Fundamentals of Acoustics
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Lecture – 53
Fourier Series

Hello, welcome to Fundamentals of Acoustics. Today is the 5th day of the current week which is the 9th week of this course and just yesterday we started discussing time and frequency domain representations of signals and what we had shown was that if we can depict a signal into its; if there is a signal which is composed of sins and cosines only, then we can depict that signal either in a domain or in a frequency domain and from such representations those graphical representations especially the frequency domain representations, we can extract the frequency and phase associated with each frequency component of the signal.

So, what we plan to do today is figure out how do we mathematically convert a periodic signal into its frequency domain representation.

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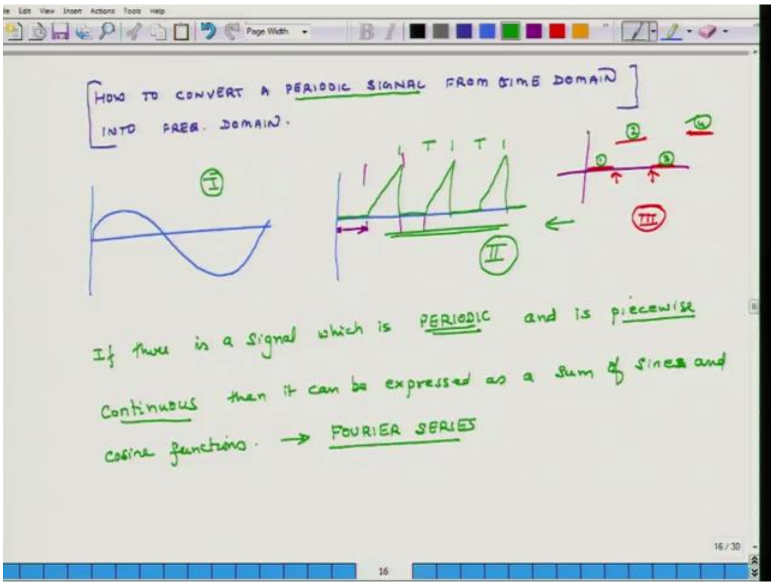

So, I will explain that. So, what we will learn today is how to convert a periodic signal from time domain into frequency domain? This is our aim. So, please note that I have use the term periodic signal so that means, that the signal has a specific period, it may not be harmonic in nature. So, what does that mean? I will draw couple of periodic signals, this is one periodic signal another periodic signal is this and so and so forth. So, this signal repeats itself after every T seconds and the nature of the signal is identical after every T seconds. So, this is another periodic signal. So, this is signal type 1 which is purely harmonic, I can express right away just by looking at it in to its sin and cosine terms, but this second signal is periodic and it will be really nice if we can somehow convert this type of periodic signal into a sum of sins and cosines because the moment I convert it into some of it is sins and cosines, I can then use that conversion and get a frequency domain representation of the signal.

So, our problem is how do we convert especially this type two signal into a sum of sins and cosine, because the moment I convert it into sin and cosine then I have amplitude of each sin and cosine term and phase and then that I can plot it on a graph for an amplitude plot and a phase plot. How do we do that? So, you must have learnt this technique that if there is a signal which is periodic it is important, which is periodic and is piecewise continuous. So, what is piecewise continuous mean? See this signal is continuous in this piece and it is also continuous in this region and it is also continuous in this region, another periodic signal could be this. So, it is like this, this, this, this. So, at these points the signal is not continuous. So, this is type three signals.

So, signal 1 and 2 are continuous because the value is continuous at all points, it has only single value, but in signal 3 at these transition points the value is not continuous if I look at it from right side from the left side the value is 0, if I look at it from the right side the value is nonzero. But even type 3 signal is continuous because it is continuous in piece 1, it is continuous in 2, it is continuous in piece 3, it is continuous in piece 4. So, that is what the meaning of piecewise continuous. So, if there is a signal which is periodic. So, type 3 signal is also periodic same is type 2 same is type 1 and if it is piecewise continuous, then we have learnt in our earlier years of mathematics and engineering that this. So, and if it is piecewise could then it can be expressed as a sum of sins and cosine functions. So, it should be sine, it is a sum of sine and cosine functions and this representation of the signal is known as Fourier series.

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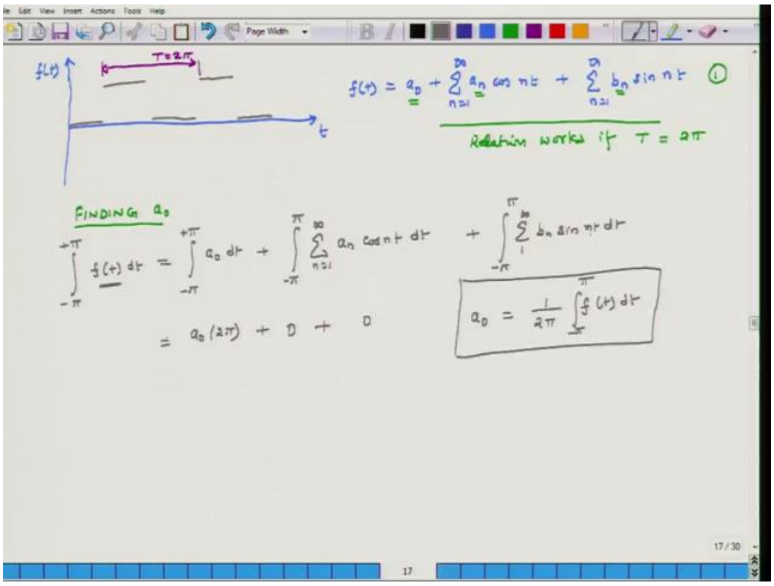

So, what does that mean? If there is a periodic signal, suppose this is the time domain representation. So, this is f t and let say the signal looks something like, this is a periodic signal, then this f t I can represent it as a sum of 3 parts a constant, a naught, plus a sum of cosines a n cosine n t, plus b n sin n t where n equals 1 to infinity and equals 1 to infinity. So, if there is a periodic signal represented by ft, then it can be expressed as a naught plus a sum of cosine terms plus a sum of sin terms. Now this relation works if the time period of this signal. So, time period is equal to 2 pi seconds. So, what does that mean? So, how do I depict time period on this? This is the time period of the signal.

So, this is the simplest form of a periodic signal, later we will generalize it that whatever is the time period we can develop an expression for this. So, that is our time period. So, if the time period is equal to 2 pi, then I can represent the signal as a naught, plus a n cosine n t, plus b n sin nt, where n is an integer and it does not start from 0, it starts from 1 and it can go up to infinity. So, then our aim is how to find what are the values of these parameters, a 0, a n, b n because the moment I know a 0, a n, b n I will also know n, n is just changing 1 to infinity. So, I can express f t as it is Fourier series.

So, a 0, finding a 0; how do I find it? What I do is I multiply this expression. So let us call this expression 1. So, I multiply it is left side and right side by 1 or basically essentially I integrate the left side as well as right side with respect to time over the time period and when I equate them I will see that I end up with an expression for a 0. So, let us do that. What is the time period 2 pi. So, I integrate it from minus pi to plus pi f t d t and that equals integral of minus pi to plus pi a naught d t, plus minus pi to plus pi sum of a n cosine n t d t, plus third integral minus pi to pi, b n sin n t d t and here n is equal to 1 to infinity here also it is 1 to infinity.

So, if I know this function f, then I can evaluate the left side and what is the right side this is equal to a 0 times 2 pi plus let us look at this. So, the integral of cosine n t is sin n t divided by n, and sin n t we evaluated at pi is 0 and sin n t evaluated at minus pi 0. So, this is 0 plus integral of sin n t is cosine n t divided by n into minus 1 and the value of cosine of pi is same as value of cosine of minus pi. So, when I take the difference this also becomes 0. So, a 0 equals 1 over 2 pi f t d t minus pi to pi. So, this is my expression for a 0.

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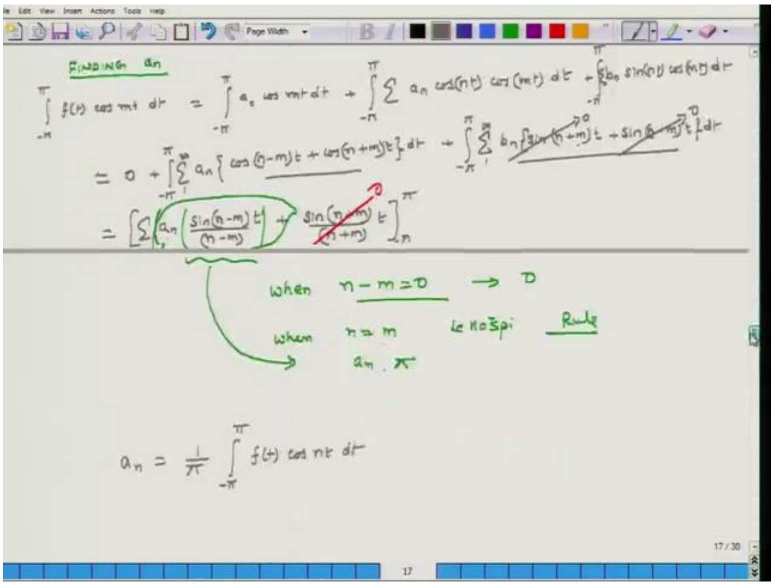

So, the next name is that we want to find a n. So, what do we do if we have to find a n I multiply equation 1 on both sides by cosine mt instead of n I put another index m, m is some dummy number and I do the integration and let see what happens. So, minus to pi f t cosine mt d t this equals minus pi to pi a 0 cosine mt d t, plus minus pi to pi sum of a n cosine n t cosine mt d t, plus integral minus pi to pi b n there is an integral sin also here is a summation sin n t cosine mt dt. So, this equals what is the integral of cosine mt, sin n t when I evaluated pi and minus pi I get 0. So, the contribution from the first term is 0, plus now cosine n t times cosine mt, there is trigonometric relation that cosine a times cosine b is equal to what cosine a plus b time plus cosine a minus b. So, we do that.

So, this is summation of I will actually erase it, first write a n cosine n minus m t plus cosine n plus m t d t plus integral of minus pi to pi, summation also this goes from 1 to infinity 1 to infinity b n and this is sin n plus m t, plus sin n minus m t dt. So, what is the contribution from this term? So, sin n plus m t if I integrate it because minus cosine n plus m t divided by n plus m, if a time t is equal to pi and time t is equal to minus pi the values are going to be same because cosine theta equals cosine minus theta. So, contribution from this term is 0, contribution from this term is 0 for the same reason. So, the only thing which is going to contribute is this term. So, we will a n and this is in parenthesis, sin n minus m t divided by n minus m, plus sin n plus m t divided by n plus m and this is evaluated at minus pi and pi.

Now, look at the second term, sin of the denominator of the sin n plus m t is always a nonzero number, n plus m is nonzero. So, the denominator will always be non 0; the numerator when t is equal to pi and n and m r integers, it will be 0 when n and m r and for n and m being integers when t is equal to minus pi also it will be 0. So, this is 0 your denominator is always 0.

Now, look at this guy, this is interesting when n minus m. So, when n minus m is not 0, then there will be the denominator will be non 0 number, what will happened to numerator? Consider the case, so the numerator will always be 0, numerator is always 0 let say n is 2 m is equal to 1 sin of pi minus sin of minus pi it will be 0. So, this green term is 0, when n is equal to m then what happens? I have 0 in the numerator and I have 0 in the denominator. So, I cannot calculate it is value. So, how do I calculate the value? I do I calculate it is value using le hospital rule and we find that when n equals m, then this term in the green is not 0, we find that it is not 0.

So, in that case if I can calculate this term. So, when n is equal to m, for all values of m when n is not equal to m it is this thing will be 0, when n is equal to m only for that n I will calculate the value of a n, all other parameters they will go away. So, I can write it as that a n is equal to. So, this is equal to and when n is equal to m, when we use the le hospital rule we find that this entire thing it becomes. So, actually this entire thing this entire thing it becomes a n over 2 times pi this is what we get out of it.

So, what do I get? So, a n excuse me. So, a n equals 1 over pi integral minus pi to pi f t cosine n t d t, this is what I get out of it actually this 2 should not be here and if I do the similar mathematics. So, if I have to calculate dn what do I do? I multiply this original equation one in both sides left and right side by d by sin mt and I do the integration and I will find that I am left with only one term and using that I get the expression b n equals 1 over pi minus pi to pi f t sin n t d t.

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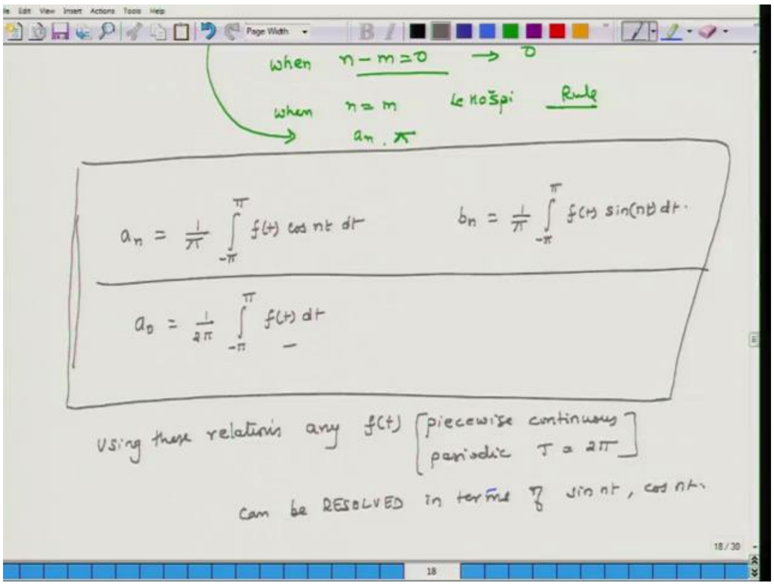

So, these are the expressions and then a 0 we had already shown is equal to 1 over 2 pi minus pi to pi f t d t. So, these relations work, using these values of a, we can break any function. So, using these relations any f t and this f t has to be piecewise continuous, it has to be periodic such that it is time period is 2 pi. So, any function which is piecewise continuous and periodic with the periodicity T is equal to 2 pi and a relation f t can be resolved in terms of sin n t and cosine n t and the moment we get each component each sin n t and cosine n t for each of these frequencies, I can calculate it is magnitude and it is phase.

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How do I calculate magnitude by a n square plus b n square the whole thing under square root and how do I calculate phase? It is tan inverse of b n over a n. So, the moment I get this type of a representation, I can calculate for each frequency it is phase and amplitude and those phases and amplitudes I can plot on a in frequency domain in a amplitude plot in a phase plot.

And I can represent the function as in the frequency domain. So, that is what why we are doing all this. Now we but here the only limitation is that t has to be 2 pi. Now what we will do in the next class is we will expand our scope and we will say how we can represent the signal if it is periodic, but the time period need not be 2 pi seconds, then how do we transform it and then we will do an example. So, with that we close the discussion for today and I look forward to seeing all of you tomorrow.

Thank you.