Turbulent Combustion: Theory and Modelling
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Lecture-30
Turbulence (Contd...)

Welcome back, let’s continue the discussion on this linear stability analysis. So, what we have done, we have already looked at different type of analysis and we are going to look at two different kind of normal mode analysis here. One is the K-H instability, another is linearized equations for distribution where we come across the Sommerfeld equation. So, for K-H instability, we have taken situation and derived the linearized perturbation equation.

So, this is the system that we have taken and we defined all the characteristics of this particular system and other properties using that,

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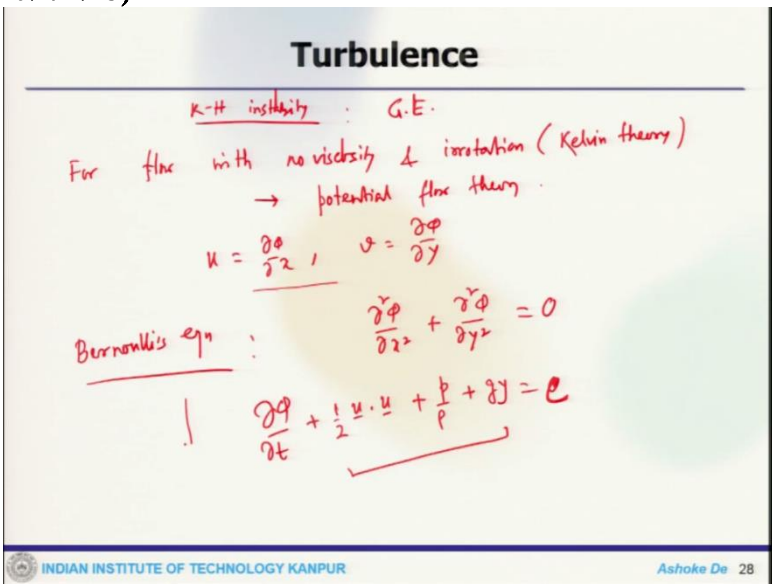

We bring back our potential flow theory and got the model equations.

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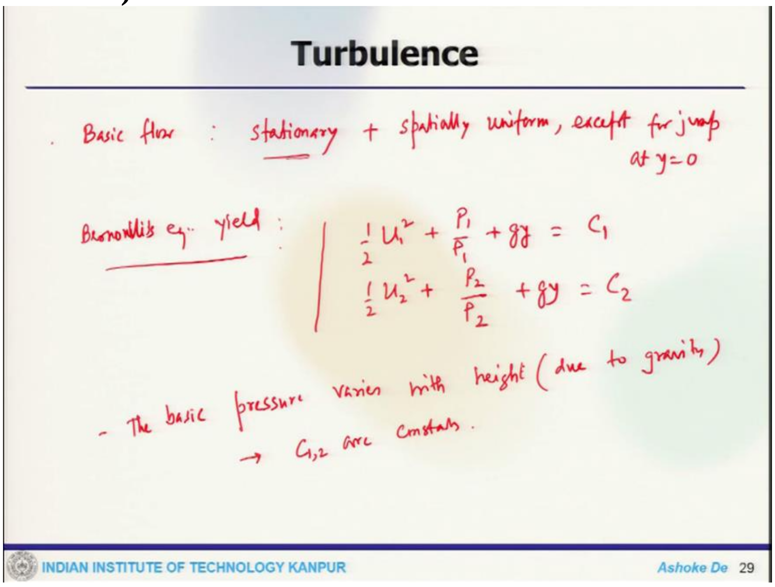

Now, for base flow which is stationary and spacially uniform except for that location where y equals to zero there could be a jump, we get this equation and basic pressure varies with height.

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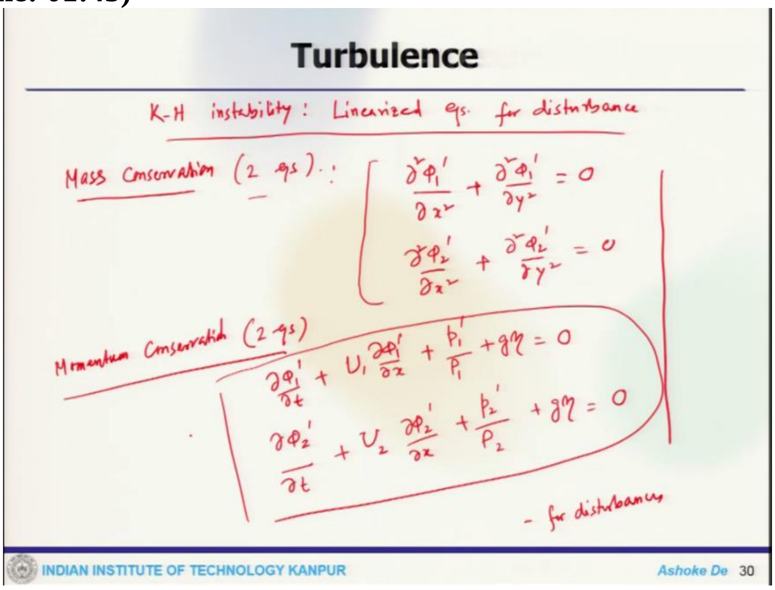

There are four equations, 2 equations from the mass conservation and the 2 equations from the momentum conservation.

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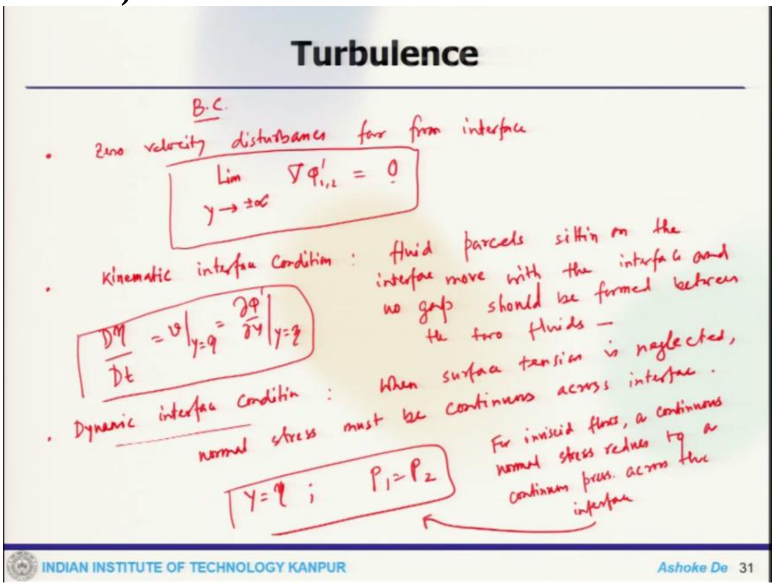

Now, to solve this problem, we need boundary conditions and there are three different set of boundary conditions which are required. One is the zero velocity disturbance at $y = \pm\infty$, one is the kinematic condition and another one is the dynamic interface condition which says that when the surface tension is neglected, the normal stress must be continuous across the interface. Now for inviscid flows a continuous normal stress reduces to a continuous pressure across the interface. So, that essentially gives me this condition.

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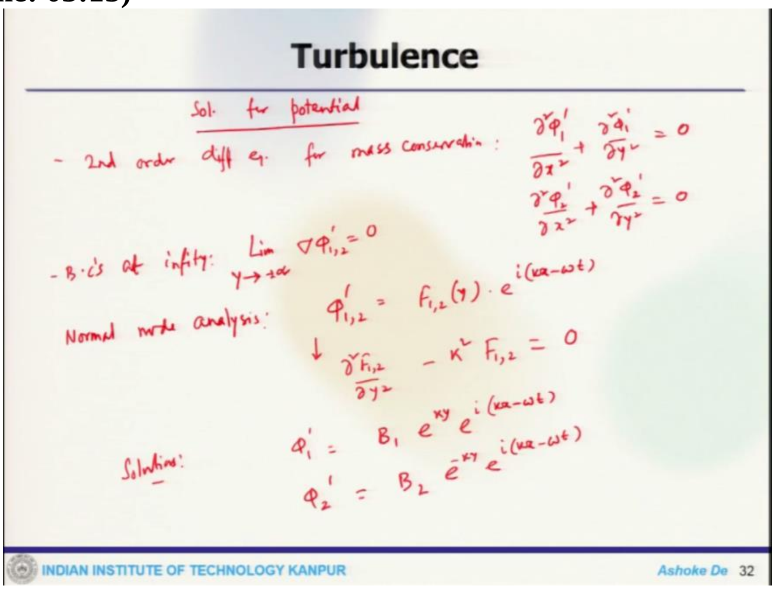

Now, how do we get the solution for potential flow. Now, we have second order differential equation for mass conservation. So, which are:

$$\frac{\partial^2 \varphi_1'}{\partial x^2} + \frac{\partial^2 \varphi_1'}{\partial y^2} = 0$$

$$\frac{\partial^2 \varphi_2'}{\partial x^2} + \frac{\partial^2 \varphi_2'}{\partial y^2} = 0$$

Now we have boundary conditions at infinity, which is:

$$\lim_{y \to \pm\infty} \nabla \varphi_{1,2}' = 0$$

Now we have normal mode analysis that means:

$$\varphi_{1,2}' = F_{1,2}(y).\, e^{i(K\alpha - \omega t)}$$

So, that is how, now this equation this reduces to:

$$\frac{\partial^2 F_{1,2}}{\partial y^2} - K^2 F_{1,2}$$

So, once these we put it back there this is what now the solution what you can have. That is:

$$\varphi_1' = B_1 e^{Ky} e^{i(K\alpha - \omega t)}$$

And

$$\varphi_2' = B_2 e^{-Ky} e^{i(K\alpha - \omega t)}$$

So, this is what we get for the solution of the potential. Now, we apply the second boundary condition.

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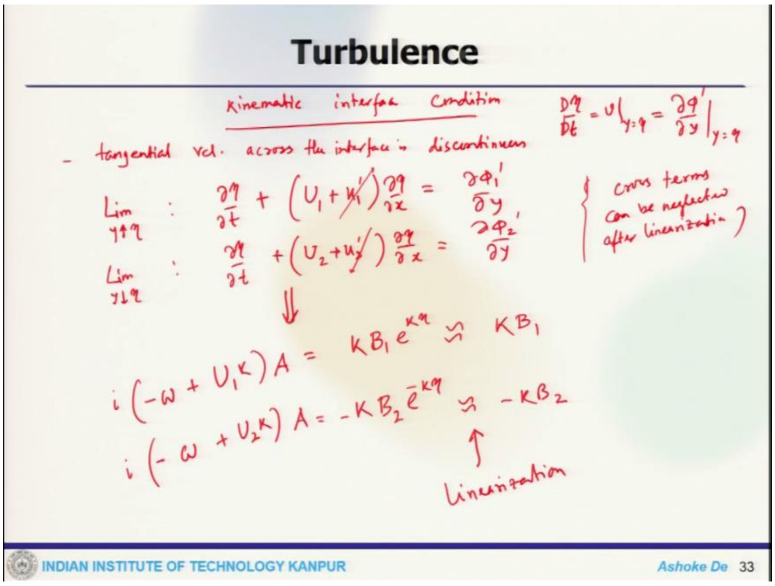

So, that is kinematic interface condition. So, kinematic interface conditions gives us:

$$\frac{D\eta}{Dt} = |v|_{y=\eta} = \left|\frac{\partial \varphi'}{\partial y}\right|_{y=\eta}$$

So, this is my kinematic interface condition. Now, if our tangential velocity essentially discontinuous. So, tangential velocity across the interface is discontinuous. So, we get 2 set of equations:

$$\lim_{y\uparrow\eta} = \frac{\partial \eta}{\partial t} + (U_1 + u_1')\frac{\partial \eta}{\partial x} = \frac{\partial \varphi_1'}{\partial y}$$

And second:

$$\lim_{y\downarrow\eta} = \frac{\partial \eta}{\partial t} + (U_2 + u_2')\frac{\partial \eta}{\partial x} = \frac{\partial \varphi_2'}{\partial y}$$

Now, cross terms can be neglected like this term or this term. So, these are the cross terms can be neglected. After linearization, so which get us:

$$i(-\omega + U_1 K)A = KB_1 e^{K\eta} \approx KB_1$$

Similarly,

$$i(-\omega + U_2 K)A = -KB_2 e^{-K\eta} \approx -KB_2$$

So, essentially this is my linearization. So, the linearization get me these 2 equation.

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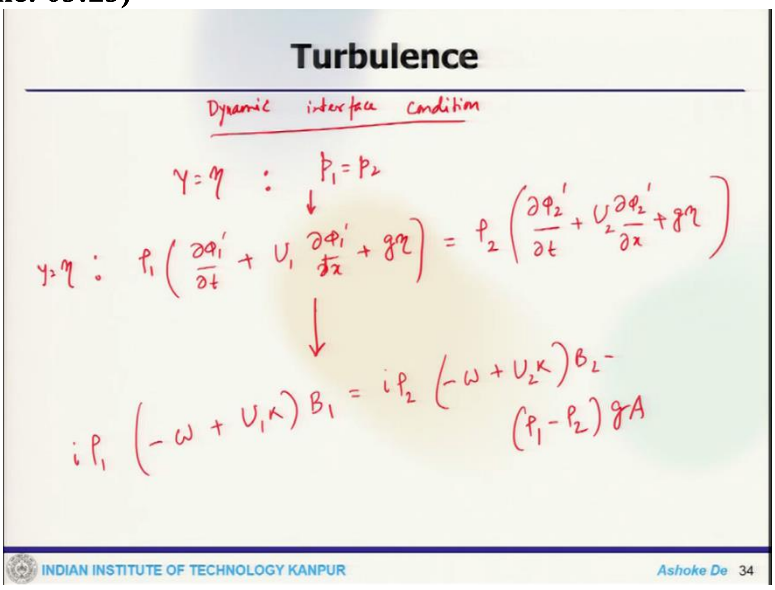

Now we apply the third boundary condition that is the dynamic interface condition, which says that at $y = \eta$ and $p_1 = p_2$ so means at $y = \eta$:

$$\rho_1\left(\frac{\partial \varphi_1'}{\partial t} + U_1\frac{\partial \varphi_1'}{\partial x} + g\eta\right) = \rho_2\left(\frac{\partial \varphi_2'}{\partial t} + U_2\frac{\partial \varphi_2'}{\partial x} + g\eta\right)$$

So, this is what we get. Now, from there we get:

$$i\rho_1(-\omega + U_1 K)B_1 = i\rho_2(-\omega + U_2 K)B_2 - (\rho_1 - \rho_2)gA$$

So my dynamic interface condition provides this. So once we apply all these three boundary conditions that means.

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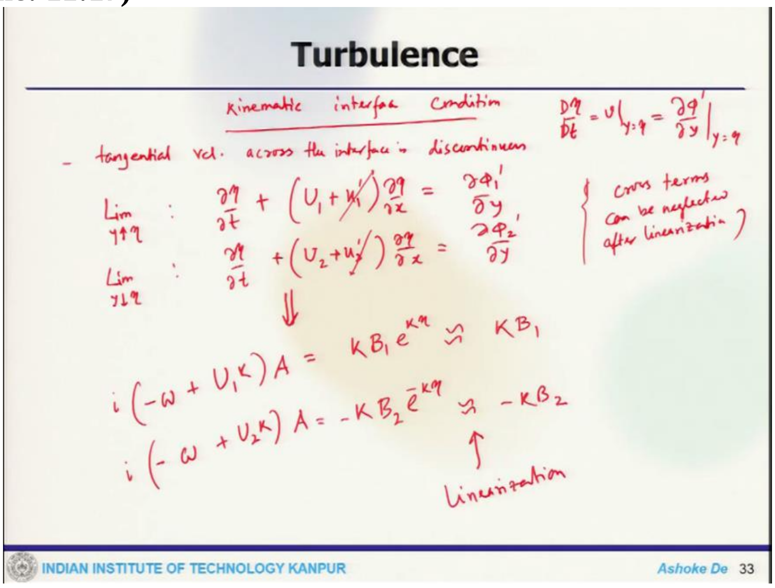

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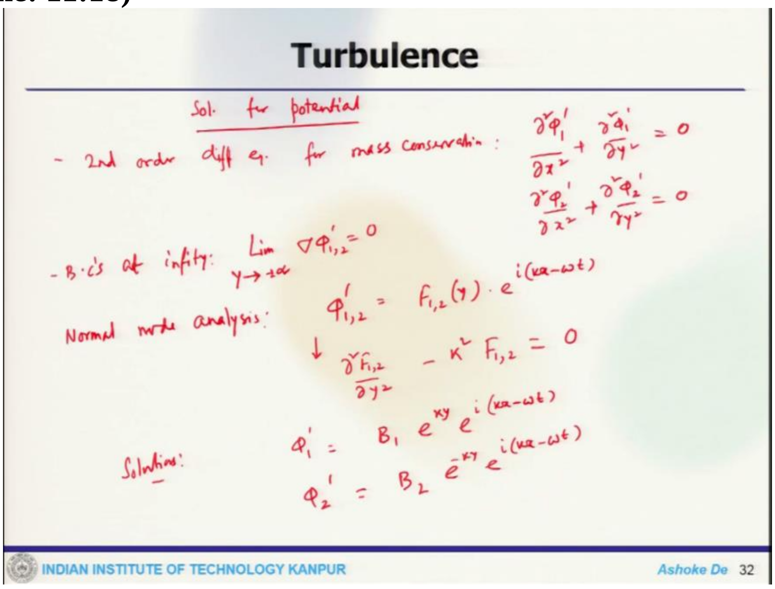

First the boundary condition at the infinity, where $y \to \pm\infty$. So, you get these two solutions. Then we put the kinematic interface condition,

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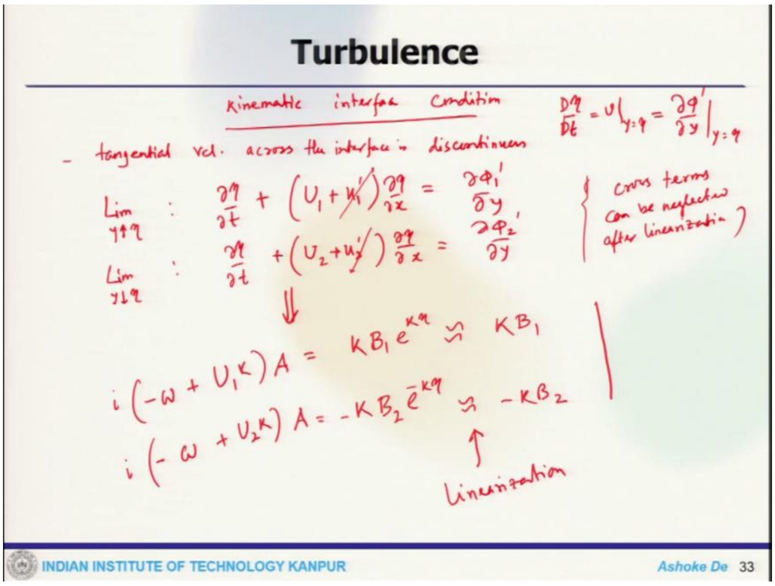

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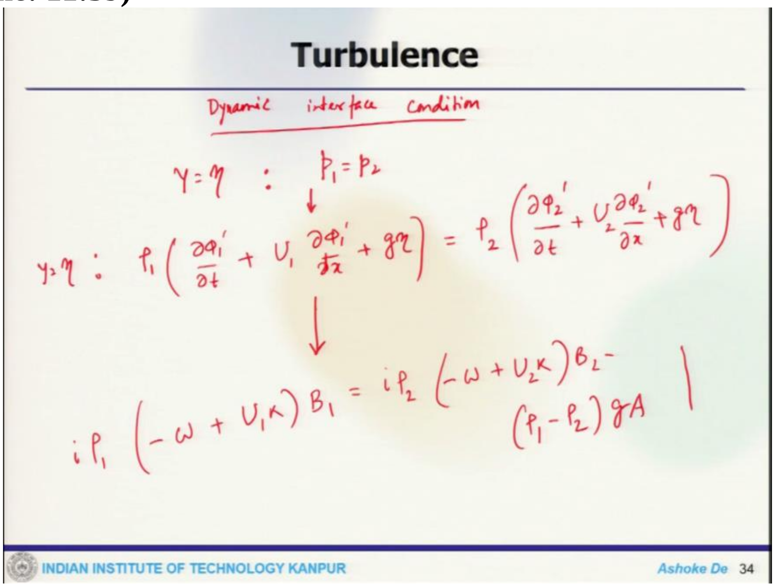

So, that after linearization, we get this and then finally, we get the dynamic interface condition which after application it may this. So, now, if we put the whole system of equation after applying the boundary condition in the linear system or in the matrix system, how the matrix would look like?

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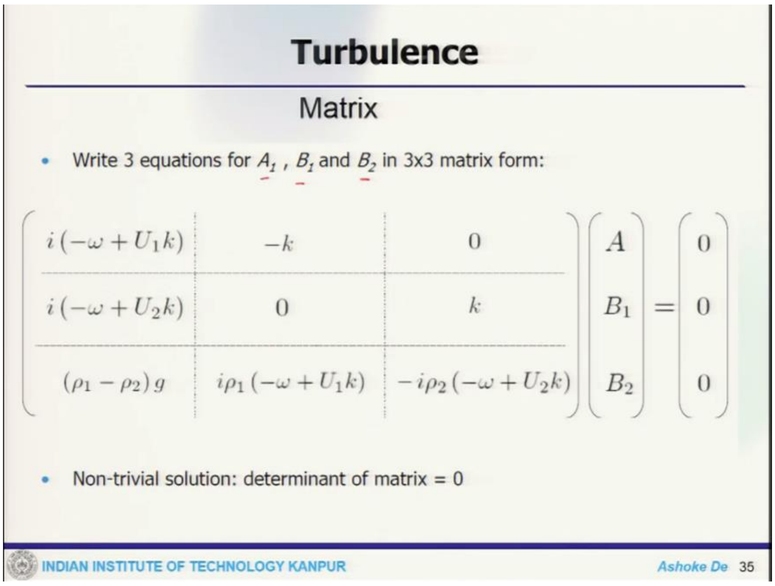

So, we have 3 × 3 system and the equations are for A_1, B_1 and B_2. And if we look back, this is for A beyond B_2 coupled.

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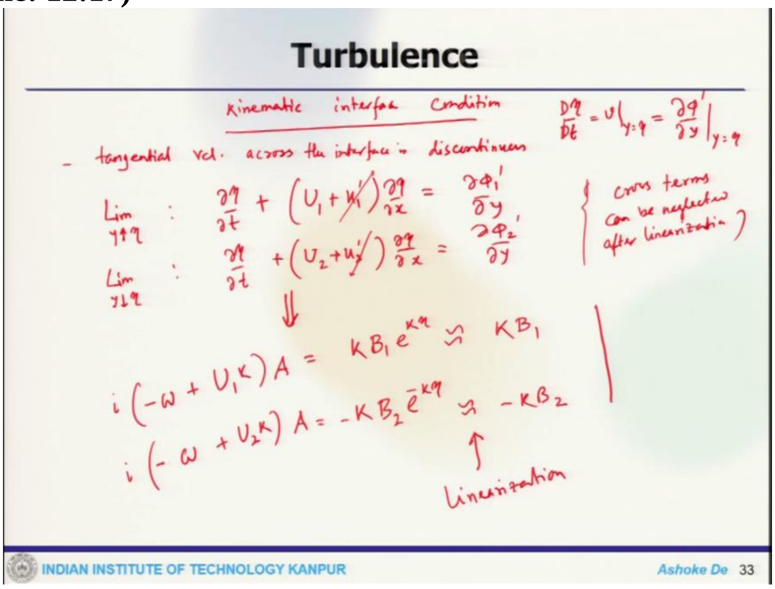

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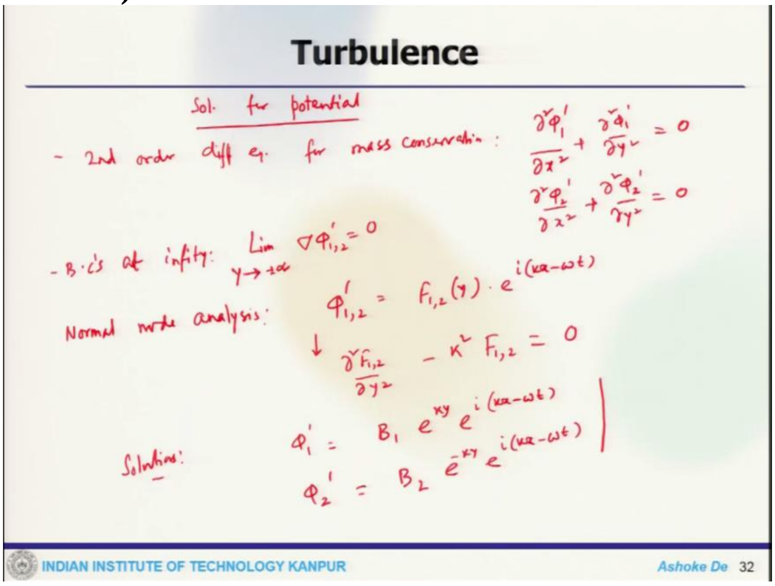

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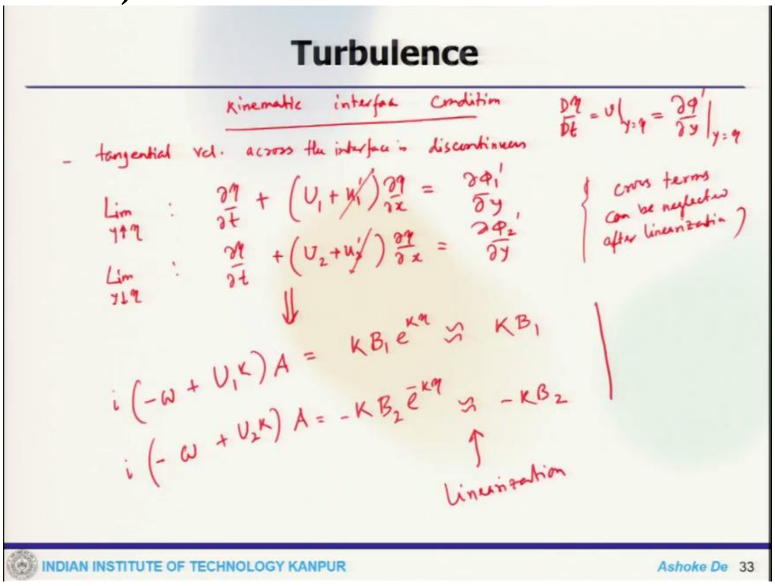

So, once we do that, this is for A, B_1 and B_2 these are essentially you can think about this is A x b. So, we are solving for A x equals to zero.

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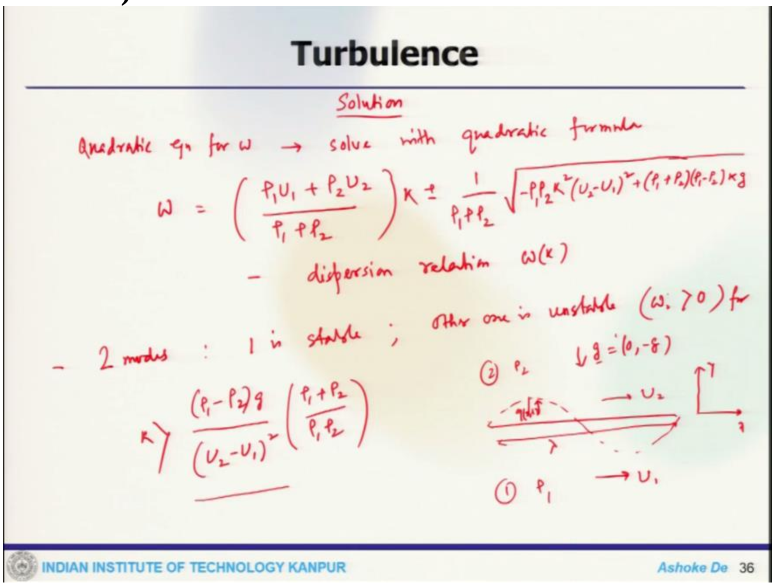

So, you put things together you get that particular situation and the non-trivial solution is that because of determinant of the matrix is zero. Now, if you look at the solution, so, there is a quadratic equation for ω which we solve with quadratic formula. So, that will get:

$$\omega = \left(\frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2}\right) K \pm \frac{1}{\rho_1 + \rho_2}\sqrt{-\rho_1\rho_2 K^2 (U_2 - U_1)^2 + (\rho_1 + \rho_2)(\rho_1 - \rho_2) K g}$$

So, this is one can think about it send dispersion relation of omega k. Now there would be a quadratic solution. So there could be two modes. Obviously, one is stable mode and the other one is unstable which is greater than zero. Now, for that is unstable case. So for unstable:

$$K > \frac{(\rho_1 - \rho_2) g}{(U_2 - U_1)^2}\left(\frac{\rho_1 + \rho_2}{\rho_1 \rho_2}\right)$$

So with, that’s λ. That is my η(x,t). This is state two, this is state one, this is my axis and this is my gravity. So, if we neglect the surface tension, then you can see what happens that one can work out also. So, the solution which will tell you one stable mode, one unstable mode. Now, we can take some specific and very specific case.

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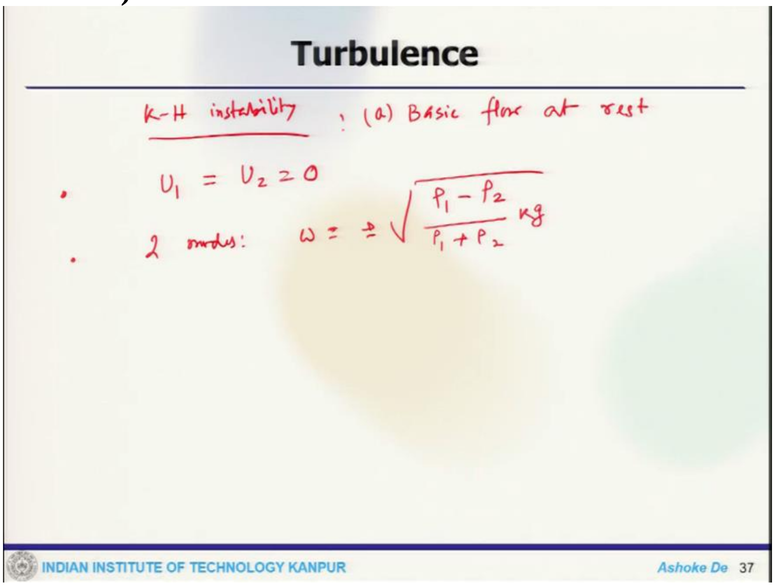

And see what happens now, first one look at basic flow at rest. So, which means my U_1 U_2 these are 0. So, the 2 modes that will get that would be:

$$\omega = \pm\sqrt{\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}Kg}$$

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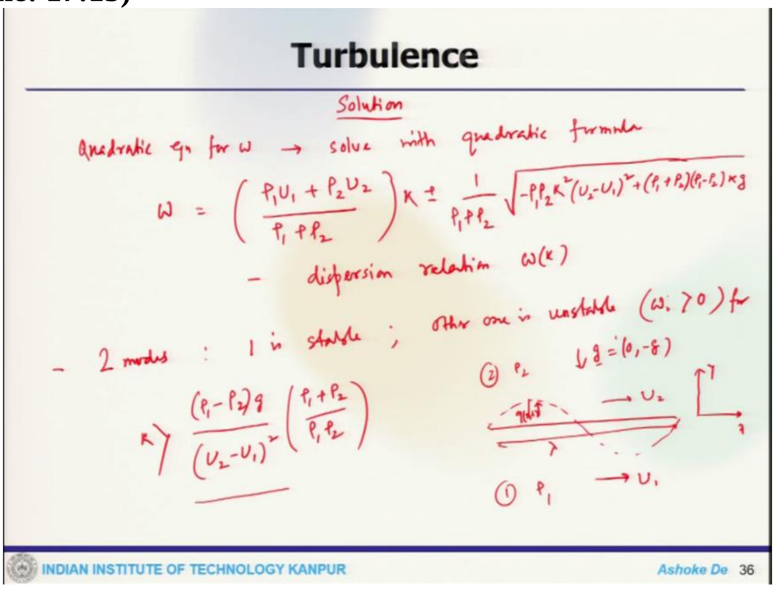

So, if you look at this picture here, these are the basic flows and they are at rest, then we get these two modes. Now, if ρ_1 get up then ρ_2 that means, this is ρ_1 and this is ρ_2.

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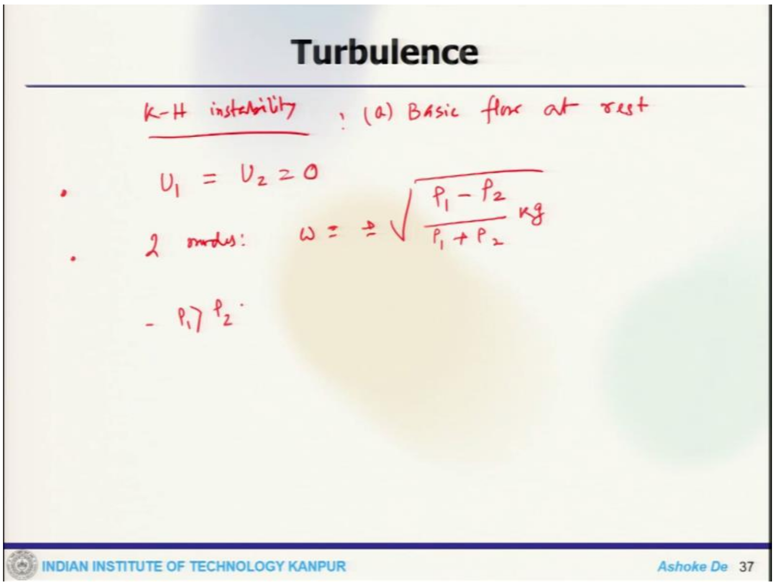

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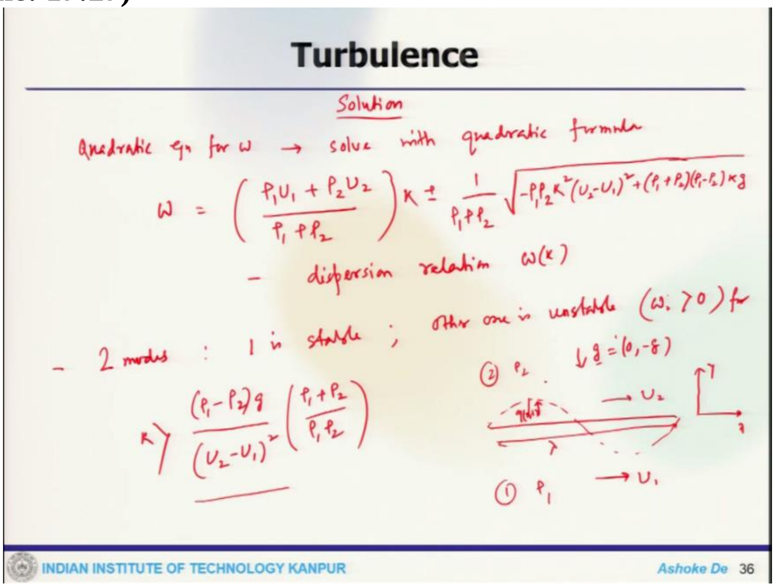

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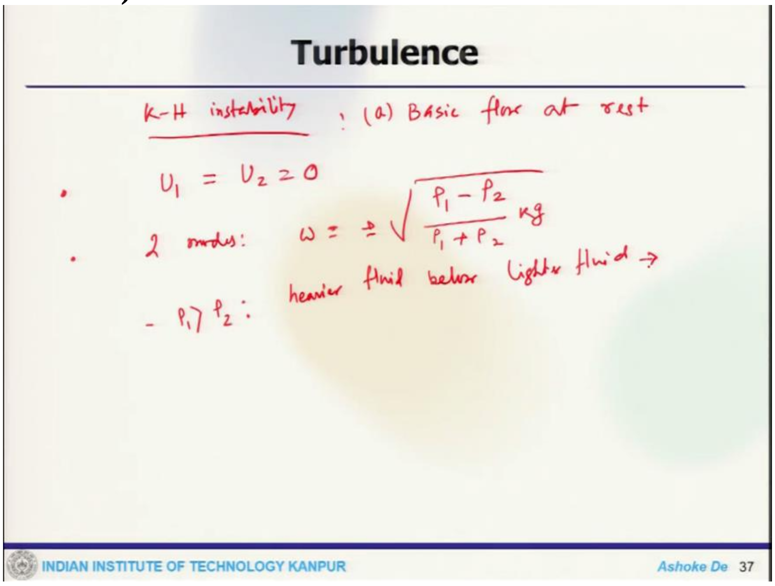

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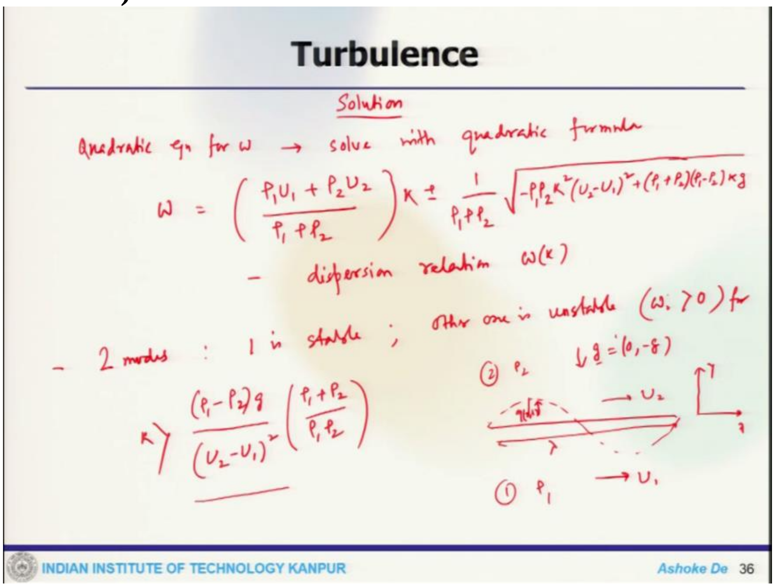

So, which tells that heavier fluid below lighter fluid, which means, the lighter fluid is top of that plate and heavier fluid is below which is essential a normal flow characteristics and that gives you a stable system, which nothing but my gravity wave.

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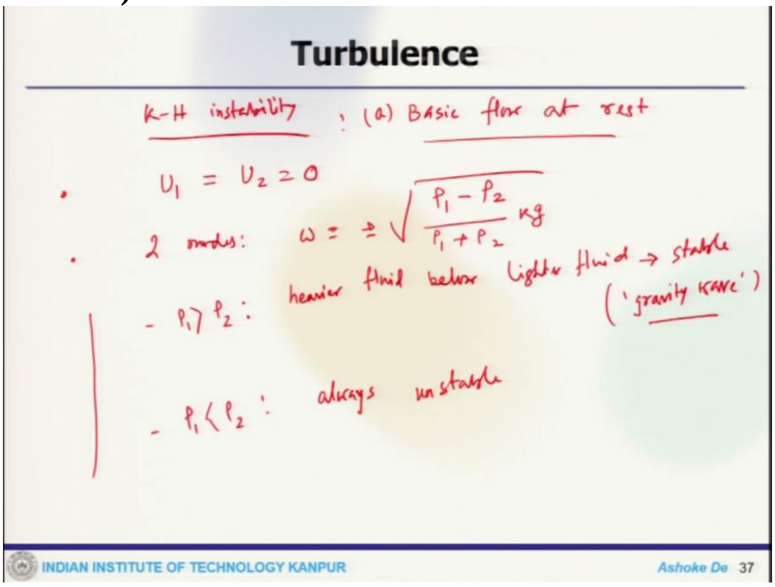

Other situation when ρ_1 is less than ρ_2 that means lighter fluid is in the bottom. So, this shows always unstable behaviour. So, in this particular case where you see on the basic flow is that raised the stability and the instability regime can be identified now.

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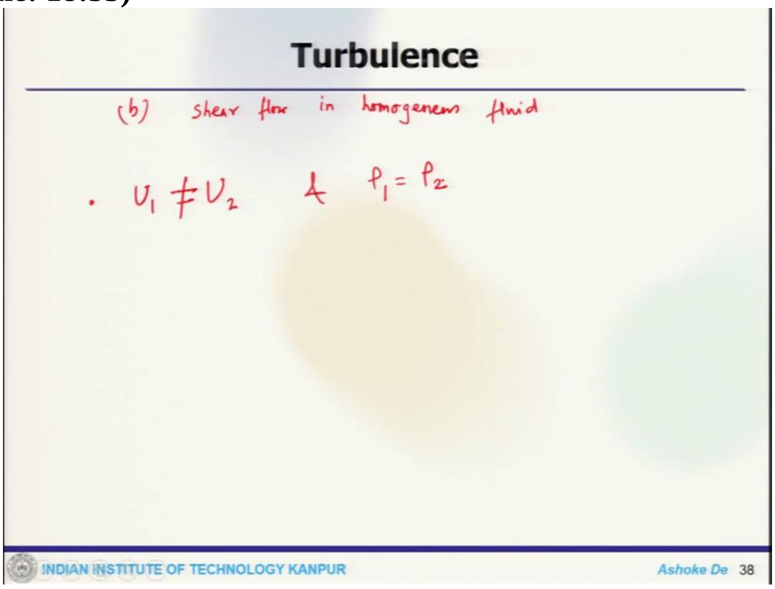

Now the second case we can consider, shear flow in homogenous fluid, so which will tell me that, U_1 is not equal to U_2 and ρ_1 equal to ρ_2. If you look at this,

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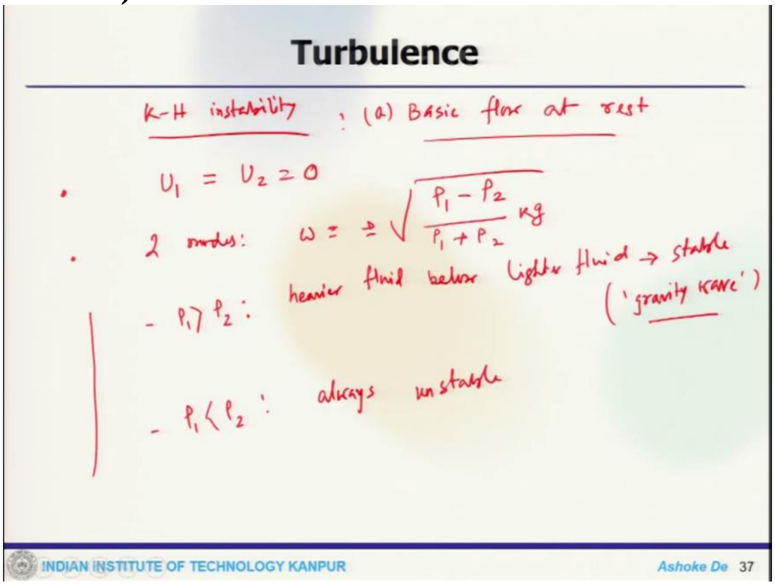

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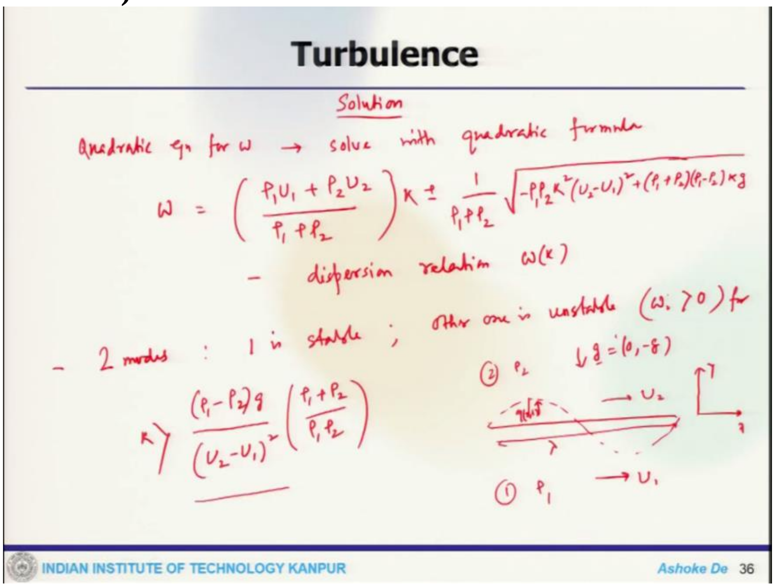

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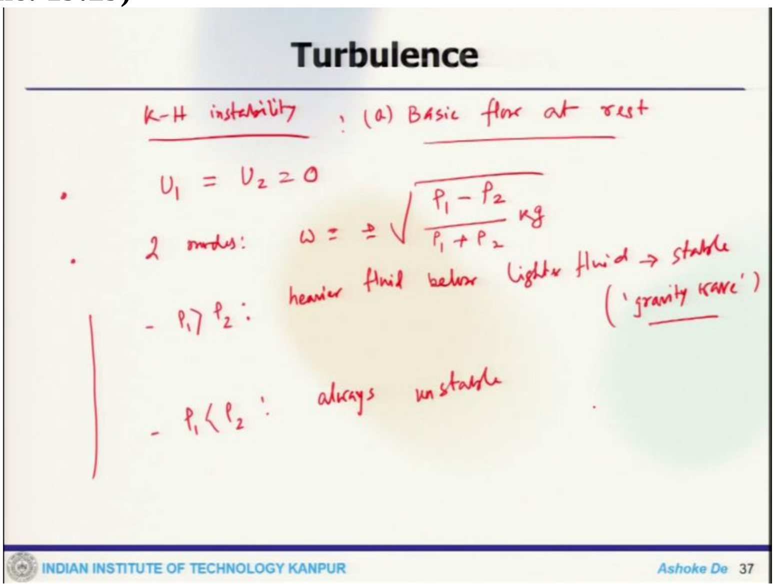

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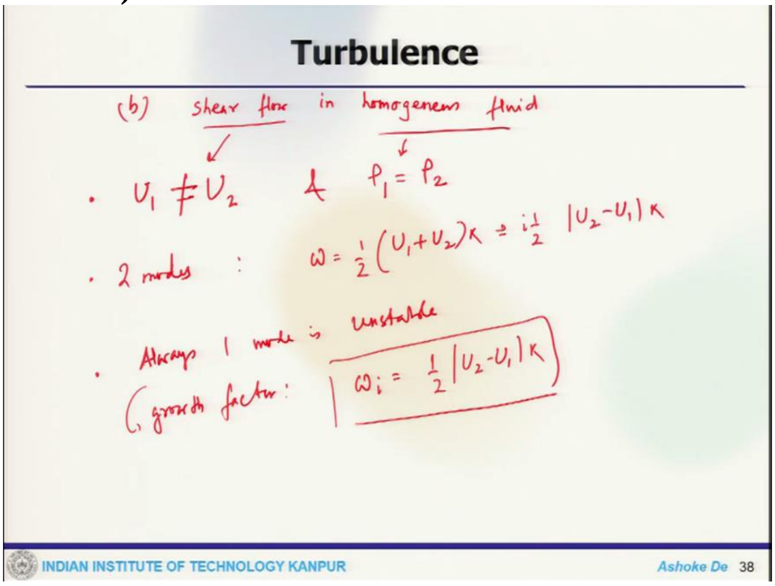

U_1 is not equal to U_2 and ρ_1 and ρ_2 are same, because it is an homogeneous fluid. So that gives me this condition which is shear flow that gives me this condition. So my modes of that solution, the two modes should be:

$$\omega_i = \frac{1}{2}|U_2 - U_1|K \pm i\frac{1}{2}|U_2 - U_1|K$$

So, again this case is also one mode. So, always one mode is unstable. And the growth factor would be growth factor which is:

$$\omega_i = \frac{1}{2}|U_2 - U_1|K$$

So, that is my growth factor. So, one mode, would be stable and another mode would be always unstable with growth factor to be like this.

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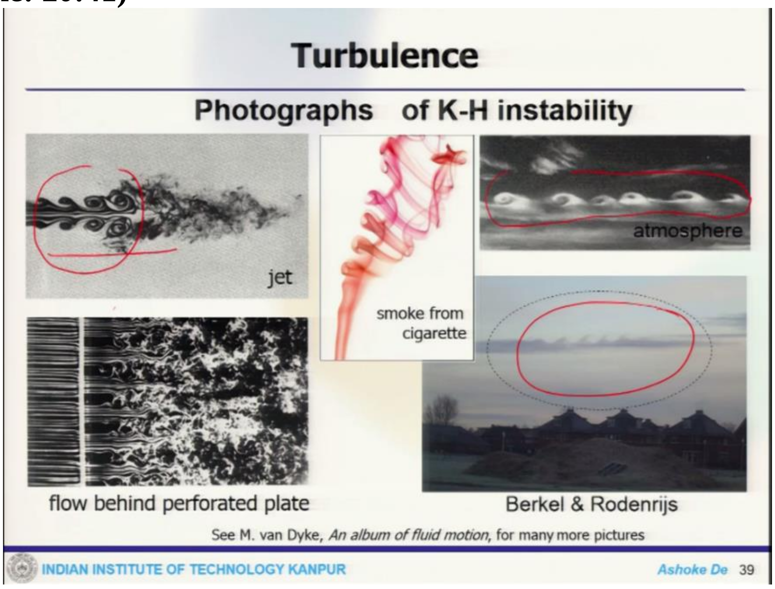

So, we can see some photographs of K-H instability. So, there are two types of instability you can see. So if there is a shear, which essentially you can see here in the atmosphere, this is primary due to the gradient of the density. Now in the jet also, you see this K-H instability, this could be due to velocity gradient, this could be due to density gradient then there is a flow behind perforated plate. So, you can see the K-H instabilities, there are nice picture of instabilities in cloud, smoke from cigarette all these photographs actually.

So, some sort of K-H instability. So, essentially if one would like to see these kind of instability we can see in regular life quite often, whether we term it as K-H instability or not that is a different issue, but we get to see this kind of because these are the pictures which you often see.

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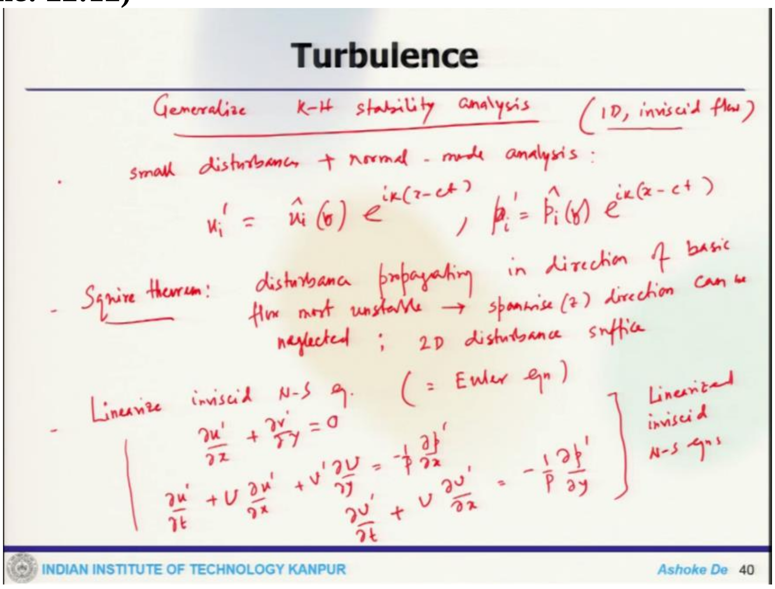

Now, we try to generalize K-H stability analysis. So, if we try to generalize, we again assume one dimensional inviscid flow. So, the first thing we have small disturbances plus the normal mode analysis. That will give us:

$$u_i' = \widehat{u}_i\,(y)e^{iK(x-ct)}$$

And second one would be:

$$p_i' = \widehat{p}_i\,(y)e^{iK(x-ct)}$$

So, there is a nice theorem, which is called squire theorem which says that disturbance propagating getting in direction of basic flow in direction of basic flow is most unstable.

That means, the span wise direction can be ignored. So the span wise direction can be neglected. So we will get 2D disturbances. So that will suffice the whole business. Now, if you write the linearized inviscid N–S equation, which is some sort of an Euler equation. We will get:

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0$$

Then,

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + v'\frac{\partial U}{\partial y} = -p\frac{\partial p'}{\partial x}$$

And the other component is:

$$\frac{\partial v'}{\partial t} + U\frac{\partial u'}{\partial x} = -\frac{1}{p}\frac{\partial p'}{\partial y}$$

So, these are the set of linearized equation. So, these are linearized inviscid N-S equation. So, first one from continuity other two from the momentum. So, generalize this K-H instability equation where we again assume it is in only inviscid flow. So, we have a small disturbance and we assume that the disturbance is only in the direction of the basic flow which will become more unstable.

And so that we can neglect the span of direction and we get this equation in the linearized form. So, now we will see how we can obtain solution for this particular form and this linearized system. We will stop here and continue in the next lecture.