Turbulent Combustion: Theory and Modelling
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Lecture 31
Turbulence (Contd…)

K-H Instability

Welcome back. So, let us continue the discussion on turbulence. we are looking at the generalization of the stability analysis. And we started with the K-H instability.

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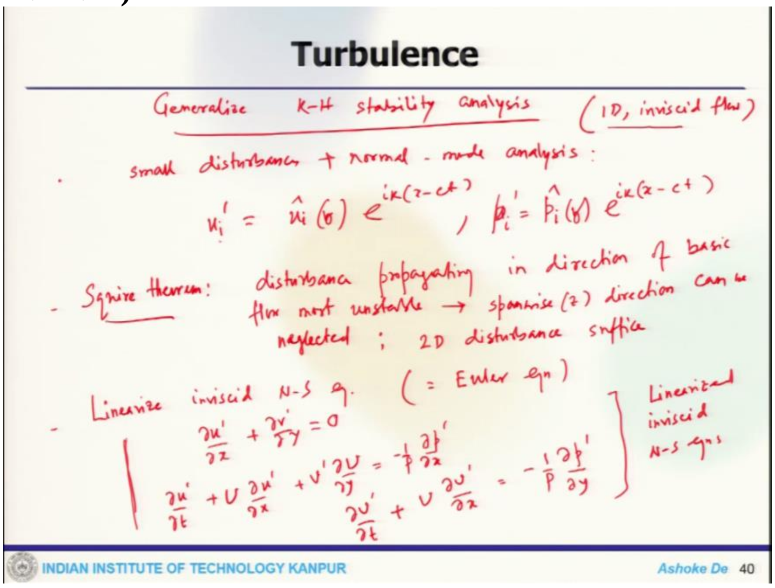

So, this is where we actually stopped in the last class. If you look at this generalization of this K-H instability. This is what we got with an assumption that it is a 1D inviscid flow, and then we add some disturbances for normal-mode analysis, then our u_i', p_i' and then finally, if we linearize the system of equations we get these following linearized inviscid equations.

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0$$

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + v'\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{\partial p'}{\partial x}$$

$$\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} = -\frac{1}{\rho}\frac{\partial p'}{\partial y}$$

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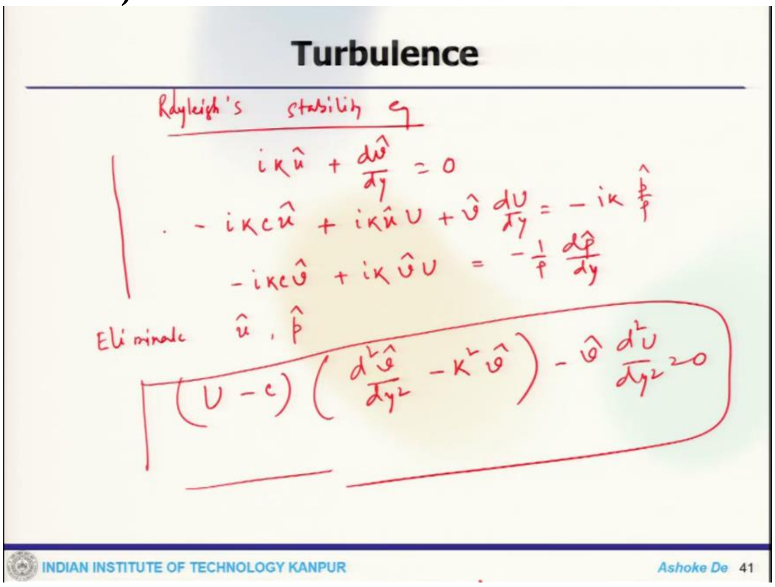

Rayleigh's stability equation

Now, we look at the Rayleigh's stability equation. So, from those equations, when you put

$$u_i' = \hat{u}_i(y)e^{ik(x-ct)}$$

$$p_i' = \hat{p}_i\,(y)e^{ik(x-ct)}$$

We get,

$$ik\hat{u} + \frac{d\hat{v}}{dy} = 0$$

$$ikc\hat{u} + ik\hat{u}U + \hat{v}\frac{dU}{dy} = -ik\frac{\hat{p}}{\rho}$$

$$ikc\hat{v} + ik\hat{v}U = -\frac{1}{\rho}\frac{\partial \hat{p}}{\partial y}$$

Now from these set of equation, we can actually eliminate. what to eliminate we can eliminate $\hat{u}$. So, we eliminate $\hat{u}$ from the first equation and $\hat{p}$ from the second and third equation. So, we get:

$$(U-c)\left(\frac{d^2\hat{v}}{dy^2} - k^2\hat{v}\right) - \hat{v}\frac{d^2U}{d^2y} = 0$$

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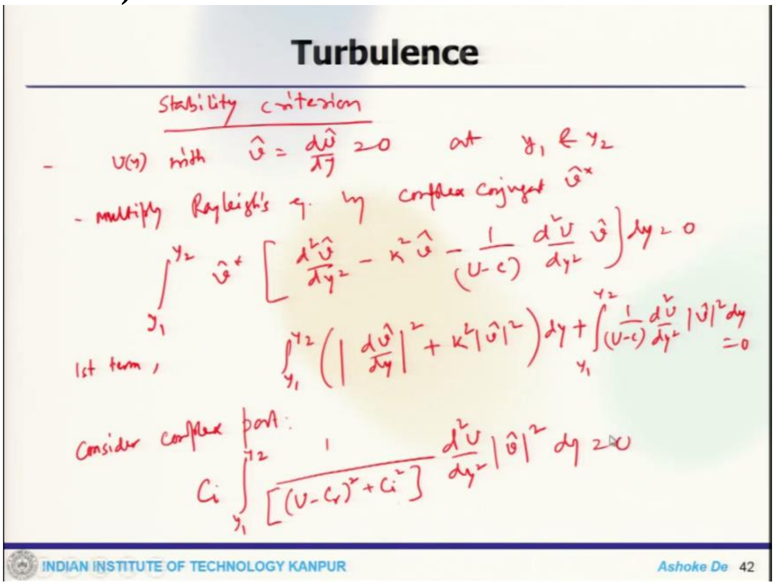

So, now we can put the stability criteria. Let us consider the base flow $U(y)$ with $\hat{v} = \frac{d\hat{v}}{dy} = 0$ at y_1 and y_2. Then multiply Rayleigh's equation by complex conjugate $\hat{v}*$ and integrate over y. So that will get us:

$$\int_{y_1}^{y_2} \hat{v}^* \left[\frac{d^2\hat{v}}{dy^2} - k^2\hat{v} - \frac{1}{(U-c)} \frac{d^2U}{d^2y} \hat{v} \right] dy = 0$$

Now the first term if we integrate it with partial integration, we get:

$$\int_{y_1}^{y_2} \left(\left| \frac{d\hat{v}}{dy} \right|^2 + k^2 \left| \hat{v} \right|^2 \right) dy + \int_{y_1}^{y_2} \frac{1}{(U-c)} \frac{d^2U}{d^2y} \left| \hat{v} \right|^2 dy = 0$$

Now you can consider the complex part:

$$c_i \int_{y_1}^{y_2} \frac{1}{(U - c_r{}^2) + c_i{}^2} \frac{d^2U}{d^2y} \left| \hat{v} \right|^2 dy = 0$$

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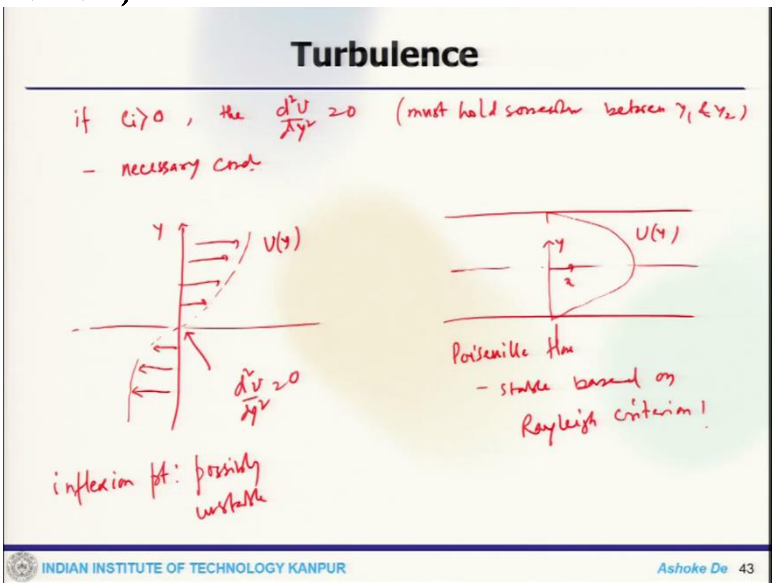
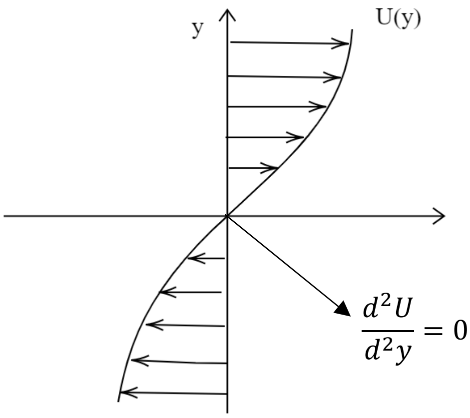
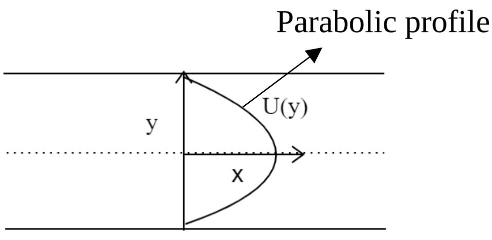


Now we can have, if $c_i > 0$, then $\frac{d^2U}{d^2y} = 0$ (This must hold somewhere between y_1 and y_2. So, this is a necessary condition, but not sufficient condition in general. Let us say we can have a profile shown as below.



inflexion point which is possibly unstable



Poiseuille flow

stable based on Rayleigh criterion

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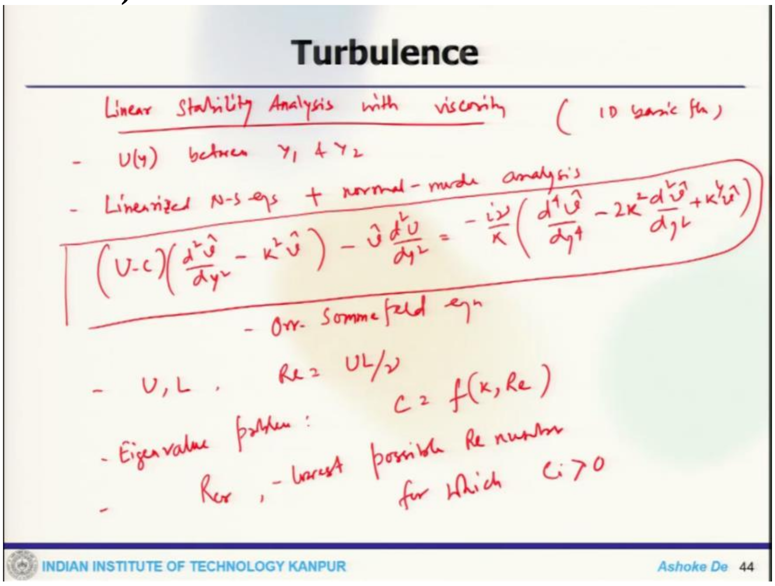

Linear stability analysis with viscosity

So, now, we can do the linear stability analysis with viscosity. So, again we can have 1D basic flow and we have flow $U(y)$ between y_1 and y_2. So, we can write linearized N-S equation + normal mode analysis which will get us:

$$(U-c)\left(\frac{d^2\hat{v}}{dy^2}-k^2\hat{v}\right)-\hat{v}\frac{d^2U}{d^2y}=-\frac{i\nu}{k}\left(\frac{d^4\hat{v}}{\mathrm{dy}^4}-2k^2\frac{d^2\hat{v}}{\mathrm{dy}^2}+k^4\hat{v}\right)$$

This is a well-known equation Orr-Sommerfeld equation. Now in dimensional form, if we use the velocity U and length scale L. We can use $Re = \frac{UL}{\nu}$. So, the eigen value problem essentially becomes $c = f(k, Re)$. Critical Reynolds number Re_C is the lowest possible Re number for which c_i would be positive.

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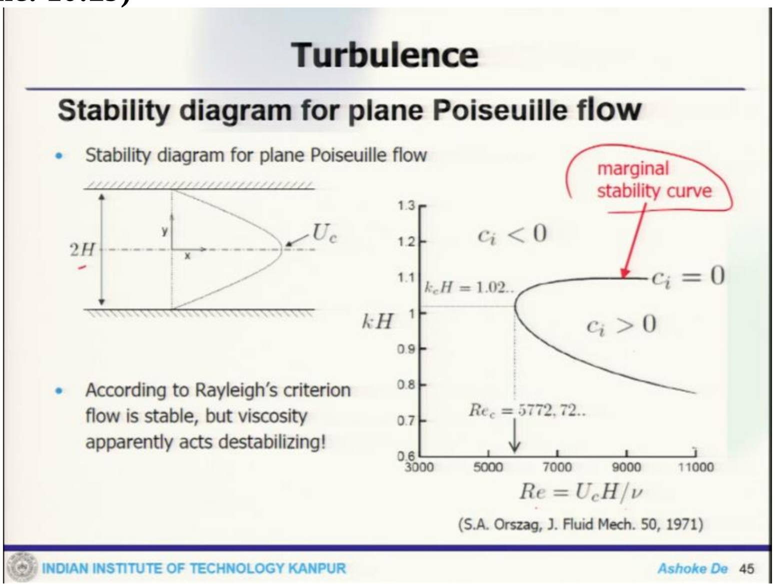

We can look at plane Poiseuille flow in the above slide. This is the stability diagram for the plane Poiseuille flow between two parallel plates and the distance between that is $2H$ and U_c is the centre line velocity. Now, if you look at the diagram where Reynolds number is plotted vs kH and the curve shown is for $c_i = 0$, that means a marginal stability curve. The zone in which $c_i > 0$ is the stable zone and for $c_i < 0$ is the unstable zone. So, we get critical Reynolds number $Re_C = 5772.22..$ for the plane Poiseuille flow.

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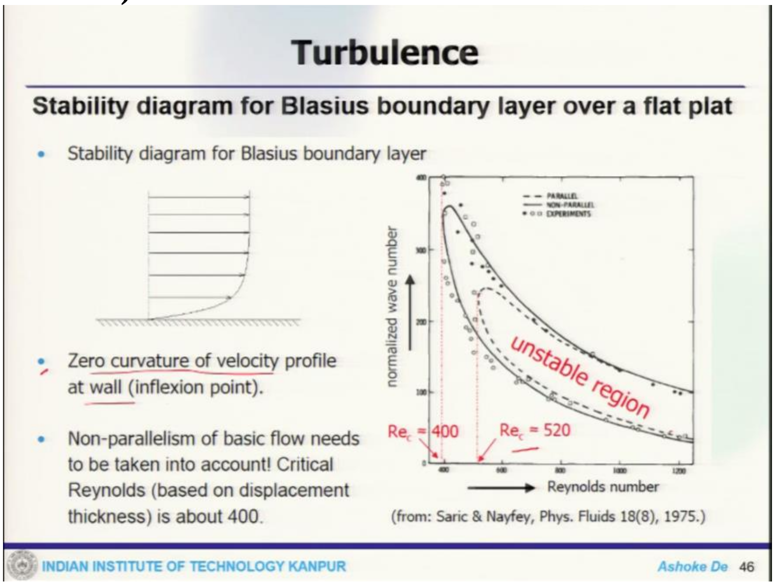

Now, similarly, stability diagram for boundary layer over a flat plat is shown in above slide. The Blasius boundary layer is shown on the flat plate. So, one important point here to be noted here is that, zero curvature of velocity profile at the wall that means at the inflexion point. So, non-parallelism of basic flow needs to be taken into account. Critical Reynolds number is taken, $Re_C = 520$ for parallel and $Re_C = 400$ for non-parallel. The zone enclosed by the parallel curve with $Re_C > 520$ is the unstable zone.

If you look at the Reynolds number, you can take different flow regions and whether it is a simple flow between two parallel plates or it is a boundary layer flow, you can demarcate the zone through the linear stability analysis as stable zone and unstable zone. And the Reynolds number which is going to be at the point that will be the critical Reynolds number which tells you what could be the situation for that zone.

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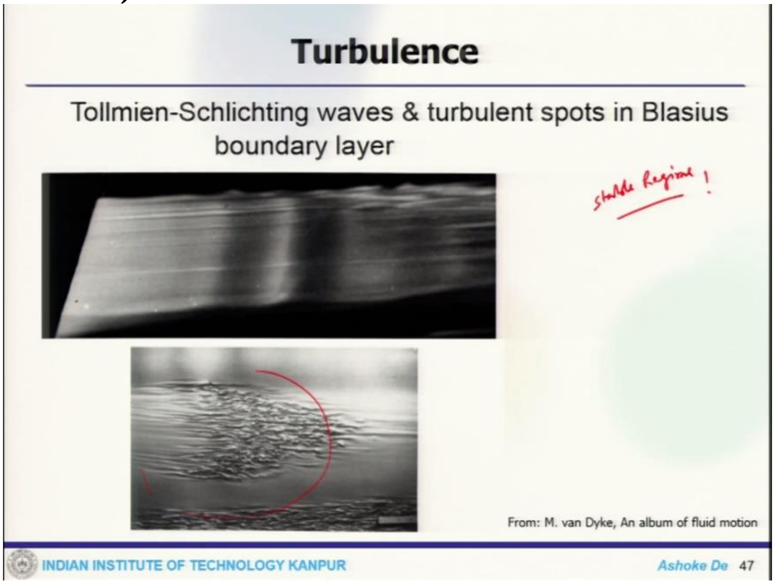

Now this is another very well-known situation or picture which is so, this is Tollmien-Schlichting waves and turbulent spots in boundary layer. Over a flat plate if you look at the boundary layer, these are the wave that you see. The flow goes through transition to become turbulent. In the second image you can see the stable regime and how far it is stable and then when it is becoming unstable. These are some of the basic features that we can look at.

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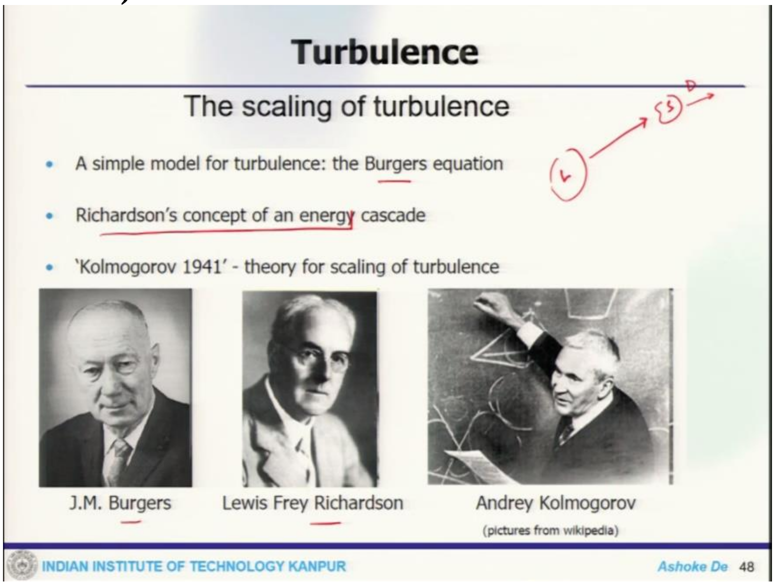

The Scaling of turbulence

Now, the important aspect of the turbulence is the scaling, that means, we have to somehow quantify the turbulence. Scaling is very important in turbulence and these are the famous scientists J.M. Burgers, Lewis Frey Richardson and Andrey Kolmogorov who have proposed different theories. One of the important theories of Richardson is the energy cascade, which talks about that the energy transform from large eddies to small eddies and at the end dissipated as heat.

The theory proposed in 1941 by this famous scientist Kolmogorov, talks about the scaling laws. These quantification and scaling models started with the work of Burger, that is why the equation is called is the Burger's equation.

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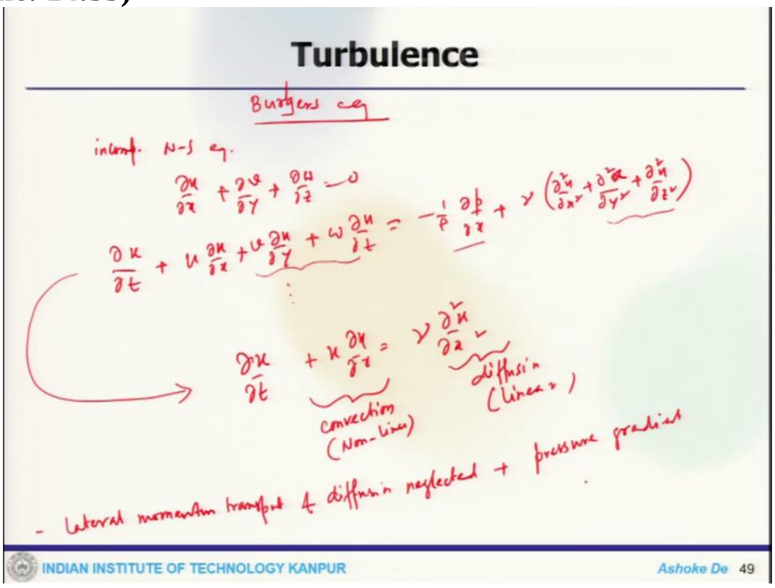

Burger's equation

We will look at the Burger's equation. In Cartesian coordinate system the incompressible Navier Stokes equation:

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

X-momentum equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

Similarly, for other directional momentum equations.

For simple model to study the characteristics of turbulent flow, we take the equation:

$$\underbrace{\frac{\partial u}{\partial t}}_{\text{Unsteady (Linear)}} + \underbrace{u\frac{\partial u}{\partial x}}_{\text{Convection (Non-linear)}} = \underbrace{\nu\left(\frac{\partial^2 u}{\partial x^2}\right)}_{\text{Diffusion (Linear)}}$$

Here the lateral momentum transport and lateral diffusion both are neglected plus if you can see, pressure gradient term is also neglected. So, once we neglect these terms, we get this equation which is known as the Burger’s equation.

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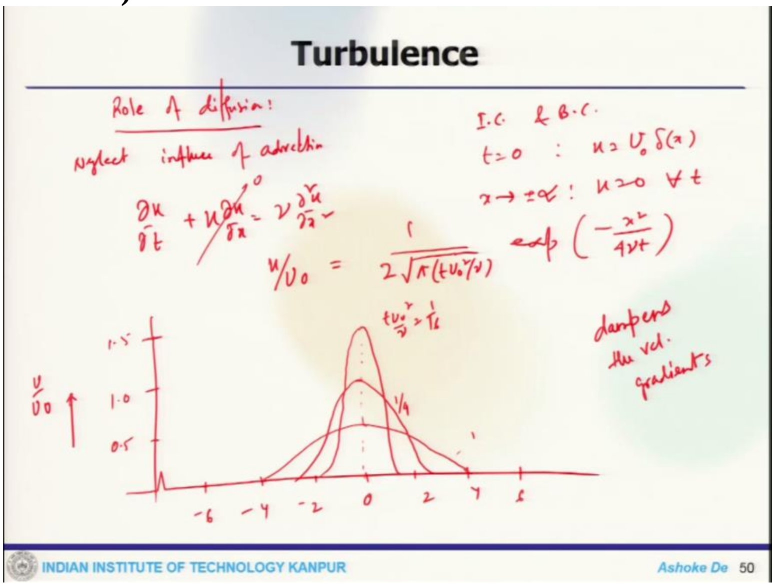

Role of Diffusion

Now we see the role of diffusion which essentially dampens the velocity gradient. We neglect the influence of advection or convection. The equation becomes:

$$\frac{\partial u}{\partial t} = \nu \left(\frac{\partial^2 u}{\partial x^2} \right)$$

The initial conditions and boundary conditions are:

$$t = 0 \qquad : u = U_0 \delta(x)$$

$$x \to \pm\infty \quad : u = 0 \quad \forall t$$

The solution will look like:

$$\frac{u}{U_0} = \frac{1}{2 \cdot \sqrt{\pi \left(t {U_0}^2 / \nu \right)}} exp \left(\frac{-x^2}{4 \nu t} \right)$$

The solution is visualised the above slide with curves for different values of $t{U_0}^2/\nu$. So that is how you can see the impact of the diffusion, essentially the diffusion dampens the velocity gradient. So that is what it does.

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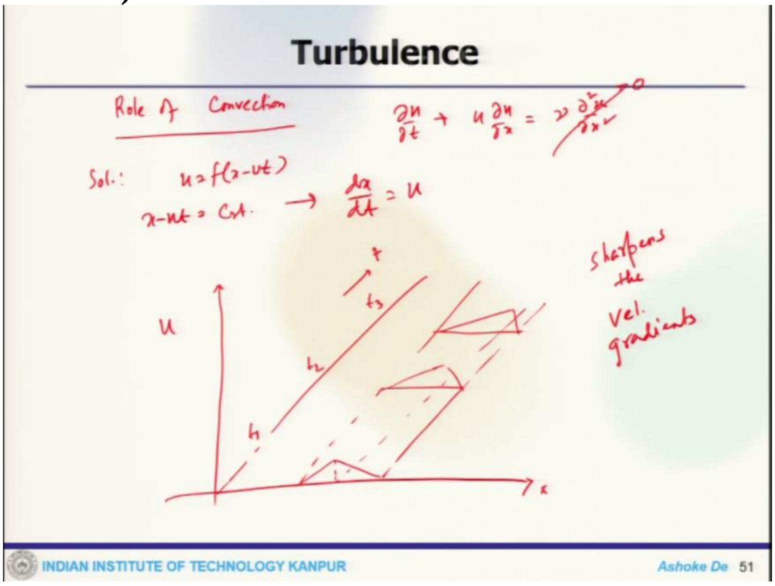

Role of Convection

Similarly, for the role of convection. We neglect the influence of diffusion term. The equation becomes:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

Solution is of the form:

$$u = f(x - ut)$$

So, the propagation along the characteristics $x - ut = constant \rightarrow \frac{dx}{dt} = u$.

The solution is plotted in $u, x\ and\ t$ dimensions for different values of t as shown in the above slide.

So, essentially the role of convection term is that it sharpens the velocity gradients.

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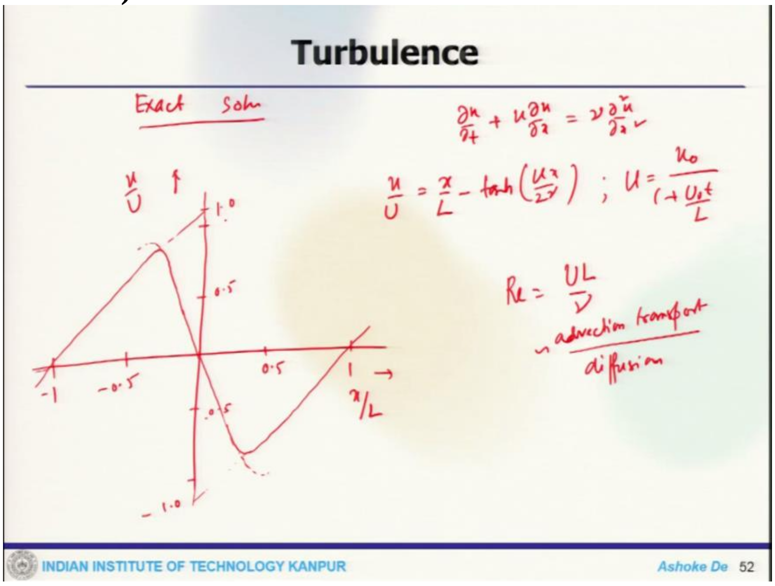

Exact solution of the Burger's equation

Now, we can find out the exact solution of the Burger's equation.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\left(\frac{\partial^2 u}{\partial x^2}\right)$$

The solution which satisfies the Burger's equation would be of the form:

$$\frac{u}{U} = \frac{x}{L} - tanh\left(\frac{Ux}{2\nu}\right) \qquad \text{Where } U = \frac{u_0}{1+(U_0 t/L)}$$

The solution is plotted with axes $\frac{x}{L}$ and $\frac{u}{U}$ as shown in the above slide. The velocity scale U and length scale L which will get the Reynolds number $Re = \frac{UL}{\nu} = \frac{Advective\ transport}{Diffusion}$.

This is an exact solution of this simple system; one can just take some points and recreate this plot in MATLAB.

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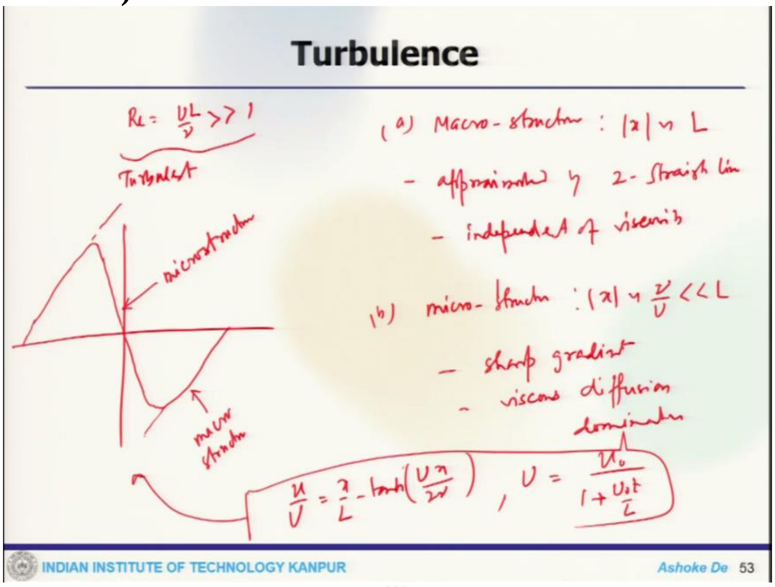

Now, when Reynolds number which is $Re = \frac{UL}{\nu} \gg 1$ (really high) that will lead to turbulent situation. Now the solutions, they could be of two types, one could be macro structure where this would $|x| \sim L$. So, it can be can be approximated by two straight lines and this is independent of viscosity. we can actually provide solution by two straight line where it is a macro structure as shown in the slide.

And then, we have micro structure where $|x| \sim \frac{v}{u} \ll L$. There would be sharp gradient and viscous diffusion dominates. The solution of micro structure is indicated in the plot shown in the above slide. So, the single solution which we have

$$\frac{u}{U} = \frac{x}{L} - tanh\left(\frac{Ux}{2\nu}\right)$$

where $U = \frac{u_0}{1+(U_0 t/L)}$

So, this solution is plotted here and there could be two types of structure which will influence that.

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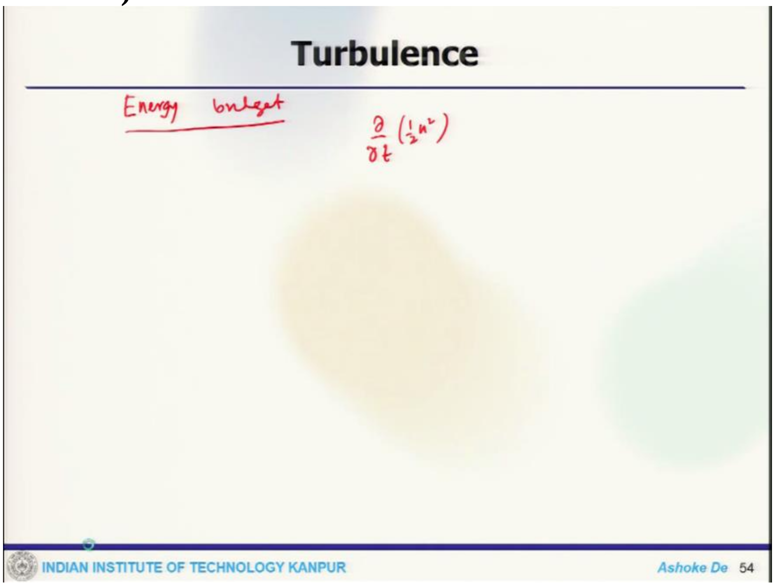

Now we can basically account for the energy budget. For the energy budget, we can multiply the Burger’s equation by velocity u and we can obtain the equation for the kinetic energy. Ok, we can stop it here and look at the energy budget in the next lecture.