rho\overline{u'_k{}^2}\delta_{ij}}_{\substack{\text{Isotropic part}\\ \text{(Turbulent Pressure)}}} + \rho\underbrace{\left(-\overline{u'_i\,u'_j} + \frac{1}{3}\rho\overline{u'_k{}^2}\delta_{ij}\right)}_{\substack{\text{Deviatoric part}\\ \text{(Turbulent Shear stress)}}}$$

Now, boussinesq hypothesis based on the similarity with molecular stress, we can write

$$\rho\left(-\overline{u'_i\,u'_j} + \frac{1}{3}\rho\overline{u'_k{}^2}\delta_{ij}\right) = \rho\upsilon_t\left(\frac{\partial\bar{u}_i}{\partial x_j} + \frac{\partial\bar{u}_j}{\partial x_i}\right)$$

υ_t is the eddy viscosity or turbulent viscosity.

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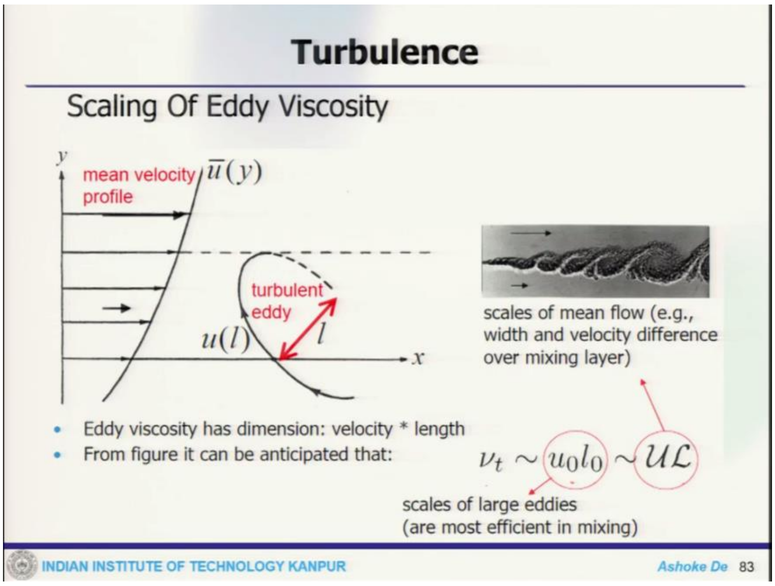

Now, the scaling of eddy viscosity. If l is the size of the turbulent eddy, which is having a velocity scale of $u(l)$ and the picture of the shear layer is shown above. The Eddy viscosity is order of $\upsilon_t \sim u_0 l_0$and for the large scale it is $\upsilon_t \sim U\ L$, that is for macro structured scale for the large eddies, which allows more efficient mixing.

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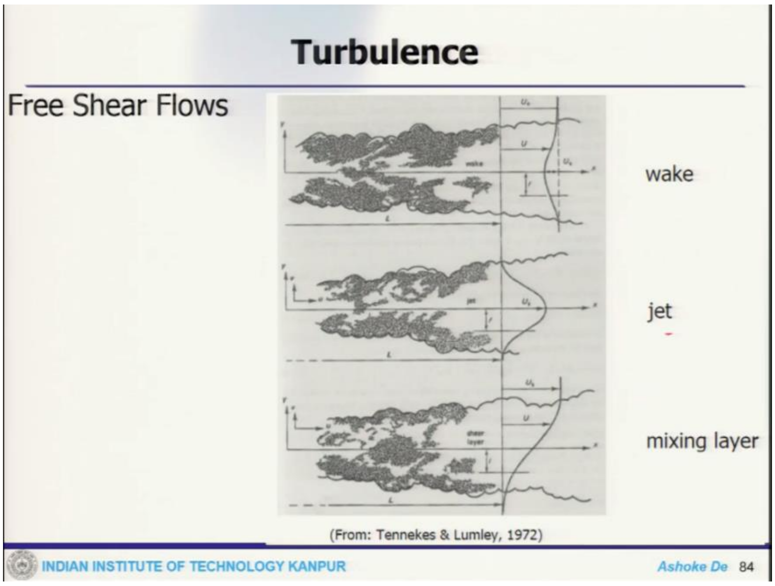

Now, this is an example of free shear flow where you see a mixing layer where two different layers they are mixing. Different kind of eddies that form and the velocity profile is shown. These are some example of free shear flows.

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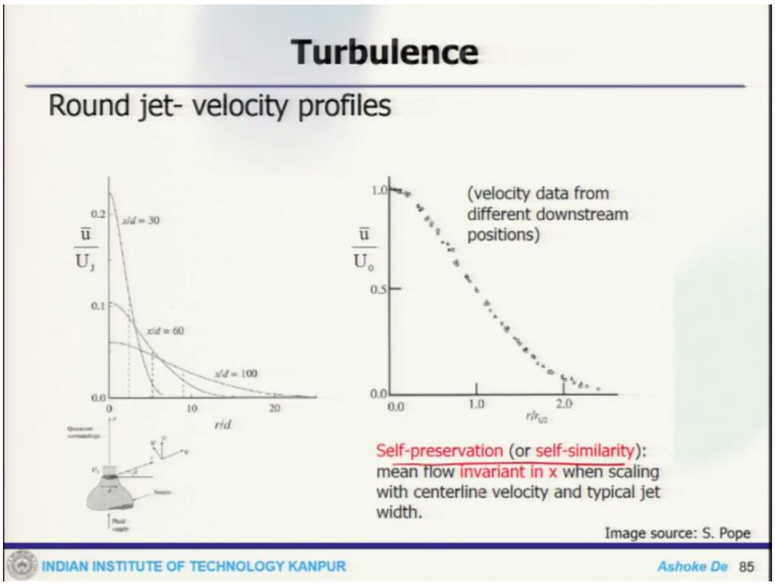

When we look at the round jet, as we go along with the axial distance, the velocity profile is shown for this. This actually gives an idea that this is self-similarity: mean flow in variant of x when scaling with centreline velocity. This is an important hypothesis.

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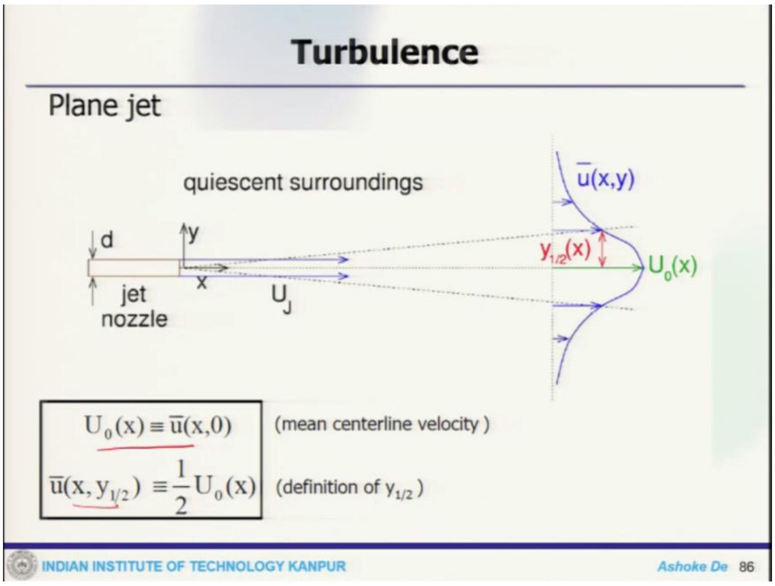

Now, we see a planar jet, there is a jet nozzle and from where the jet is injected and further downstream, we can see how the profile looks like.

Mean centerline velocity

$$U_0(x) = \overline{u}(x, 0)$$

Definition of $y_{1/2}$

$$\overline{u}(x, y_{1/2}) = \frac{1}{2} U_0(x)$$

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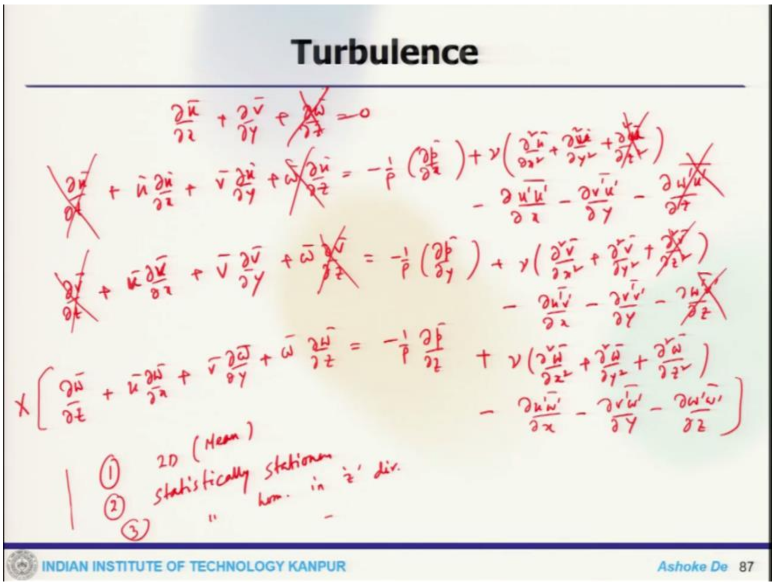

Starting with the governing equation which are in cartesian coordinate system.

Continuity equation

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}=0$$

X-momentum equation

$$\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}+\nu\left(\frac{\partial^2 \bar{u}}{\partial x^2}+\frac{\partial^2 \bar{u}}{\partial y^2}+\frac{\partial^2 \bar{u}}{\partial z^2}\right)-\frac{\partial \overline{u'u'}}{\partial x}-\frac{\partial \overline{v'u'}}{\partial y}-\frac{\partial \overline{w'u'}}{\partial z}$$

Similarly, for other directional momentum equations.

Y-momentum equation

$$\frac{\partial \bar{v}}{\partial t}+\bar{u}\frac{\partial \bar{v}}{\partial x}+\bar{v}\frac{\partial \bar{v}}{\partial y}+\bar{w}\frac{\partial \bar{v}}{\partial z}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}+\nu\left(\frac{\partial^2 \bar{v}}{\partial x^2}+\frac{\partial^2 \bar{v}}{\partial y^2}+\frac{\partial^2 \bar{v}}{\partial z^2}\right)-\frac{\partial \overline{u'v'}}{\partial x}-\frac{\partial \overline{v'v'}}{\partial y}-\frac{\partial \overline{w'v'}}{\partial z}$$

Z-momentum equation

$$\frac{\partial \bar{w}}{\partial t}+\bar{u}\frac{\partial \bar{w}}{\partial x}+\bar{v}\frac{\partial \bar{w}}{\partial y}+\bar{w}\frac{\partial \bar{w}}{\partial z}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}+\nu\left(\frac{\partial^2 \bar{w}}{\partial x^2}+\frac{\partial^2 \bar{w}}{\partial y^2}+\frac{\partial^2 \bar{w}}{\partial z^2}\right)-\frac{\partial \overline{u'w'}}{\partial x}-\frac{\partial \overline{v'w'}}{\partial y}-\frac{\partial \overline{w'w'}}{\partial z}$$

Now we can have some assumption to simplify the system and carry out this analysis. One of the simplifications can be done is that

1. mean flow is 2D
2. flow is statistically stationary
3. flow is statistically homogeneous in z direction.

With these assumptions we get a simplified system of equations:

Continuity equation

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0$$

X-momentum equation

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \nu\left(\frac{\partial^2 \overline{u}}{\partial x^2} + \frac{\partial^2 \overline{u}}{\partial y^2}\right) - \frac{\partial \overline{u'u'}}{\partial x} - \frac{\partial \overline{v'u'}}{\partial y}$$

Y-momentum equation

$$\overline{u}\frac{\partial \overline{v}}{\partial x} + \overline{v}\frac{\partial \overline{v}}{\partial y} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \nu\left(\frac{\partial^2 \overline{v}}{\partial x^2} + \frac{\partial^2 \overline{v}}{\partial y^2}\right) - \frac{\partial \overline{u'v'}}{\partial x} - \frac{\partial \overline{v'v'}}{\partial y}$$

Now we can use this simplified system to find out the solution. We will stop here and discuss the solution procedure in the subsequent lecture.