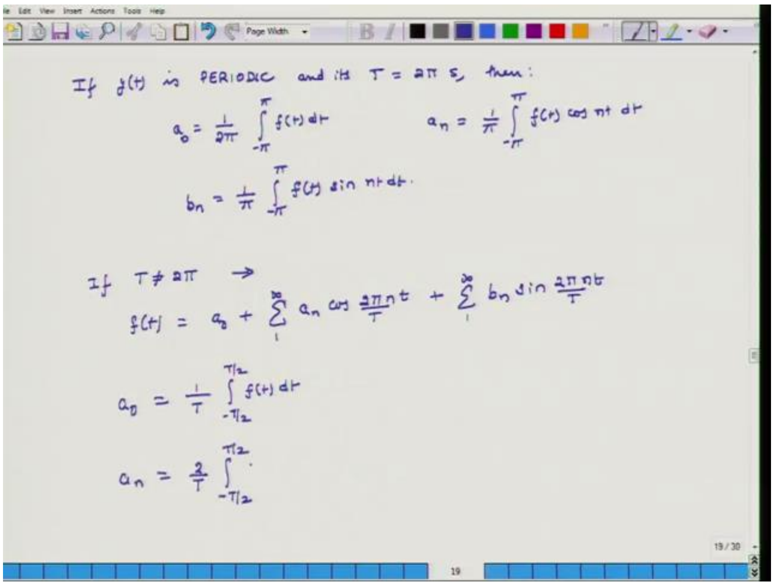
Fundamentals of Acoustics
Prof. Nachiketa Tiwari
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Lecture - 54
Fourier Series

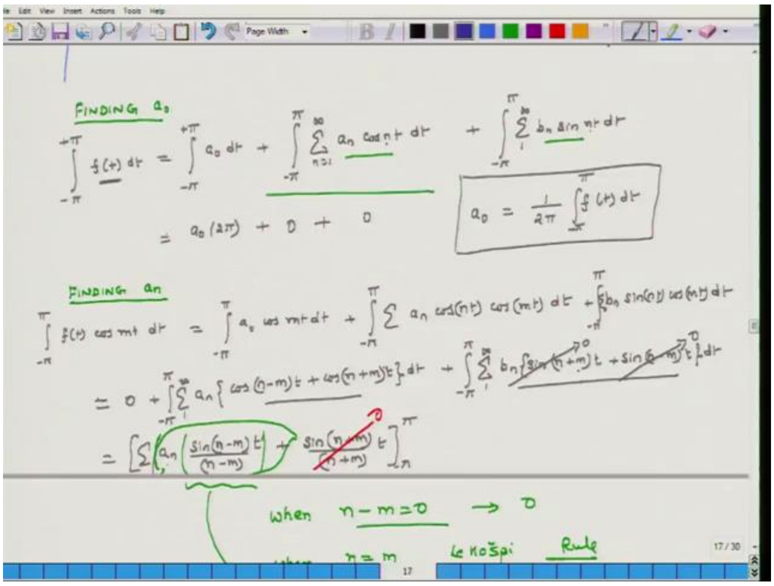
Hello, welcome to Fundamentals of Acoustics. Today is the last day of the ninth week of this course and yesterday we had developed expressions which will enable us to decompose or resolve a times signal into its Fourier domain. If the signal was periodic and the periodicity of that signal was 2 pi seconds, what we planned to do today is we will generalize that relationship and then we will also do a couple of examples so that we get a better feel of Fourier analysis.

(Refer Slide Time: 00:58)



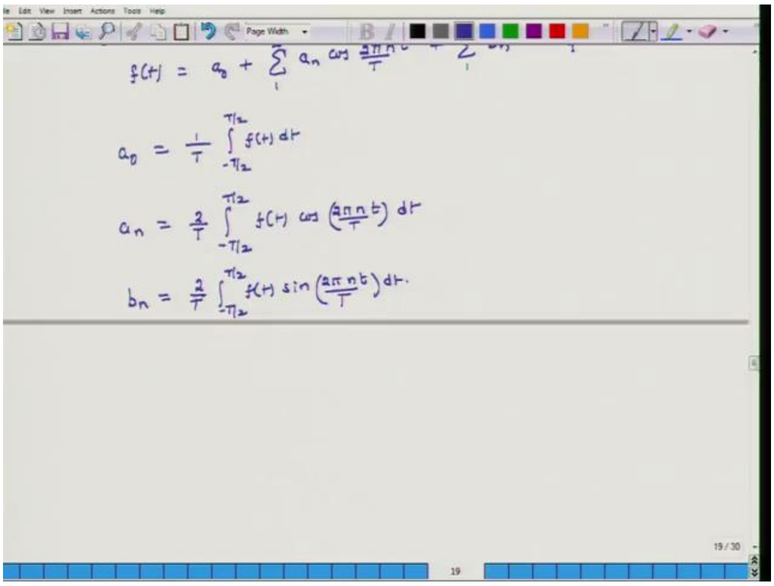
So we had said that if y t is periodic and its time period equals 2 pi seconds then a 0 equals 1 over 2 pi minus pi, 2 pi f t, dt; a n, so that is a 0 equals 1 over pi minus pi to pi f t cosine nt dt and b n which are coefficient of sin terms that equals 1 over pi minus pi to pi, f t sin nt dt.

(Refer Slide Time: 02:21)



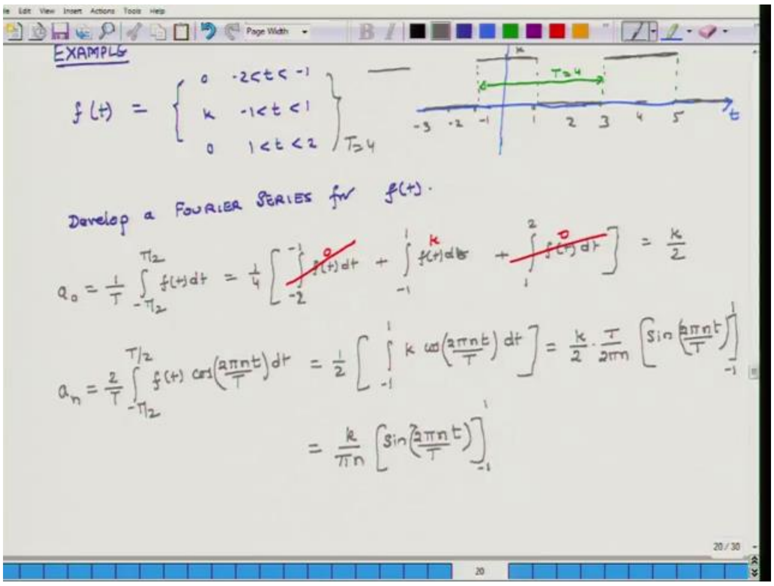
Now, if t is not equal to 2 pi then what do we do and the answer is pretty straight forward essentially you the overall expression in that case will not be cosine nt times sin nt right, rather what you can if T is not equal to 2 pi then f t can be expressed as a 0 plus sum of a n cosine, 2 pi n t over T plus b n; sin 2 pi n over t 2 pi nt divided by T and n varies from 1 to infinity, here also it goes from 1 to infinity and here the definitions of a naught a n and b n get slightly modified. So, what is a n? Instead of 1 over 2 pi, I just replace T, so wherever I see 2 pi, I replace that 2 pi by factor of T. So, here I get minus T over 2 to T over 2, f t dt a n equals 2 over T minus T over 2 to T over 2; f t cosine 2 pi n t over T dt and b n equals 2 over T minus T by 2 to T by 2, f t sin 2 pi nt over T dt.

(Refer Slide Time: 04:00)



What we do is we will now do an example which will make things clear.

(Refer Slide Time: 04:33)



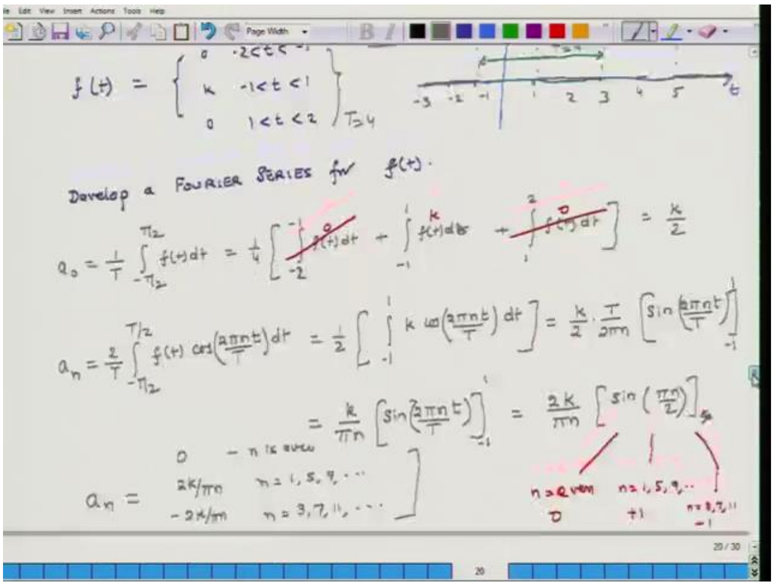
So example, so I say that consider a function f t and this function has a time period of 4 and its; so every 4 second it repeats itself and during that time period of 4, its value is 0 when time is between minus 1 second and minus 2 seconds, its value is k when time is between 1 second and minus 1 second and it is 0 when time is between 2 and 1 seconds.

So, let us first get a feel of this function, so how does this function look like. So, this is my time axis, so I am going to develop a time domain representation of the signal and let us say this is minus 1, 0, 1, 2, 3, this is minus 2, minus 3, so that signal is k. So, let us say this is k, so between so this is minus 3, minus 2, minus 1, 1, 2, 3 as we let us make it 4 and this is 5 and then from here to here it is 0, from minus 2 to minus 1 it is 0. So, from minus 1 to 1 it is k, then minus further 1 to 2 it become 0. So, this is how the signal looks like then it is again k, it k made it a little longer and then again it becomes, no actually its region is 2. So, it is from 4 to 5 and again from here it becomes 0 and it is 0 here and it is 2 here.

So what is its time period? Its time period is 4 seconds, so t is equal to 4 seconds and how do we depict it here we say that the time distance between 4 seconds. So, what is our aim? So develop a Fourier series for f t, develop a Fourier series for f t. So, first we calculate a naught; a naught equals 1 over T, minus T over 2 to T over 2 f t, dt. So, this equals to what is time period 4 seconds, so it is 1 over 4 and then we know that this function is continuous only; it is not continuous totally, so we break this integral into 3 regions. So, first we integrate f t from minus 2 to minus 1 then from 1 minus 1 to 1 and from 1 to 2. So, this is minus 1 to no, it is minus 2 to minus 1; f t d t and what is the value of f t here 0 plus minus 1 to 1 f t, dt and the value of f t is k here and plus 1 to 2; f t, dt, so here the value is 0. So, because the value is 0 and these guys I just drop these guys out, so I am really interested in integrating it from minus 1 to 1; all other components will be 0. So, this is equal to, so k is constant; so k over 4 integration of dt from minus 1 to minus t into 2 is over minus 1 to 1, so this becomes k over 2, so a naught is k over 2.

Let us look at a n, so a n is 2 over T minus T by 2 to T over 2, f t cosine 2 pi n t over t, dt and T is 4 seconds so I get half and once again I do not have to worry about the integral from minus 2 to minus 1 because f t is 0, same thing for my 1 to 2, so I am only interested in minus 1 to 1; so minus 1 to 1 f t is k cosine 2 pi nt over T dt. So, this gives me k over 2 times T over 2 pi n; sin 2 pi nt over T evaluated from minus 1 to 1. So, this is equal to, so t is 4 seconds, so t and 2 and 2 cancel out. So, I get k over pi n and I have sin 2 pi nt over T evaluated between the limits minus 1 to 1. So, let us look at this number, so this number will have 3 sets of values; n is changing from 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 like that.

(Refer Slide Time: 13:00)

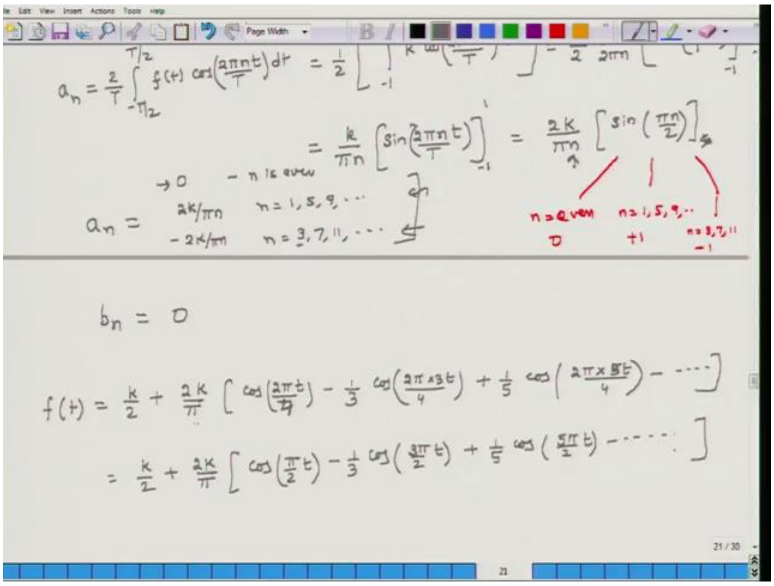


When n is even then what happens, so actually let us make 1 more simplification and then we will; so what I will do is at t is equal to 1 it becomes 2 pi n over capital T, at t is equal minus 1 it becomes sin of minus the same number right and sin of positive theta is equal to minus of minus of sin of. So basically when I do this, I get 2 k over pi n and then sin; 2 pi n over T evaluated at for different values of n.

Now let us look at this parameter when n is even, when n is n; one more thing, so the denominator is 4, so 2 over 4 is pi over 2. So, what I get is 2 pi divided by 4 pi will 2 pi n over 2 that is what I get; pi n over 2. So, when n is even then what do I get, n is equal to even this term in the parentheses sin is 0 when n is equal to 2 then 2 pi divided by 2 sin pi, when n is equal to 4 then it is sin 2 pi sin 3 pi and so on and so forth. When n is equal to 1, 5, 9 and so on and so forth, then this term is positive 1; consider n is equal to 1 sin of pi by 2 is 1, n is equal to 5 then it is, n is equal to 5 means what? It is 2 pi plus pi by 2, again it is 1, so this is 1 and when n is equal to. So, the last condition is n is equal to 3, 7, 11 and so on and so forth then this number is minus 1.

So a n equals 0, 2 k over pi n and minus 2 k over pi n; for n is even, n is equal to 1, 5, 9 and n is equal to 3, 7, 11 and so on and so forth. So, this is how we have calculated a n and using similar method we can calculate the value of b n and what we will find that b n is 0. One way to look at a function is that if the function is even function which means that if its function at time plus t and minus t, if they are the same then this b n terms they are 0 if the function is purely odd then a n’s are 0, but even if we do not know this and we use this relation which we have developed here and we can try to calculate b n we will find that b n will come out to be 0.

(Refer Slide Time: 17:00)

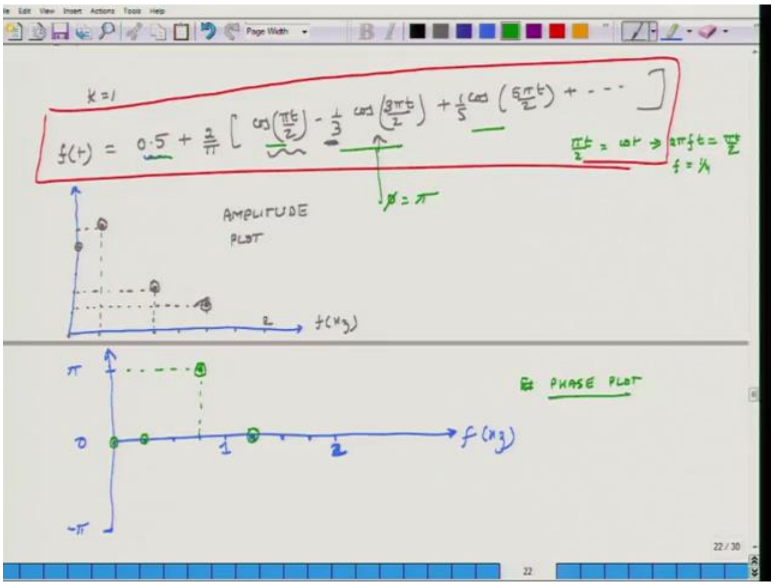


So overall function f t equals a naught which is k over 2 plus 2 k over pi, cosine 2 pi t by T, so this second term is 1 over 3 whereas, 1 over 3 come in because here I have 1 over n 2, 2 k; 2 k is already here 2 k divided by this is already there, but no sorry we have 2 k’s here, in the denominator I have pi, but I also have in the denominator n. So, in first term n is 1, so 1 over 1 is 1; in the second term n is 3, so I have to have minus 1 over 3 because it is from this row; times cosine 2 pi n, n is.

Student: 3 (Refer Time: 18:09)

3 t divided by 4, here also t is 4; next term is positive because I go back to this one, so n is equal 4 it is 0. So, I do not have to worry about it, n is equal to 5; it is a positive number. So, it is 1 over 5 cosine; 2 pi into 5 t divided by 4 minus and so on and so forth. So, this is equal to k over 2 plus 2 k over pi, cosine pi over 2 t minus one-third cosine, 3 pi over 2 t plus one-fifth cosine, 5 pi over 2 t minus so on and so forth, so that is our f t.

(Refer Slide Time: 19:31)



Assume that k is equal to 1, then our f t becomes 0.5 plus what? 2 over pi, cosine pi t over 2 minus one-third, cosine 3 pi t over 2 plus cosine one-fifth 5 pi t over 2 and so on and so forth.

Now, what we will do is we will close this discussion for today but before we do that, we will actually use this expression to develop its representation in the frequency domain. So in the frequency domain I have 2 plots, now this entire expression has only cosine terms, so for each of the frequency component, the phase is 0, right; it has only cosine terms, so phase is 0. So, the phase plot is very easy to construct, this plot is very easy to construct; on the x axis I plot frequency and on the y axis I plot phase in radiance, what is the first frequency in this system, it is this thing 0.5, 0.5 is associated with how many hertz?

Student: 0 hertz.

0 hertz, so that is the phase at 0 hertz; the second frequency is, so what is this; pi t over 2, pi t over 2 means what?

Student: (Refer Time: 21:36).

So, it means pi t over 2 means omega t, it means 2 pi f t is equal to pi t over 2 or f is equal to one-fourth hertz, pi is one-fourth hertz; is that right. So, let us say this is one-fourth hertz, f is 1 over 4 hertz from here. So, I had the second phase at 1 over 4 hertz is, so this is 1 over 4 hertz 0, what is the next frequency component; 3 pi over 3 times that; pi over 2 the next frequency component is 3 times that. So, this is 2 times that, this is 3 times that actually this 0 should not be there. This is the first frequency component f 0 hertz, this is the second frequency component which is one-fourth hertz this is the third frequency component, which is three-fourth hertz then I have the next component will be.

Student: (Refer Time: 23:07).

4 and then 5, this is the 4th frequency components; 5 by fourth hertz and so on and; and all at all these frequencies the phases 0. So, similarly I can make, so this is 0 hertz one-fourth, two-fourth, three-fourth, four-fourth, five-fourth, six-fourth, seven-fourth and so on and so forth. So, this is f in hertz, so what is this? One-fourth, two-fourth, three-fourth, four-fourth, so this is 1 hertz 2, 3, 4 this is 2 hertz. So, at 0 hertz what is the amplitude?

Student: (Refer Time: 24:00).

Half - so it is half; so let us make this half, at this is the first frequency component the amplitude is 2 over pi, so what is 2 over pi? Let us say pi is about 3, so about 0.6. So, this is 0.1, 0.2, 0.3, 0.4, 0.5 let us say this is 0.6; this is the amplitude of the second frequency, amplitude of the third frequency corresponds to three-fourth hertz and that is what 2 by 3 pi. So, pi is 3 pi is about 10, so it is about 0.2 roughly; actually we made a mistake in the plot for phase because the phase of this is not 0 what is a phase of this; phase is equal to 10 inverse what.

Student: b.

B over a b is 0, but a is negative. So, phase should have been at three-fourth hertz; phase should have been pi radians right, so this graph is not correct. So, what we will do is, we will actually fix this problem will draw it later but we have to remember that if cosine is negative and there is no sin component, it means phase is 0; a phase is pi radiance.

So, that is the third frequency component and the fourth frequency component corresponds to 5 over 4 hertz and here phase amplitude is 2 divided by 5 pi. So, this is what 2 over 5 is 0.4 divided by 3.1, let us say roughly it is 0.4 divided 3 is 0.133

something like that, so this is 0.1, 0.133 is a little more than that 1 2 3. So, actually I made a error here also, so because the second frequency component was third one was at 3 over pi, so this is 1 over pi, 2 over pi, 3 over pi and made it at 4 over pi .

Then the next frequency component is this guy, so this is the amplitude plot and the line between these 2 plots; these 2 points does not exist because these are discrete frequencies. Now very quickly, we will draw the phase plot which we could not we did wrong which we incorrectly plotted earlier. So, phase plot, so on the phase actually we should always do minus pi to pi, so that is our frequency; this is phase.

So let us say this is pi, this is minus pi, this is 0 and so this is 1 over fourth, 2 over fourth 3 over fourth and this is 1 hertz, 5 over fourth, 6 over fourth, 7 over fourth and this is 2 hertz. So, phase for a first frequency is 0, phase for the second frequency which corresponds to 1 over fourth hertz is 0, phase for the third frequency 1, 2, 3 is what pi hertz, pi radians.

Student: (Refer Time: 29:00).

[FL] and the phase for the fourth frequency is once again 0, phase for this frequency, this frequency, this frequency they are 0, for this phase is pi radians. So, this is our phase plot and in such a way we can construct phase and amplitude plots once we have done the Fourier analysis of a signal which is periodic in nature. So now what we will do is next week, we will continue this discussion and we will discuss functions which are not periodic and for those functions how can we construct frequency domain representations. So, with that discussion we close our class for today and we will meet once again in the next week.

Thank you, have a great day bye.