Turbulent Combustion: Theory and Modelling
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Lecture-28
Turbulence (Contd…)

Welcome back, so, let us continue our discussion of the turbulent flow. And we are slowly getting into the details of characterization of the turbulent flow and first thing we have looked at the some initial features of the turbulent flow, how they would look and then discuss the properties of the flow.

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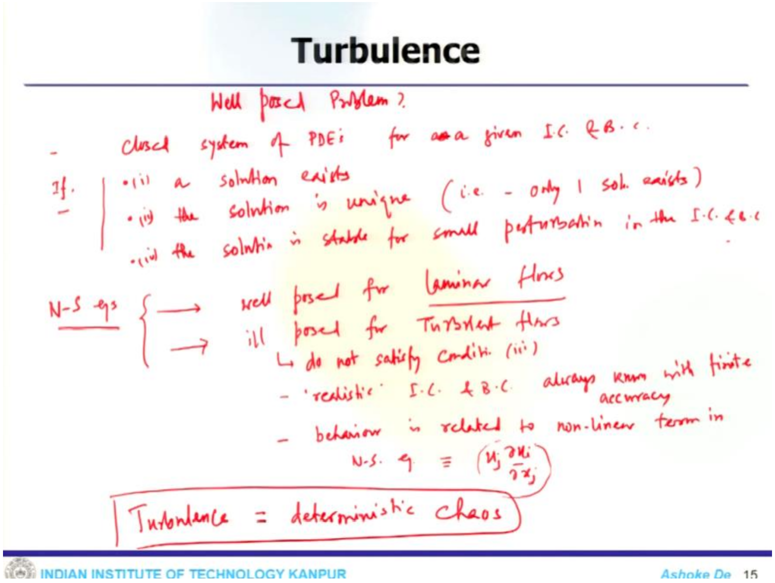

Now, where we start here is the situation where the system is a well posed problem of the ill posed problem and we said a system of PDE's or closed system of PDE's for a given initial and boundary condition. If a solution exists the solution is unique that means, the only one solution exists and the solution is stable for small perturbation in the initial. Now, the important question which arises here is what about the Navier Stokes equations? Are they ill posed or well posed? So, this is well posed for laminar flows.

So, that is a very important point. Now, that means, we mean to say this is ill posed for turbulent flows. Now, when we say it is an ill posed that means, we can say what it does not do. So, the turbulent flows do not satisfy this last condition of the third condition that means, if you say this is 1, 2, 3. So, this guy do not satisfy condition 3 which is that solution is not stable for a small perturbation in the initial and the boundary condition. So, which means the realistic initial condition and boundary condition always known with finite accuracy and also the behavior is related to non-linear term in Navier Stokes equation which is $u_j \frac{\partial u_i}{\partial x_j}$. So, this is the nonlinear term and this behavior is sort of related to this non-linear term in the Navier Stokes so, that means, which is one thing which is absolutely clear here is that for laminar flow.

It becomes Navier Stokes equation becomes an ill posed problem that means, it satisfied all these 3 conditions solution exist, solution is unique and solution is also stable. But for turbulent flow it does not satisfy the third condition that is why it becomes ill posed and the behavior is somehow related to the nonlinear term. So, that means, we can conclude saying that the turbulence which could be in deterministic chaos. So, this is what one can say about turbulent flows.

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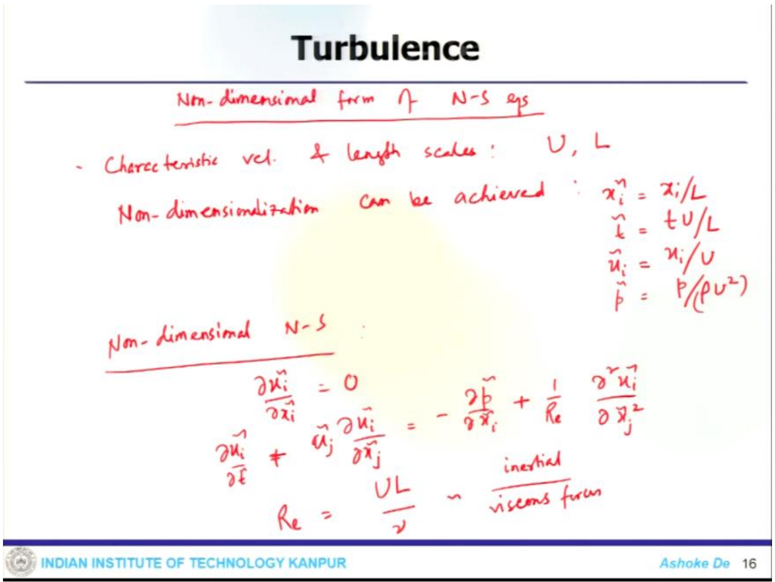

Now, we will look at slightly in details. So, first thing that we will do the non-dimensional equation so, non-dimensional form of in this equation so, first thing that will do will non-dimensional analyze the system so, for that we need some characteristic velocity and length skills. So, you say U and L and the non-dimensional analysis can be done can be achieved through this where we can define: $\tilde{x}_i$ is ${x_i}/{L}$, $\tilde{t}$ is ${tU}/{L}$, $\tilde{u}_i$ is ${u_i}/{U}$ and $\tilde{p}$ is ${p}/{(\rho U^2)}$.

So, that is what we do then if we write the non-dimensional Navier Stokes this will look like your continuity equation will look like 0 and momentum equation will be:

$$\frac{\partial \tilde{u}_i}{\partial \tilde{t}} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{1}{Re} \frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j{}^2}$$

Now here the Reynolds Number Re is defined as characteristics velocity length scale by ν which is our ratio of 2 forces, one is inertial divided by viscous forces. So, that is what we get.

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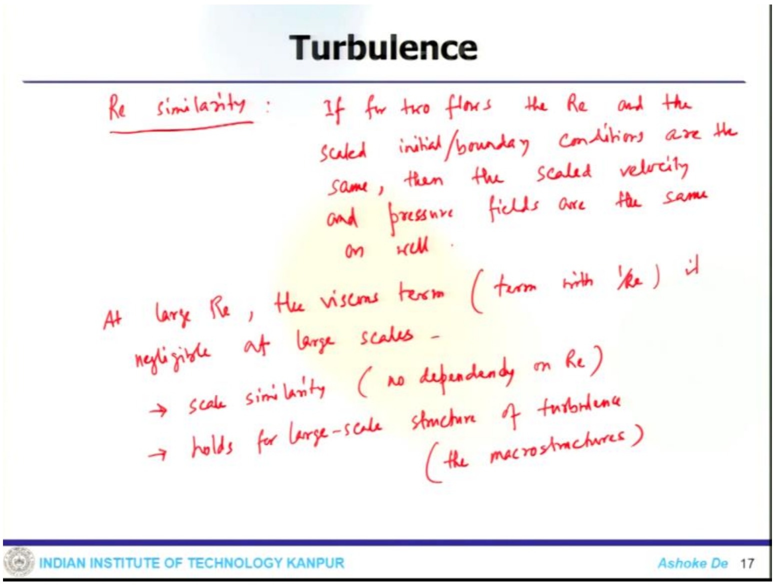

Now, look at the scale similarity. So, already we have got the non-dimensional form. So, there are 2 similarity one is the Reynolds Number similarity. So, Re similarity that is one important thing which says that if for 2 flows the Reynolds Number and the scaled initial or boundary conditions are the same then the scaled velocity and pressure fields are the same as well. So, that is how you say about Reynolds Number similarity. Now at large Reynolds number, the viscous term essentially that term with 1 by Re is negligible at large scales.

So, which means there is a scale similarity which on the other hand has no dependency on Re and other one holds for large scale structure of turbulence that is nothing but the macrostructure.

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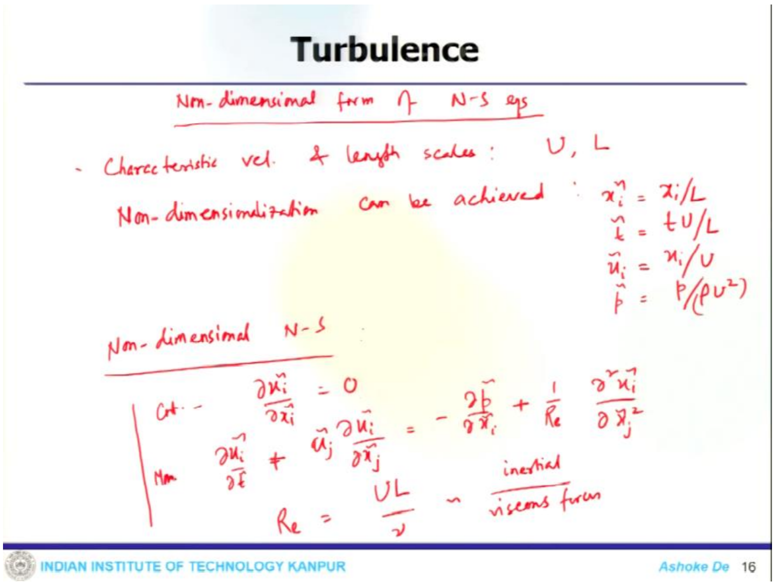

So, this is what from our non-dimensional Navier Stokes equations which is continuity this is continuity and this is momentum, we can say there are both the scales similarity and Reynolds Number similarity and the Reynolds Number similarity if for two flows are Reynolds Number and the scale initial and boundary conditions are the same then the scaled velocity and pressure fields are the same as well. So, that talks about your Re similarity. Now, when you move to a large Re, the term which has the viscous term that what we said terms with one by Re is negligible at large scale, which means, there is a scale similarity that means there is no dependency on the Re and this holds good for large scale structure of turbulence that means the macro structure. So, these 2 things are your Re and the scale similarity.

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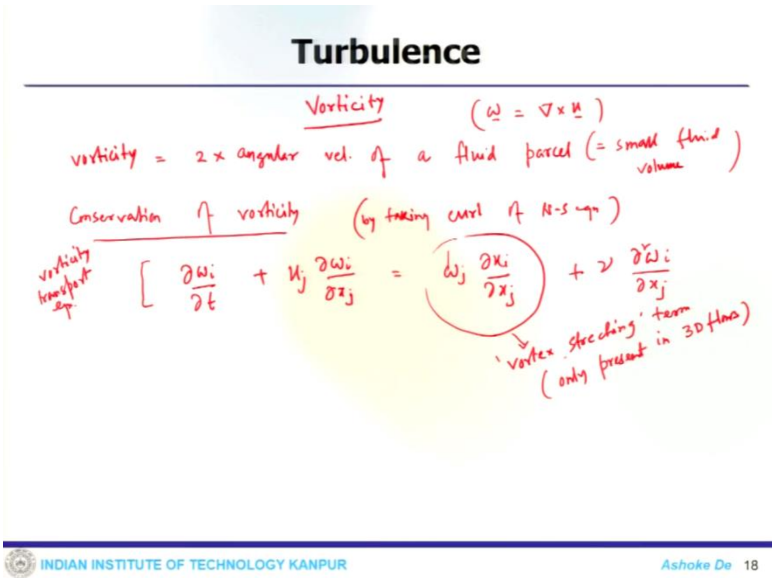

So, now that we need one more thing which to be interesting to quantify is the vorticity, so the definition of vorticity says that if you say $\omega = \nabla \times u$, which means the vorticity is 2 multiply by angular velocity of a fluid parcel so, that is small fluid volume so, that certainly takes a curl of velocity vector will get to the vorticity in mathematical term. This is the mathematics so, we can derive the conservation of vorticity so, that you get by taking the curl of N S equation. So, we have got this N S equation either in non-dimensional form so, you can or dimensional form you take the curl of that what we get:

$$\frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} = \omega_j \frac{\partial x_i}{\partial x_j} + \nu \frac{\partial^2 \omega_i}{\partial {x_j}^2}$$

Now, here this is our vorticity transport equation or conservation of vorticity equation or one can say vorticity transport equation. Now, this particular term here these terms it is called the vortex stretching term and this only present in 3D flows. Now, what it does is a very interesting phenomena which is associated with this particular term. So, vortex stretching mechanism causes change in vorticity of a fluid parcel by stretching and rotation of the parcel. So, this is very important concept here. The vortex stretching mechanism actually causes the change in both the vorticity of fluid parcel by stretching and the rotation of the parcel. So, it stretches and also rotate the parcel. So, one can think about vorticity in global sense this is a characteristic feature of turbulence.

So, that is how we, now we are looking at different aspect of this particular flow, we looked at this large scale similarity, Reynolds Number similarity and macro structure. Now we are looking at the vorticity, not only vorticity also the vorticity transport equation where we have a vortex stretching term and vortex stretching mechanism. So, that is one of the characteristics feature of the turbulent flow.

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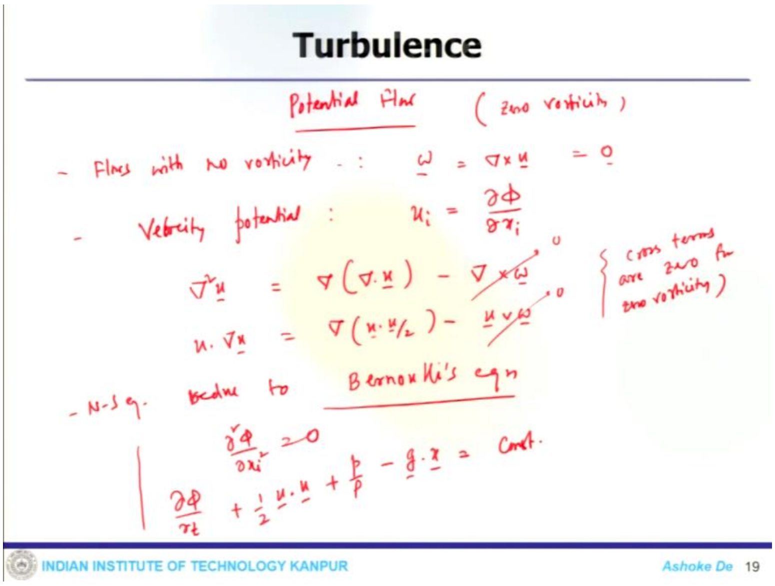

Now, we will go back a little bit and start with some simple system like potential flow, so which means that is 0 vorticity so, the potential flow means flows with no vorticity so, that means there is no stretching of fluid parcel neither the rotation of the fluid parcel so, which means ω is 0 so, you can have a velocity potential and one can define u_i as:

$$u_i = \frac{\partial \varphi}{\partial x_i}$$

Once you do some algebra here so,

$$\nabla^2 u = \nabla(\nabla . u) - \nabla \times \omega$$

So, the these goes to 0 and:

$$u. \nabla u = \nabla\left(u. {}^{u}/_{2}\right) - u \times \omega$$

This also goes to 0. So, there is cross terms are 0 for 0 vorticity. So, for potential flow the Navier stokes equation actually reduced to another famous equation called Bernoulli's equation which is:

$$\frac{\partial^2 \varphi}{\partial {x_i}^2} = 0$$

And:

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2} u . u + \frac{p}{\rho} - gx = constant$$

So, these are the set of equation that one may have for potential flow which does not have a 0 vorticity that flow field assumed to be not getting straight neither having any rotation.

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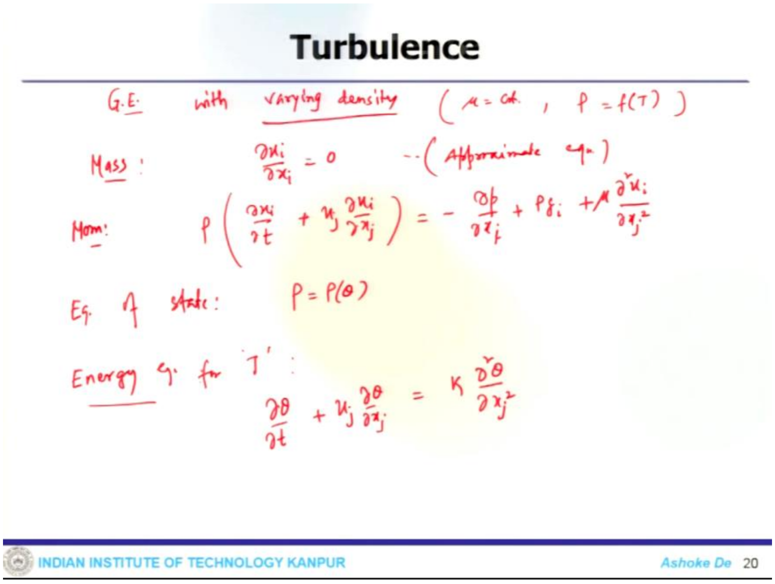

Now, we will look at the equation or the governing Equation with varying density. So, density is no more constant, but still we maintain dynamic viscosity constant and density is a function of temperature only. So, that means the density is no more constant, density is varying. So, we can have some approximate equation for conservation of mass which will be:

$$\frac{\partial u_i}{\partial x_i} = 0$$

So, one can note here approximate equation when density is varying, we cannot directly write this what we have written for incompressible flow. So, this is an approximate equation now, the momentum conservation will have the variation of the density into the system. So, that will look like density with:

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \rho g_i + \mu\frac{\partial^2 u_i}{\partial {x_j}^2}$$

Now, we use one more equation because density is a function of temperature. So, we use the energy equation and equation of state. So, equation of state would give me ρ is a function of θ and my energy equation will get me energy equation for temperature. So, that will be:

$$\frac{\partial \theta}{\partial t} + u_j\frac{\partial \theta}{\partial x_j} = \kappa\frac{\partial^2 \theta}{\partial {x_j}^2}$$

So, that is what we get. So, this is the Governing Equation that one can have for the varying density condition.

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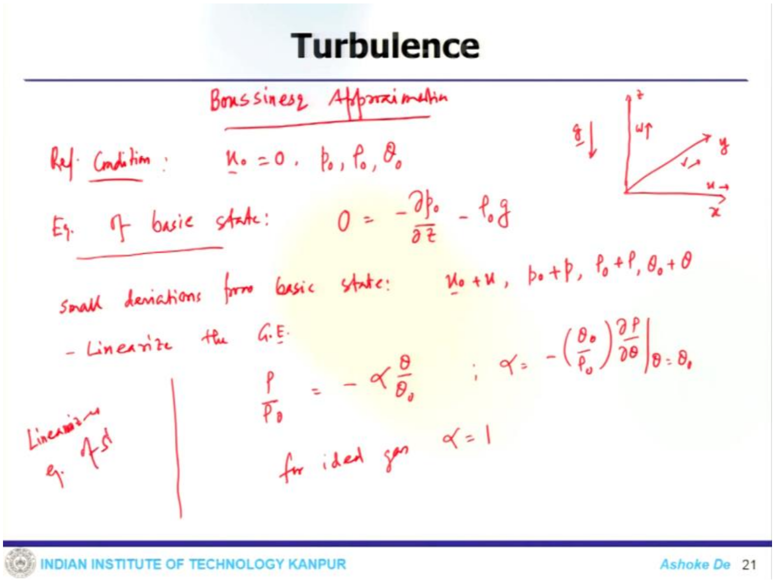

Now, we will use one important approximation is known as Boussinesq approximation. So, we can define our reference system like this. So, this is my x. So, u in this direction and this is my y, v will be in this direction, this is my z, so, my w will be this direction, so, my g will be acting on this direction. Now, first we will have some reference condition. So, the reference condition is 0, p_0, ρ_0, θ_0. So the equation of basic state would be:

$$0 = -\frac{\partial p_0}{\partial z} - \rho_0 g$$

Now you want to slightly deviate from the basic state that means small deviations from basic state. So, how do you get that? So, we will say that $u_0 + u$, $p_0 + p$, $\rho_0 + \rho$, $\theta_0 + \theta$. There is a small deviation from the basic state now, we linearize the governing equation. So, the governing equation that we have written here mass momentum energy for temperature. So, the linearized system would look like that:

$$\frac{\rho}{\rho_0} = -\alpha \frac{\theta}{\theta_0}$$

So, this is where:

$$\alpha = -\left(\frac{\theta_0}{\rho_0}\right) \left|\frac{\partial p}{\partial \theta}\right|_{\theta=\theta_0}$$

For ideal gas, α would be 1. This is my linearized equation of state.

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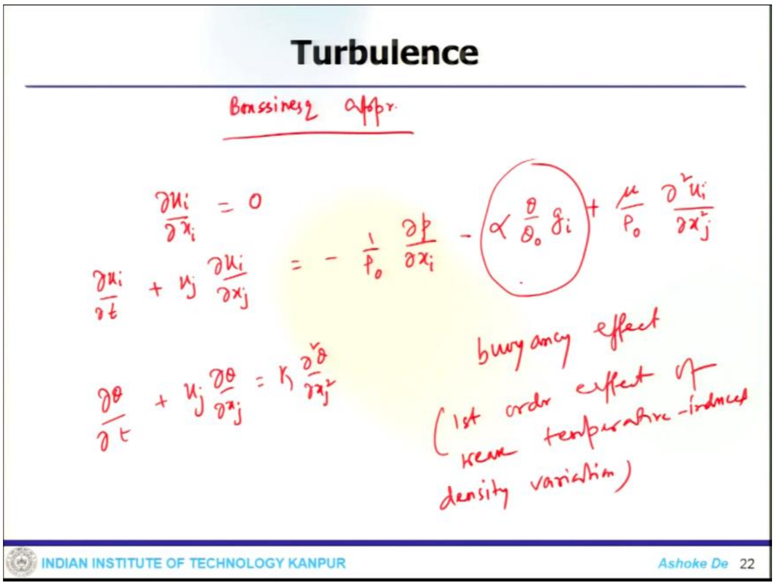

Now, my Boussinesq approximation will get me:

$$\frac{\partial u_i}{\partial x_i} = 0$$

And my momentum equation becomes:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} - \alpha \frac{\theta}{\theta_0} g_i + \frac{\mu}{\rho_0}\frac{\partial^2 u_i}{\partial {x_j}^2}$$

And we get:

$$\frac{\partial \theta}{\partial t} + u_j \frac{\partial \theta}{\partial x_j} = \kappa \frac{\partial^2 \theta}{\partial {x_j}^2}$$

And this is the term what with the negative sign is the buoyancy effect. So this is first order effect of weak temperature induced density variation. So, the density dependent on temperature due to that you get to see some variation in the momentum equation. We will stop here and continue the discussion in the next lecture.