Principles of Vibration Control
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Lecture-19
Controllability and Observability of System

Welcome to the course on principles of vibration control. We have reached the fag ends of our journey on vibration control in this course. Now in the remaining lectures I want to focus about the active vibration control.

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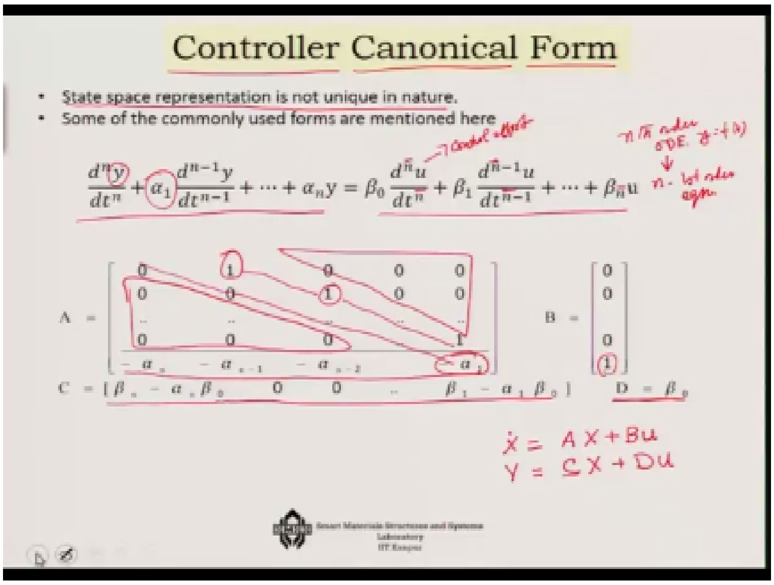

So towards that direction we are walking on states space control and for state space control we have to first of all we have talked about that how to develop the state space model but today we want to talk about certain canonical forms which will allow us to develop in the modern control that means full state feedback control for vibration control itself.

Now the first thing we have to keep in our mind and I will actually proof you that after a few more slides that state space representation state space representation is not unique in nature. Remember I told you that depending on the choice of the states the state space matrix will change. But that does not mean that the (()) (01:20) equation will change and the dynamics will change.

We will prove it because one of the most important till criteria towards that is the Eigen values of the system will not change. So no matter whatever will be your choice of state factors we have to finally cross check that. Now since let us first of all assume that this state space representation is not unique in nature. So we are looking for certain forms which we are calling to the canonical forms. canonical in good old days has to has a relationship with cannon itself anything which will be said by charge with a cannon sound would be considered to be the last word in that direction.

But as far as our things are concerned this are certain very uniform forms that is used all over the world and hence this canonical form term came and this we will learn corresponding to 3 cases one is for controller, another is for observer and the other one is called Jordon canonical form. There were few more but these 3 are generally most useful. So we will first talk about the controller canonical form.

Now if we consider the wording equation of the system in a big generalized manner, you can see that this is the Jordon equation of the system where d in a order ordinary differential equation and this is Y is the dependent variable and time is the independent variable. So left hand side gives us that up to dny dtn in it order and right hand side we have this u which is the control effort.

So right hand side also i get up to in its order or you know up to that kind of a I mean you need not go up to n here, you can go up to something like n bar or there was something like that were n and n bar need not be the same. So that is what is you know is our Jordon equation of the system where left hand side talks about the states, right hand side talks about the control effort.

Now this see this is having an nth order let say ODE of in terms of y so as a function of t. So this equation can be actually decomposed into n fast order equations. N fast order equation and that n fast order equation is going to give us the state space format of it. Now in that state space format the controller can canonical form should be search that all these coefficients are only coming at the last row of the a matrix.

So in the A matrix it will start with a representation of 01, 000, so if you look at it that this is your diagonal actually and then the super diagonal terms are all unity. So you can see that these are the super diagonal terms which are all unity, that is the way it is arranged and all other terms are 0 here and below also all other terms are 0 expect the last row. So this block is 0 and this block is 0.

And in the last row we are getting this coefficient back from the characteristic equation which is - alpha n then - alpha n - 1 like that it will go up to - alpha 1 calling as the last term of this. That is the way if we can represent that a matrix and then the b matrix that is the actuate or influence part to remember again that we are talking about the form x dot equals to AX + Bu.

So this B matrix all the entries will remain 0 except the last which will be unity that is what will be our B matrix and i earlier told you that we can develop an output matrix which is y equals to CX sometimes it is extended up to y equals to CX + Du, if the control effort directly affects some of the outputs. So in this case control canonical form, the C matrix will have this following form ok taking coefficients from the 2 sides and D matrix is simply beta 0.

So this is what if we develop you know these nth order OD by decomposing it fast in a nth fast order and fast order equation and then writing it in this abcd form, then we are going to get the controller canonical form of the system. So this form will be useful for that is why we have to always keep this form in our mind.

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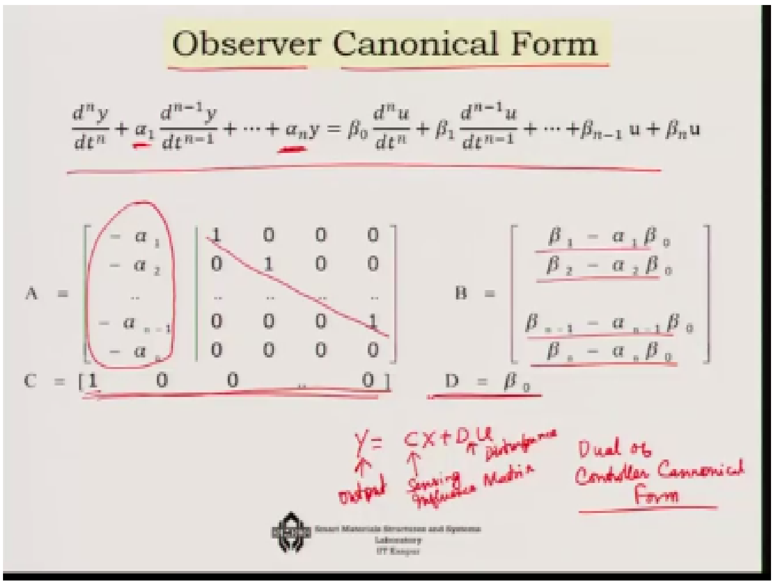

There is another which is known as the observer canonical form. So this is just the dual of the controller canonical form. And that you can see that why this is the dual part of it you can easily see each if you look at the you know structure of the A matrix here instead of the you know last row now it is the fast column where all these coefficients are appearing starting from - alpha 1 to - alpha n.

And then if you can see that once again the super diagonal terms are unity here and no other terms are coming into picture. B matrix is not 001 now, B matrix has all these based on the coefficients drawn from this given equation. C matrix is 1000 that is why that C matrix is suppose to be corresponding to the output equation y equals to CX+Du where Y is the output and that is why this is we put more focus on this that this is related to what you call the you know the sensing, ok so this is the sensing.

Like B is the actuation, this is the sensing ok, influence matrix or of course D is disturbance. So here that sensing part because observer are related to sensor wherever we cannot actually have sensors to sense the particular state we try to estimate that state and then this observer canonical form becomes useful because we have to design, we can design observer for this estimated state.

So that is why this part is simplified here as C is 1 and rest is 0 and D equals to beta0. So this is known as the observer canonical form.

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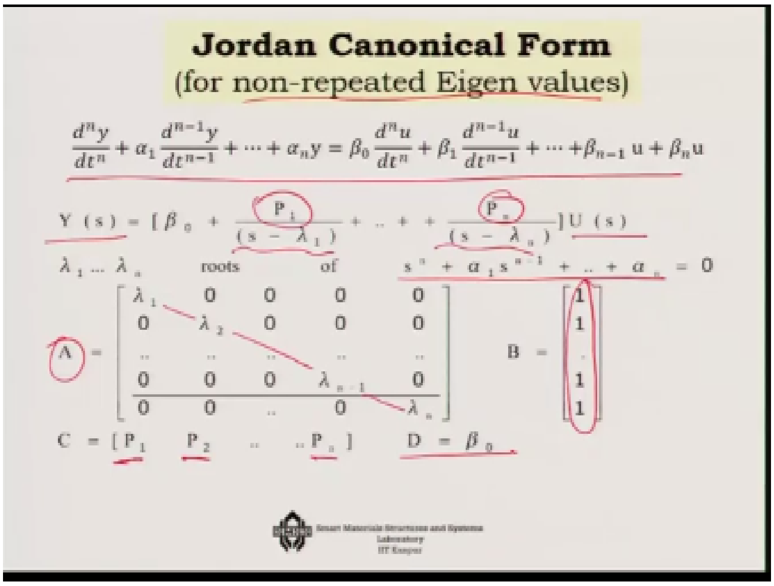

There is one more form that comes out this is known as Jordon canonical form and this is the form which is for non repeated Eigen values. So once again we have the same equation front of us and you can actually carry out this time of the entire equation, so Y will be Ys, U will be Us and you can write each one of them through a partial fraction. So these are the partial fractions, so that you actually separate it out in terms of lamda n distinct roots ok.

So that is what then becomes our characteristic equation. So now if I line up all these Eigen values which are the distinct roots along the diagonal rest is all 0. Then that is what is by A matrix in the Jordon canonical form. B matrix is all unity and C matrix has all the reseed corresponding to each one of these roots (()) (10:07) and D is beta0. SO this is the Jordon canonical form.

So these 3 canonical forms are important for controller design that is why we have first introduced theses 3 forms to you. Now this form to form this is only about it is like you know the same person who will you are putting into different dresses. All dresses corresponding to your convenience, but the person reverse the same, how do I prove that.

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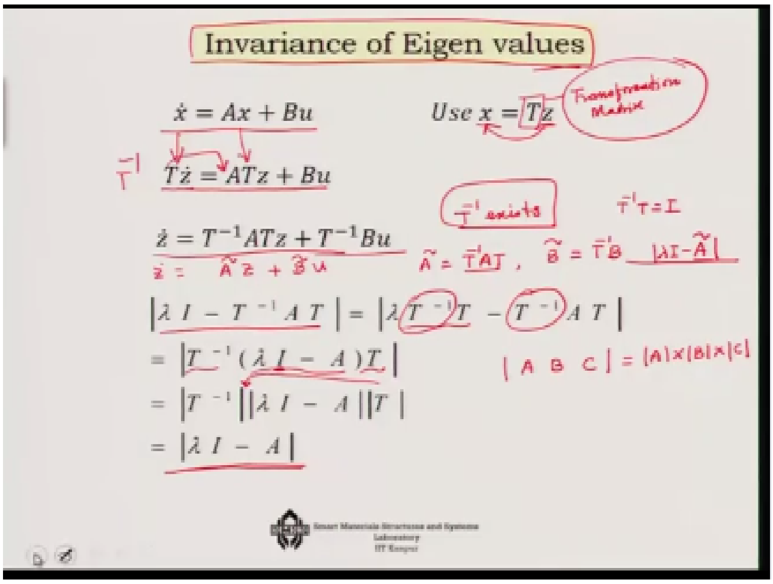

That you can prove from the proof of variance of the Eigen values. So if you look at it that our initial equation is given equation was X dot equals to Ax+Bu. Now let us say that I do not want it in these form, I want it in control canonical form or observer canonical form or the other form. So let us say corresponding to that form the state vectors are not X but Z. Now first of all we are working the linear domain.

So let us assume that this X to Z a mapping is possible through this transformation matrix D. So T is our transformation matrix. So then this entire form will look like T so I am not writing X equals to Tz so this will become Tz dot that start that Ax will become ATz and B Bu will remain as it is. That is what is my second stage. In the third stage what I will do is that I will take these T out here by pre-multiply everything both the sides by T inverse.

So if I do that then T inverse T will be identity, so Z dot will be T inverse ATz + T inverse Bu. Here I have paid an implicit assumption. What is that implicit assumption, that is T inverse exists, only they are actually these transformations would make sense. So and later on we will see that this is actually related to the controllability or observability of a system. So our new equation is a new state space equation we have obtained that is not in the form of X dot equals to Ax+Bu.

But Z dot equals to T inverse ATz + T inverse Bu. You may write that Z dot equals to something like A till Z where A tilda is T inverse AT + T tilda where A tilda is actually our T inverse AT, this is our new state matrix. And B tilda is actually T inverse B. Now if this with the case then what is the Eigen value of these A tilda, till that I can always find out if I can find out lamda I - A tilda.

So what is the determinant lamda I - A tilda. A tilda is nothing what T inverse AT, so lamda I - T inverse AT. That is what we get. Now let us say this i is the identity matrix and write it as T inverse T because T inverse T and you know the inverse of T if it exists, so T inverse T will be identity because T inverse T will be identity. So identity I can replace by these - T inverse AT.

And then I can take this T inverse you know common from both the sides, T inverse it will become then lamda i - A times T. So this identity matrix we are keeping here, just because this T I am taking to the right hand side. So now there are 3 matrix for us, T inverse, lamda i - A and T. So the determinant is the product of these 3 matrixes. Now there is a rule of determinant which says that if you have a determinant of product of 3 matrixes ABC.

Then this will be equivalent to the determinant of A times determinant of B times determinant of C. This is what we have uses here. That this will be then determinant of T inverse, determinant of lamda i - A and determinant of T. So If I do that then of course the determinants can be actually taken anywhere matrix cannot be, so I can take the determinant of T here and then this T inverse T would become identity, so that all that will be left is determinant of lambda I-A.

Now you can see that I started with lamda i-A tilda that was giving me the characteristic equation and I got lamda i-A determinant, this is giving me the same characteristic equation, so what it means is that the transformation matrix has only changes the form, it has not change the Eigen values of the system. So the Eigen values are actually in variant in this case. That is the very important conclusion that help us that from fall to fall very smoothly we can jump provided this transformation matrix exists. That is the only precondition that we have to keep with us.

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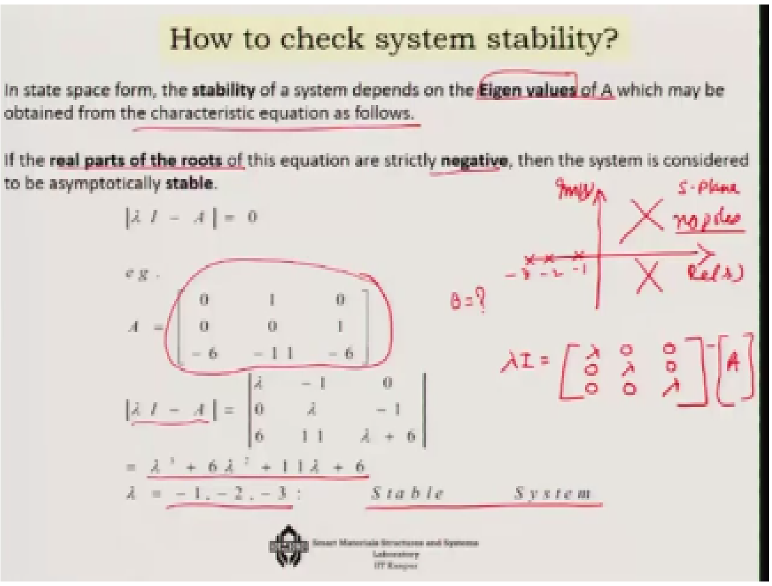

No with that we need to see a few things. For example how do we check the stability of a system. Well in state space form the stability of a system will always depend on the Eigen values of A which may be obtained from the characteristic equation. If you remember that when we are talking about these transfer function best approach then the roots I was always telling that the real parts of the roots should not go, should not become positive.

It should not go to the right hand side, that means if I plot the S plane if you remember a real part of S verses the imaginary part of S. I said that no poles can be placed in the right hand side, no poles because then it will be unstable. Now in this state space form this pole is to be replaced by the Eigen values, otherwise it remains the same. That means the real parts of the roots are Eigen values are strictly negative.

Then the system will be considered to be asymptotically stable. Let us consider you know a kind of system like say this A, this is a system if you look at it that this system looks like as it in the control canonical form, but we cannot guarantee because we have not seen the B yet. Only when we see the B also then only we can say yes it is in control canonical form.

Now anyway our job here is to find out what are the roots, what are the Eigen values of this system where we can find out what is the determinant of this system, that is lamda i-A, keep in mind that lamda i actually means I is a identity matrix, so lamda 00, 0 lamda 0, 00 lamda. So I can put that as lamda i-whatever is my A matrix. If I do that you know thing I am getting this lamda i-A.

And if I do find out the determinant I am going to get this polynomial, which is a cubic polynomial of lamda. Now by using that level by using other formulations you can actually (()) (18:13) options you can find out what is lamda and you will see that this are the 3 roots of lamda and in that means that this the stable system because the lamdas are all in the left half plane. So-1, -2 and -3. So that means this is going to be a stable system.

So that is the way we can first of all check the stability of a system. Next what we can check is actually what is this are called you know the control canonical form of the system.

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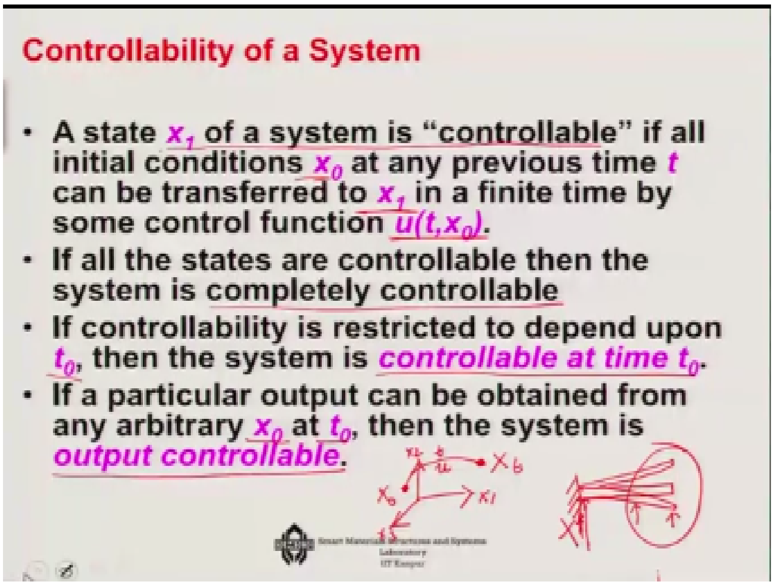

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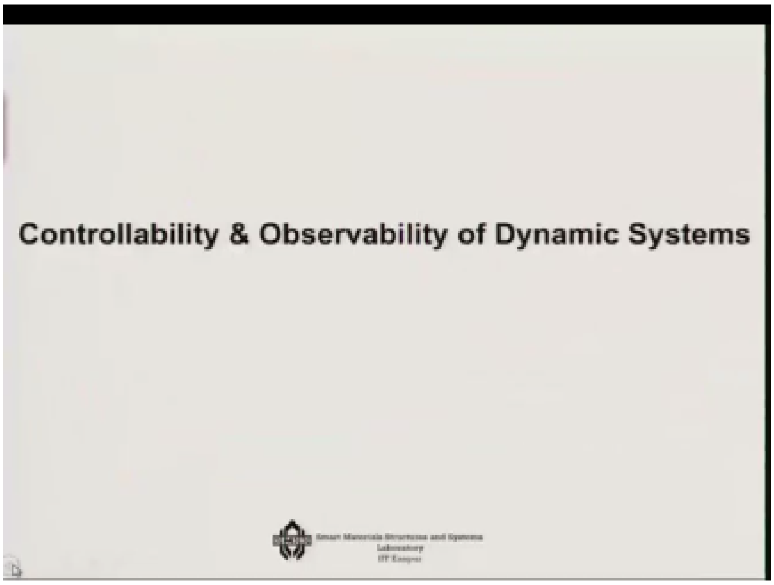

Next we want to talk about the controllability and observability of the system. Because particularly we will talk about the controllability and also in observability will check as a side by side with respect to the observer canonical form. Now let us look in to this forms. So how do you define the controllability of a system. The controllability the term itself is telling us that if we have a system were I have some actuator through which I am trying to control the vibration.

The vibration should be this actuator should be placed in such a manner that it can control the vibration. For example that we has give something like a physical significance of what we are talking about, suppose this is a cantilever beam and this beam is vibrating. Now if you want to you know control this vibration of the cantilever beam you have to place actuators. So if you place an actuator here you will be able to control.

If you place it somewhere here the effort will be increasing what still you will be able to control. If you place it very close to the structure you may be need much more force and finally if you were here on the (()) (20:16) point here itself it will be impossible for you to control. So what it means is that based on my position of actuator the controllability of the vibration mode will be actually you know determine that whether it is controllable or not.

Now for you know a particular state x1 of a system will call it controllable when if all initial conditions X0 at any previous time t, t can be 0 or any previous time can be transfer to a future x1 in a finite time by some control function utx0. And this also has to be a finite control effort. So that means suppose we are in a 3 dimensional thing let us say so I have you know a position here and this is what let say my x0 position with respect to 3 steps suppose ok.

X1 and x2 and x3. Now from here I am going to some location here in this states space which is some xt. If I can go from these state to this state within a finite time t and with a finite control effort u then only we will say that the systems controllable and this is simply not possible in this case that I have discuss that within a finite time if you place your actuator here you will not be able to control the vibration of the system.

So if all these states like that are controllable then the system is completely controllable. If controllability is restricted to depend up on t0 then it is called controllable at time t0 that is within this time it is controllable and if it is less it is not if it is more, it will be. If a particular output can be obtain from any arbitrary x0 at t0 then the system is called output controllable. So these are some of the subdivisions of controllability of a system. So that is the definition of controllability.

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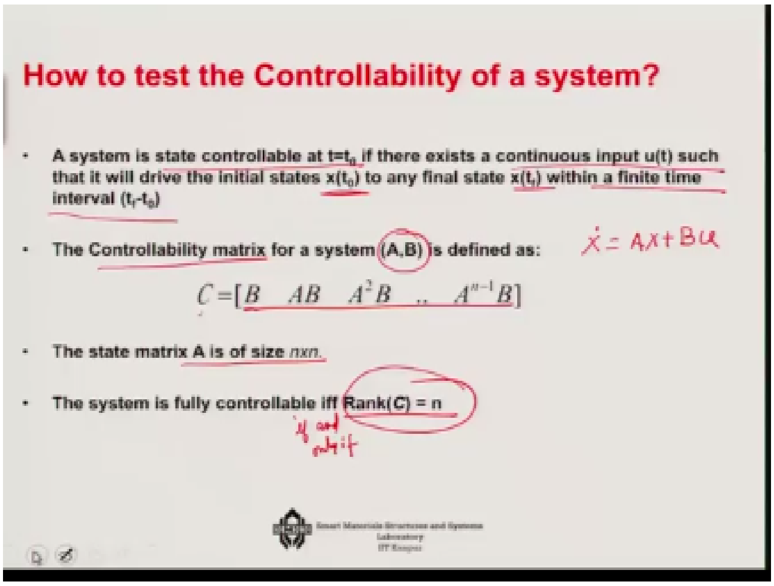

How do I test the controllability of a system and that is simple if you have a controllability matrix called AB and then you can if you have these you know A and B from these states space representation x dot equals to Ax+Bu. You can develop your controllability you know matrix from this AB with the help of these particular formation of the matrix which is B AB A square B2 An-1B where A is of size nyn and the system is fully controllable if and only if the rank of CX equals to a number of the states.

We will then say yes the system is full state controllable from t equals to T0 to any time t such that you will get a continues input ut and it will be able to drive it from Xt0 to Xta within a finite time interval and with a finite energy. So by checking these particular condition if this is satisfied we can say yes it is controllable. For designing before designing any controller in state space control is controllability must be check always.

So that is the first condition. If you get it check then the transformation the inverse of the transformation will actually exist.

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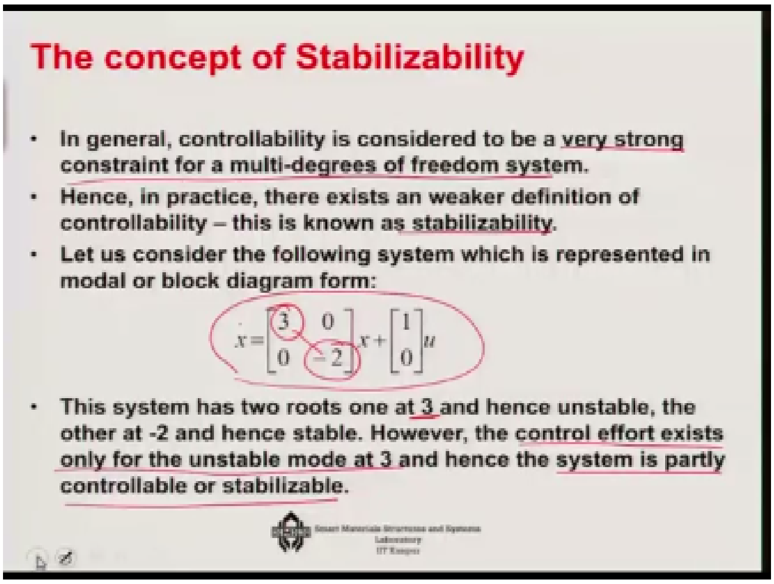

In some cases controllability is considered to be a very strong constraint for a multi degrees of freedom system. SO there is a weaker definition of controllability. We call it as stabilizability. For example if you look at this particular system X dot equals to X + Bu where A is like this 3,0,0-2 and B is 1,0. The system has 2 roots clearly it is only block diagonals one at 3 and another at -2. This root is unstable.

However, this root is stable. So the controller if the control effort exists only for the unstable mode at 3 and hence the system is partly controllable or stabilizable. So this is the you know partial controllability or stabilizability as a special subset.

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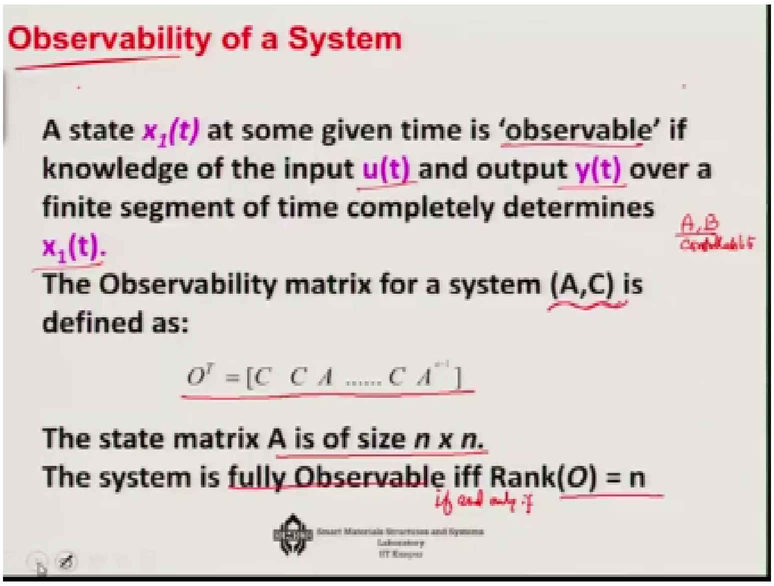

Now I told you that observer canonical form gives us that is the dual problem of it that is from the same thing prospective. That a particular state x1t at some given time is observable, if the knowledge of ut and output yt over a finite segment of time can completely determines the states x1t. So this needs this for controllability we needed the A, B for controllability if you remember.

And for you know observability we need the power of A C, we need the AC for the observability and this is define as OT equals to this particular thing you know C CA up to CA n - 1. This state matrix is considered of size n by n and the system is fully controllable if and only if the rank of O equals to n. The only this is same, is fully observable. So this are the controllability and observability are 2 dual forms.

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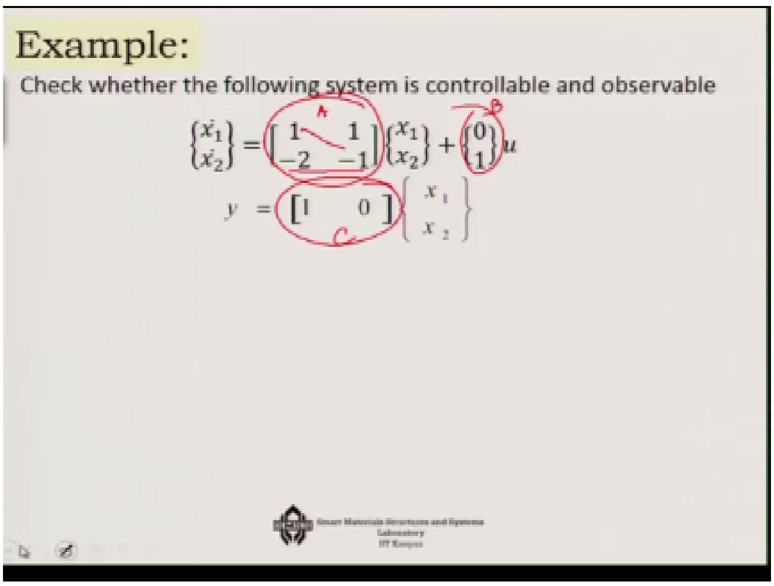

Now let us say I give you a very simple example that we have this X dot equals to Ax+Bu and this is the A matrix to y2 this is B and this is C. Ok from by looking at it we can say that it is you know which may be the control canonical form because these we have the coefficients here and only at the last row and here the super diagonals unity, rest terms are into there. So any way we can first find out that what is the controllability and then we can find out what is the observability. So this is A, this is B, and this is C. So we can find it out.

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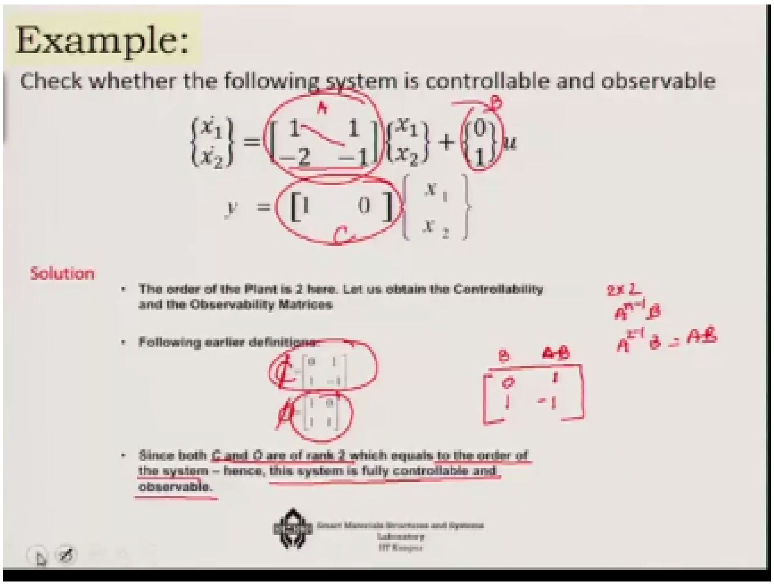

So this controllability is what it is B and then AB, so for us what is BB is 0, 1 and what is A, B, A times B that means first row first column, that will give you that 1 into 0 + 1 into 1 so that is 1 and then second row this column so -2 x0+-1 into 1 that will be 1-1x1, so that is my BAB, ok we do not have to consider the additional terms because A is of 2 by 2 size so An -1, B means actually A2-1B.

And that means AB. So we have to go only up to B AB and we get the controllability matrix, let us call it as C with a special pattern in it. So that we do not confuse it with this, so this is C. And similarly I can find out observability again the special pattern and observability is C, CAC and -1. So we get the observability matrix point.

Now if you check that for both C and O the rank is 2 which equals to the order of the system. Hence the system is both fully controllable and observable. That is just an example to actually keep tu with construability and observability. Thank you. In the next lecture we will talk about how to apply all these concepts together for full state feedback control.