Principles of Vibration Control
Prof. Bishakh Bhattacharya
Department of Mechanical Engineering
Indian Institute of Technology-Kanpur

Lecture – 10
Maxwell & 3-Parameter Model

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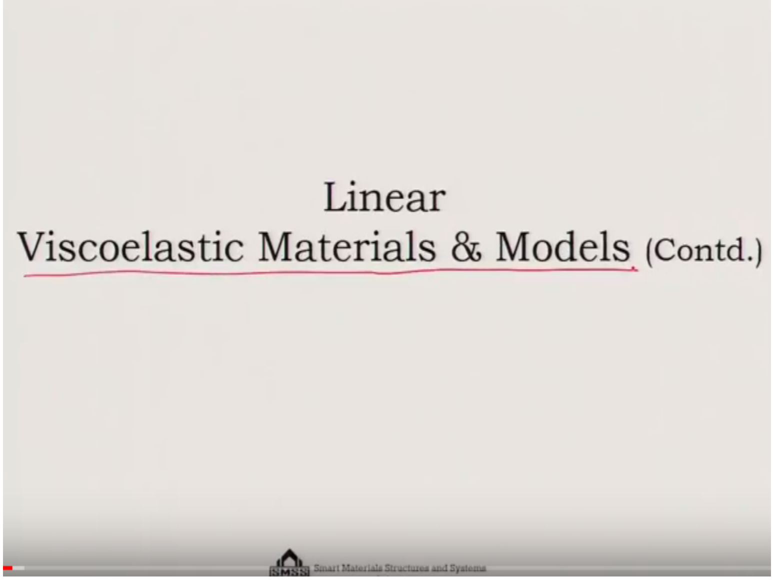

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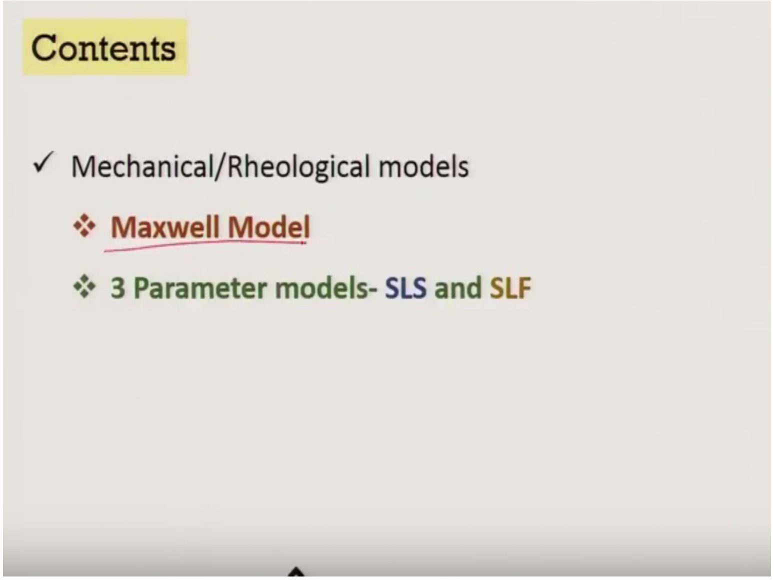

Welcome to the course on principles of vibration control and today we are in the tenth lecture, in which we will be continuing on our discussion on linear viscoelastic materials and its models, so basically we will talk about how we can develop mechanical models of viscoelastic materials. In this direction, I have already told about kelvin Voight model how you can keep a spring and dashpot in parallel to develop a kelvin Voight model.

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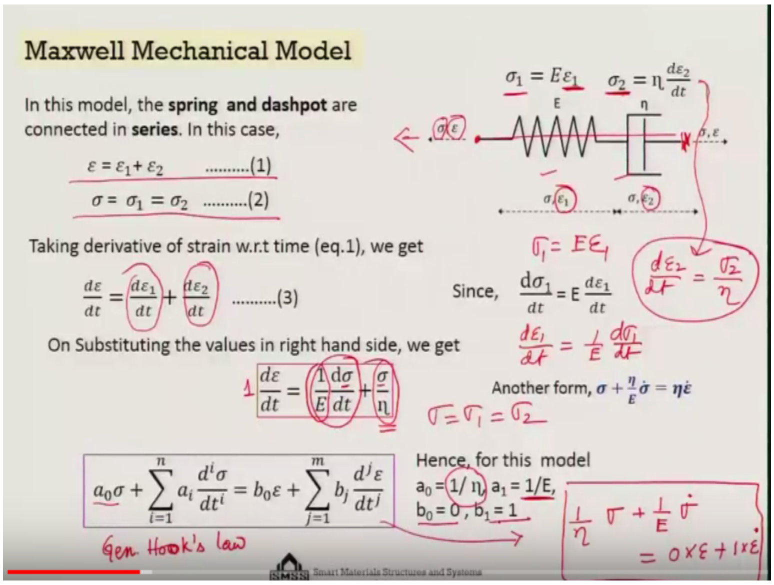

Today, we will show how you can develop a Maxwell model and then extending that further how we can develop 3 parameter models like standard liner solid and standard linear field models. SLS is our standard linear solid, short form and SLF standard linear field. So we will see how we can make all these models by following very similar principles, so first let us concentrate on the Maxwell model.

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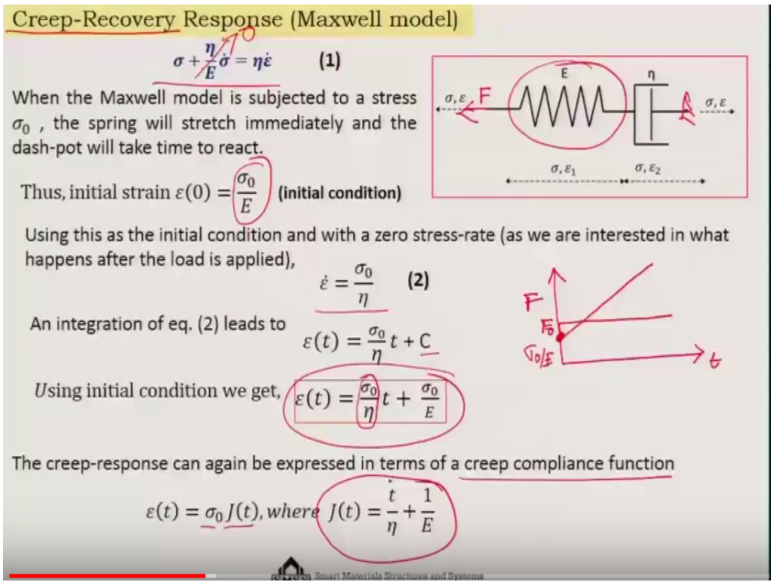

Now, in the Maxwell model, we have a spring and the dashpot, these 2 are in series with each other, so let us imagine that we have applied a force, which is causing a stress sigma and a overall strain () ok in the spring dashpot model, so suppose if I fix it one side, apply a force then the stress is sigma and the strain is Epsilon. Now, there are 2 things we need to think of it, one is that since there is no other connectivity from input to the output.

So the same stress will be there in both the elements, both in spring, as well as in the damp part, so that is what is our second equation says that this sigma is the same as sigma1 and it is the same as sigma2. The other thing is that the above the string, it is just the reverse of kelvin Voight model. In this case, the total strain epsilon is actually the strain that is shown by the spring and the strain that is shown by the damper, which is epsilon2.

So our epsilon, the total strain that is given by the first equation is actually epsilon + epsilon 2. Now, if I take the derivative of the first equation then the rate of strain is actually D epsilon DT that is the rate of strain in the spring element and D epsilon 2 DT that is the rate of strain in the damper element. Each one of them, of course we can very easily find out because we know that sigma equals to E epsilon, which means sigma1 = E epsilon1, so D Sigma 1 DT is ED epsilon 1 DT.

So that is something that we know and the other one D epsilon2 DT for that, we can use this model and from, which we can say basically that D epsilon DT is nothing, but sigma2 over ETA, so then we can actually write this equation 3 by substituting all these things in the right hand side, the D epsilon, theta DT is actually D epsilon1 DT and that is written in terms of from this one, a little bit of simplification, the D epsilon1 DT is 1 over E D Sigma 1 DT.

So that is our first part and the second part is D epsilon2 DT that we are taking for the viscoelastic material from this relationship, the D epsilon 2 DT is nothing but sigma 2 over ETA, now since your sigma is nothing but sigma1 and sigma2, the same stress is flowing through the entire system, so the stress is not changing. Hence you can instead of sigma1, Sigma2, you can denote them as sigma herself.

So you get the final relationship very straightforward, the D epsilon DT is 1 over ED Sigma DT + sigma over ETA. I already told you that the fundamental definition of generalized, so this is the generalized Hook Hooke's law, ok so this is the generalized Hooke's law and in that if I try to fit this model, then naturally our A0 sigma that A0 will be 1 over ETA that is this term and then our A1 D sigma DT only up to fast derivative we can go.

A1 will be nothing but this 1 over E, so that is what is 1 over E, and in the right hand side, B0 there is no constant term there, so that will be 0 is 0 and B1 is actually unity, so B1 is 1 here, so thus this generalized Hooke's law in this format if I have to write, then I have to write it in a manner that this equation will simply become 1 over ETA sigma + 1 over E sigma dot and that equals to where the sigma dot means D Sigma DT and that equals to 0 times epsilon.

Because the first one is constant + 1 times epsilon dot that is what will be relationship in the form of generalized Hooke's model if you apply all these parameters, so this is all what is our Maxwell mechanical model. Next, let us try to look into it that ok, we have proposed a model how this model is going to behave with respect to 2 things, one is the creep recovery and another is the stress relaxation.

Now when we will talk about creep recovery, you have to keep in mind that we are going to apply the force only once and then we will keep it steady that means it will behave like a heavy side function, so if I try to plot the T versus the load F suppose you know I fix this site and I apply the force F, then the force F will be of this type in nature that means it is like a step function, ok .So some force F0 and then it will remain constant, so our Maxwell model equation is this which we have already declared and we are applying some force.

If we apply the force what we will see is that the spring is going to deform instantly, so there will be some strain in the initial strain in the system, which can be denoted as Sigma 0 over E. Now that is my initial condition and then you will see that the stress is not changing anymore, so there is no further stress rate that will come into the picture, which means this part will not be coming into the picture.

So you can write epsilon dot equals to then whatever is the initial stress Sigma 0 over ETA and then you can integrate this equation in terms of Sigma 0 over ETA T + C that is the integration constant and apply that at T equals to 0, your epsilon 0 is Sigma 0 over E, so you apply that condition then you are C will be Sigma 0 over E that means your epsilon T Sigma 0 over ETA T + Sigma 0 over E.

So if I have to plot that with respect to time, it will be something like Sigma 0 over E as the initial value let us say somewhere here in the same plot and then it is going to be you know it comes to a uniform slope, where constant slope line and the slope is Sigma 0 over ETA that is the slope of the line with that it will continue, so this is the solution that Maxwell model is going to tell us.

In fact you can write it in terms of a creep compliance function such that this JT, so if you write Sigma 0 in one part and the rest are the time varying part, then the JT time varying part, the creep compliance function will take the shape of T over ETA + 1 by E showing thereby that it has a linear you know creep compliance function is linear in nature with respect to time, so this is the relationship for us.

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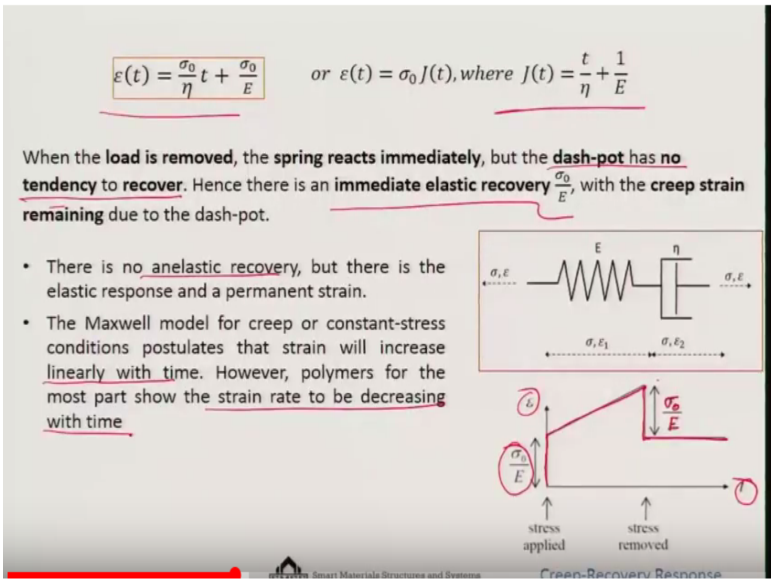

And this is the JT that we have discussed. Now what will happen if the load is removed, then the spring will react immediately, but the dashpot will have no chance, no tendency to recover that means this is a silent and time plot, so you have the initial strain Sigma 0 over E and then as I told you that it will increase after some point, if you immediately release the load that part will be recovered.

So that is the recovery part of it Sigma 0 over E and then it will continue, the plastic strain will continue, so there is an immediate elastic recovery that will happen with the creep strain remaining due to the dashpot, there will be no an elastic recovery in this system, but there is only elastic response and the permanent strain. Now, this Maxwell model therefore postulates the creep or constant stress condition that strain will increase linearly with time.

However, polymers for the most part show the strain rate to be decreasing with time, so as a result this does not match with our usual you know type of response that we find in the polymers, so Maxwell model is actually unfit as far as the creep response is considered. How about the relaxation response, so if you look at the, this is the creep recovery part of it?

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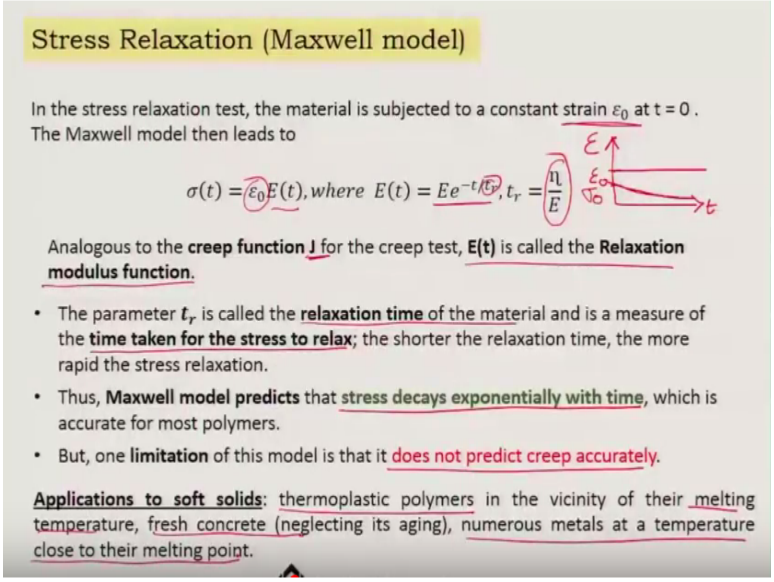

But if you look at the relaxation part maybe I will just talk about the stress relaxation part and then, there ok what we are giving is that we are you know subjecting it to a constant strain epsilon 0 at T equals to 0 and then we are actually measuring that how the stress is changing that means with respect to time if I plot now not the stress, but the strain so you are giving epsilon 0 and you are holding it.

And you are trying to measure that what is the change of stress in the system. Now, we already know from the model that Sigma T is epsilon 0 ET, where ET is E raised to the power - T by TR and this TR is actually ETA over E and in analogous to the creep function J, our function here is the relaxation function ET, the relaxation modulus function and this time TR is very critical.

The TR is called the relaxation time of the material, which is a measure of the time taken for the stress to relax and to come down to 0, so what we expect is that at T equals to 0, we expect that this E would be you know E to the power this thing will be 0, so you know E to the power 0 will be unity, so you will get a constant and you will get the epsilon 0, so that means you will get some constant value and then so if I try to plot the stress.

So it will be something like sigma 0 and then it will be constantly coming down, so that is what will be the nature exponentially it will come down, so thus Maxwell model predicts that stress decays exponentially its time and that is found accurate to be most of the polymers, so the limitation of Maxwell model is that it does not predict creep accurately, but it does predict relaxation accurately.

And this is you know observed in thermoplastic polymers in the vicinity of their melting temperature, fresh concrete as you neglect the aging or numerous metals at a temperature close to their melting point, you will observe this type of furniture of stress relaxation, so the Maxwell model can be applied in such a case.

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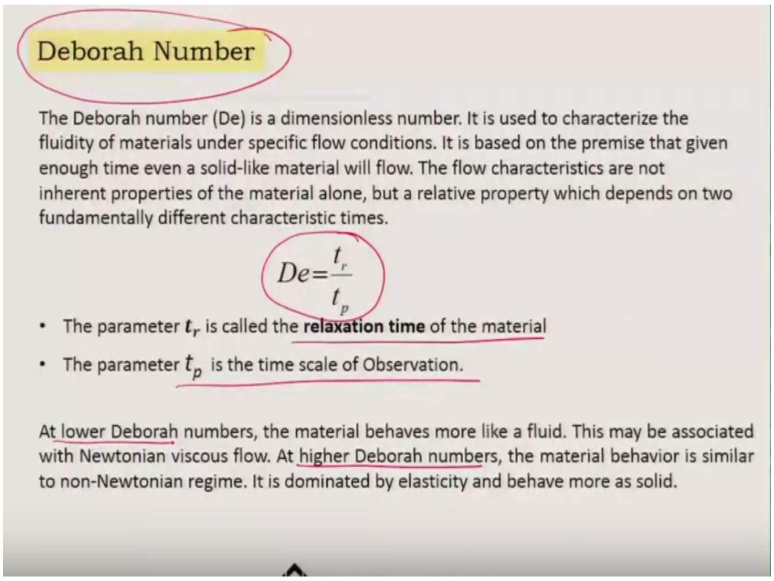

Now this relaxation time TR that can be actually further used in terms of a number that is called Deborah number, so it is a dimensionless number and it is used to characterize the fluidity of materials under specific flow conditions and it is based on the premise that given enough time every solid will behave like a flow like manner and this number is defined as the ratio of TR over TP, where TR is the relaxation time of that particular material and TP is the time scale of observation that is important.

That suppose you find something is very much solid like, so that means its Deborah number is actually very high and if you see that it is behaving like a flow that means you know the Deborah number is actually quite low, so the Deborah number will be low, which will signify if there is a low Deborah number that may signify that for the same relaxation time, your time of observation is quite high, let us say instead of measuring in terms of seconds, you are measuring in terms of months or years or thousands of years, you know.

So then you will see the flow like behavior, so this is another interpretation of the Deborah number.Now with this basic you know mechanical model, we will actually go to slightly higher order model, which is known as a 3 parameter model and you would see that the limitation of Kelvin Voight model that it does not describe stress relaxation and the Maxwell model that it does not describe creep recovery. This can be actually answered with the help of this higher parameter model, which are used to predict both the phenomena.

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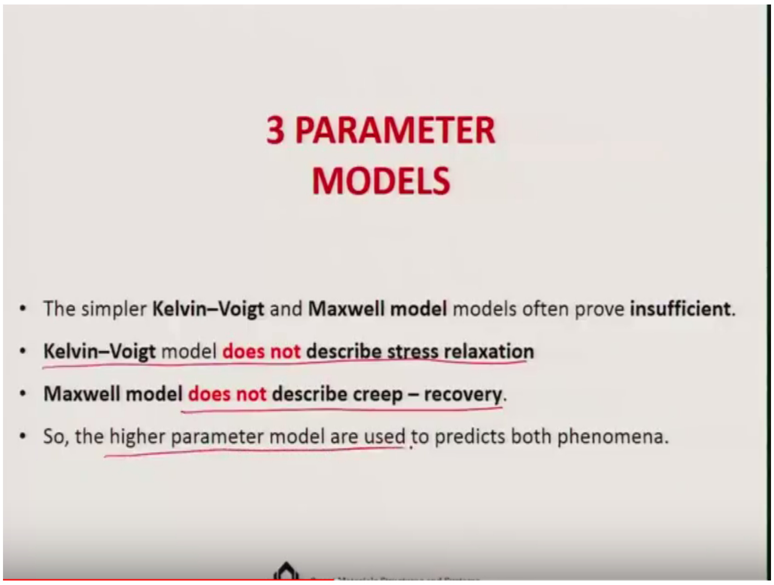

So we will try to find out the formulations of some of these models now. As 2 parameter models cannot explain creep and relaxation appropriately, so we are going for higher order models and hence we will discuss about 2 different types of models, one is 3 parameter models and another is four parameter model. You can even go for even higher order models, but at least this will be good enough for most of the observations.

Now in the 3 parameter models, we will have 2 variations of it, one is called Zener model and another is called anti-Zener model, Zener model looks something like this. There are 2 types of Zener models, one would look symbolically as 2 3 elements we have to have right, so we have a spring and we have a damper followed by a spring, so this is one variation of the Zener model.

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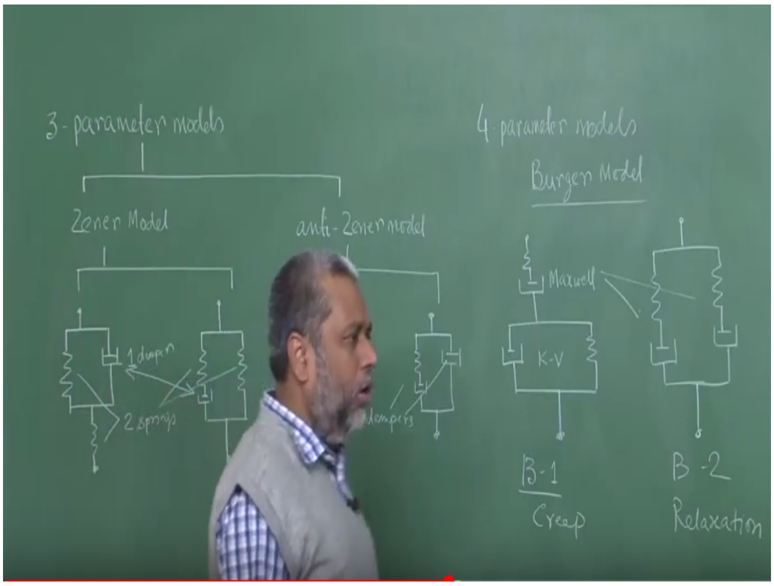

The other variation of the Zener model is something like these that we have 2 branches to begin with. One spring, one damper, so one branch is like the Maxwell model, which we have just now discussed and together it is like a Kelvin Voight model and it is coming out as the output, so one of the Zener model is like Kelvin Voight and then another spring in series and here it is Maxwell and another spring in parallel, so these are the 2 Zener models.

These are the 2 Zener models and then we will look into the anti-Zener models. This also will have 2 variations in it. One variation is that these variations are in terms of the damper now, so here in these 3 parameter models, we have used one damper, in both the cases and 2 springs in both the cases.

However, in this model, anti-Zener models we will 2 dampers and one spring, just the reverse that is why it is an anti-Zener model, so you can see that there are 2 dampers and one spring. Actually there is one more variation possible of it and that variation would look something like this.

So here also you have 2 dampers and you have one spring, so these are Zener and anti-Zener model. If I go for four parameter models, then the model that generally used is called burger model and there are 2 variations of the burger model here, a spring and a dashpot together that is the Maxwell element and then we are branching out and we are going to have one damper and one spring together, then we are coming out.

So that means we have a KV, we have a Maxwell, we are joining together to get the fist Burger model B1. The second Burger model would look like that means you have 2 Maxwell models in parallel, both are Maxwell here. You have 2 Maxwell models in parallel like as a KV form. So this B2, once again B1 model is used mostly for creep and B2 model is used mostly for relaxation. So these are the 3 parameter and four parameter models.

Now if the parameter increases, how am I going to develop the generalized Hooke's law or the constitutive relationship? Let us look into one simple case, which you can later on work for all other cases, so let us try to work on one simple case, which we will call as the standard linear solid model.

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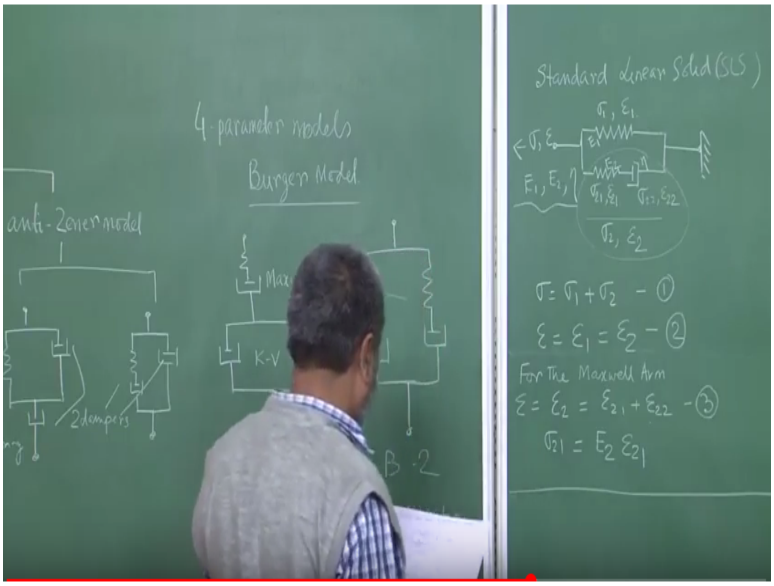

Standard linear solid SLS, it is a form of a Zener model and this will be something like let us try to draw it, so we have 3 parameters here. One spring and then you have another spring and a damper together, so this is the model. So as you can see here that you have 2 springs and one damper, which means that it will behave like this, a Zener model, which has 2 springs and one damper.

So this is the standard linear solid model, SLS model and in this model, let us say the stress here, overall stress is sigma and epsilon as I am applying a force F and let us say I am fixing it here, easier for us to visualize, the stress is going to be divided into 2 parts, one is this part, so here the stress is Sigma 1, strain is epsilon 1, and here overall the stress is Sigma 2 and epsilon 2.

But it is divided into 2 parts here, so in this case it is let us call it Sigma 21, epsilon 21 and let us call this case here as Sigma 22 and epsilon 22, so thus we have defined all the basic parameters, so let us quickly see what are the relationships that we can get from here. First of all is that this sigma is actually sigma 1 + sigma 2, so you can write it down that sigma by studying this sigma is sigma 1 + sigma 2 and what is epsilon.

Epsilon is actually the strain same because it is a parallel model, so it is epsilon 1 and that is epsilon 2, that is what is my second equation. Now for the Maxwell arm that means for this arm, let us try to close on it further, so for the Maxwell arm, one is that epsilon, the strain itself or epsilon 2 whatever we will call it, this is nothing but epsilon, we have already said and that equals to epsilon 21 + epsilon 22 because it is a Maxwell part of it.

And you also can say so let us put this as equation 3. We also can say that sigma 21, in this part it is like a spring, so it is E2 epsilon 21, so let us say that this one has a modulus of E1, this one has modulus equivalent of E2 and this one has a damper equivalent of ETA, so ultimately we have to express everything in terms of E1, E2 and ETA because these are the 3 parameters ok, E1, E2 and ETA.

These are the 3 parameters that we have, so ultimately you have to express ok, so we have expressed sigma 21 as E2 epsilon 21, so let it be our equation number 4 and also we can write sigma 22 and Sigma 22 is actually ETA epsilon 22 dot and that is our equation 5, so we have all the basic equations with us, all the basic constitutive relationships with us.

Regarding Sigma1, I did not write, but we can write that also that Sigma 1 is E1 epsilon 1, so let us try to now quickly see that how we can apply this whole thing first of all in Laplace transformation LT ok and you should be knowing that the Laplace transformation of FT is denoted as F bar S and also you should be knowing that if it is a Laplace transformation of F dot T that is a derivative with respect to time.

Then you can write it as S F bar S - F bar 0, so in this case we will use the 0 initial condition, so this will be 0 here and so, in this case if we convert these equations, for example four and five, if we use the Laplace transformation then we can write sigma 21 bar now that equals to E2 epsilon 21 bar and sigma 22 bar equal to ETA S epsilon 22 bar that means we have used the 0 initial condition here.

So sigma 22 is ETA S epsilon 22 bar, so we can use these 2 things together in our equation, so we can we now know that epsilon 2 bar, we can write now as with the help of this we can write now that epsilon 2 bar that is the overall epsilon strain, so epsilon 2 bar is actually the lap lass transformation of epsilon 2 itself, so epsilon 2 bar will be epsilon 21 bar + epsilon 22 bar, so it will be epsilon 21 bar + epsilon 22 bar.

And by using these 2 equations, we can write that epsilon 21 bar is sigma 21 bar over E2 + sigma 22 bar over epsilon 22 bar, so it is ETA S, so that is what is our epsilon 2 bar, in fact I can also write it as a little bit cleanly one over e E2 +, one over S ETA times Sigma 2 bar. Of course, we have to keep in our mind that sigma 2 bar and sigma 1 bar, so this summation is actually Sigma bar.

So we have to use these at the second stage now, so we can write here from this we can simply write that using this relationship, sigma 2 bar is actually 1 over E2, so maybe we can actually write it in one shot, you can do the algebra on your own, so we can see the reciprocal we can take and write it as S ETA E2 over E2 + S ETA just carrying out this whole thing directly times epsilon 2 bar, so this is what is our first part of the thing.

We can also see here that for this entire part epsilon 2 bar is epsilon 2 and epsilon 2 is epsilon1 and that is epsilon, so we may also write that this Sigma 2 bar is actually S ETA E2 over E2 + S ETA epsilon bar because the strain is the same, so we can use that and let us now use the last equation that is Sigma equals to Sigma 1 + Sigma 2.

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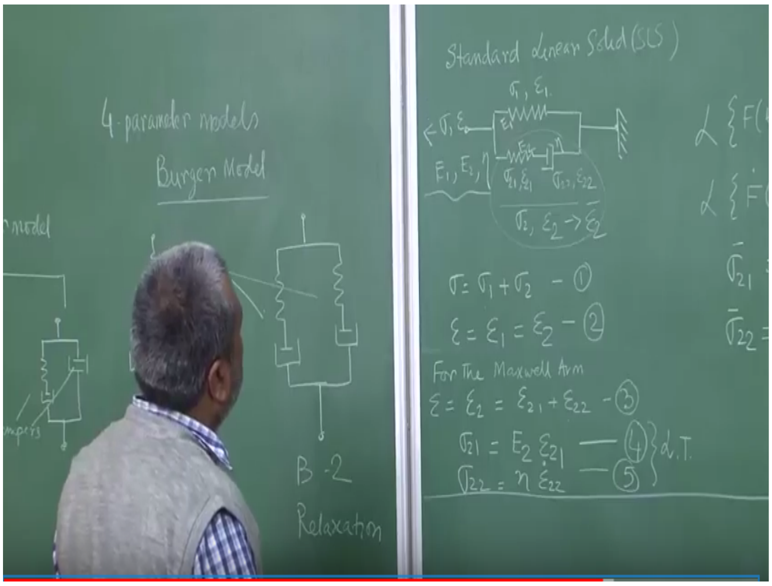

So I am just erasing this part now, the middle part because these things we have already obtained, so our one relationships we know is this relationship that is what is Sigma 2 bar and now we know that Sigma bar is Sigma1 bar + Sigma2 bar ok that is the Laplace transformation of the first equation, so from this first equation we get this.

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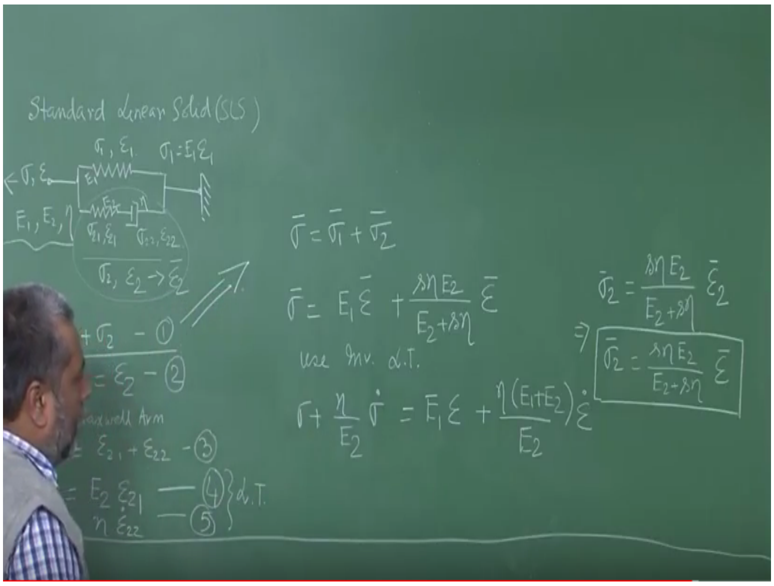

So we can write Sigma bar as what is our Sigma1 bar. Now Sigma 1 is simply E1 epsilon1 is a simple string element, so actually it is Sigma1 bar can be written as E1 epsilon1 bar and Sigma 2 bar can be written as S ETA E2 over E2 + S ETA epsilon bar. In fact, epsilon1 bar is nothing but epsilon bar, so I can even remove this one now, so we are very close now.

The constitutive relationship is already there, but it is in the frequency domain, so we can convert it to time domain now and if I convert it to time domain, so use inverse Laplace transformation then this would become we have Sigma + ETA over E2, so you can carry out this on your own, Sigma dot and this will become you want epsilon + ETA E1 + E2 over E2 epsilon dot.

So remember our generic Hooke's law was something like A0, so A0 is actually unity here and A1 is actually ETA over E2 and B0 is E1 and B2 is ETA E1 + E2, ETA E1 + E2 over E2, so we can very nicely fit the whole thing in terms of a generalized Hooke's law, so thus the same way so that means in this case all you have to know is this E1 is this E2 and is this ETA so similarly for the other model also you can do the same thing and you can find it out just by using this technique you want E2 and ETA.

And also you can carry it out for all other anti-Zener models and four parameters models. I have just shown how to do it for the one of the Zener models that is the standard linear solid model, so let us summarize the whole thing now. We will now summarize all our observations in the following Slides that this is the standard linear solid model, which we have just now derived. And compare it with the Hooke's law, so this is one model and.

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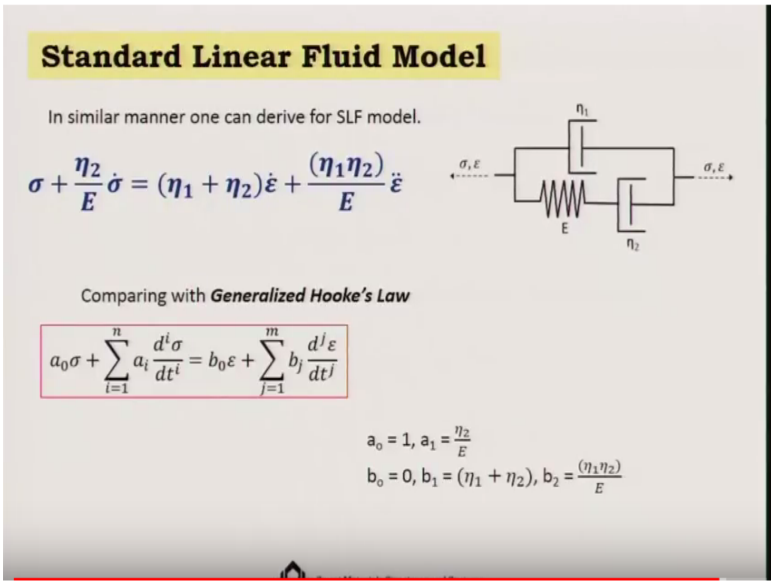

Then this is the standard linear fluid model that is where you have 2 damper and one spring, so that is something like one of the anti-Zener models and in this case, the relationships you can derive this is the way it will come out and you can fit it with the generalized Hooke's law with these values. You have to keep in mind that all the higher-order values A2, A3, these are all 0s and similarly B2, B3, all higher order terms are 0s that is for standard linear fluid model.

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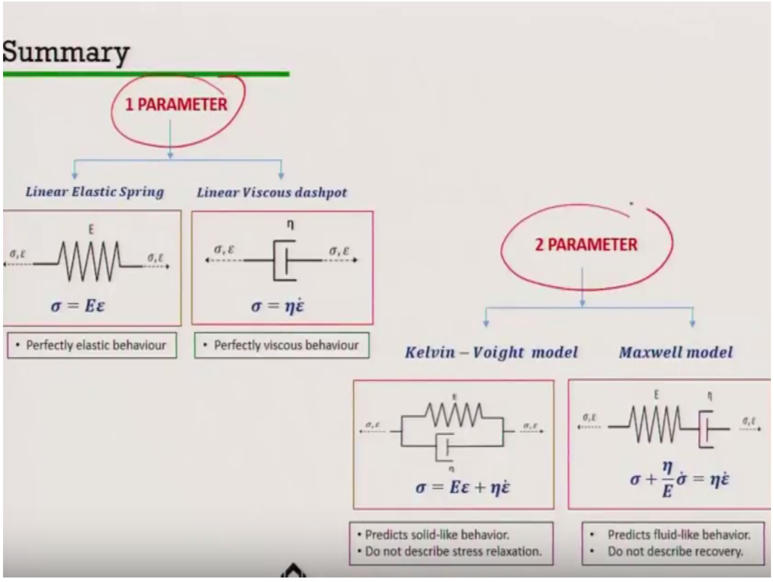

And further also, so our complete summary is that we have discussed about one parameter models, we have discussed about 2 parameter models, Kelvin Voight model and Maxwell models, so the linear elastic one was perfectly elastic, linear viscous was perfectly viscous fluid, Kelvin Voight model predicts solid like behaviour, some amount of viscosity is there, do not describe stress relaxation and Maxwell model predicts fluid like behaviour, do not describe the recovery.

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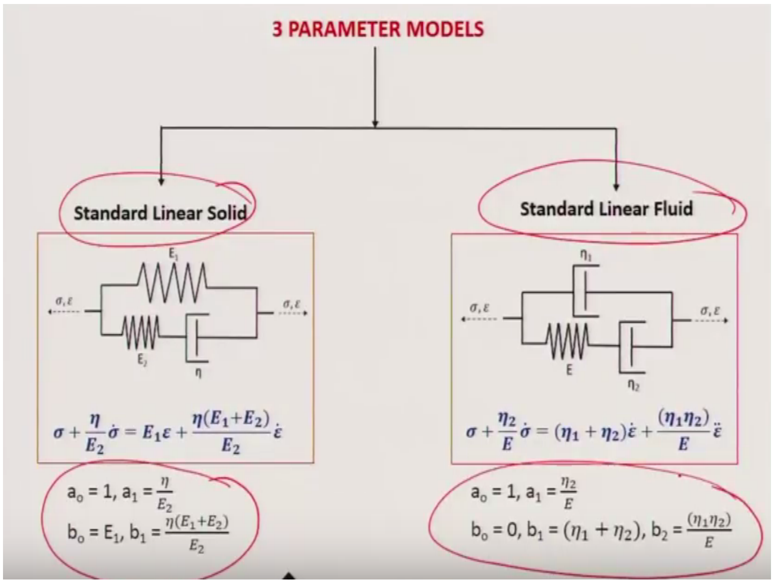

In order to do that, we have actually worked for higher order models. These are the 3 parameter models, some of them like the standard linear solid model and standard linear fit model and even if we get a nature, which is not satisfied by these things, then we will go for

Burger's model, which we have described and these are the relationships of the various parameters in these models.

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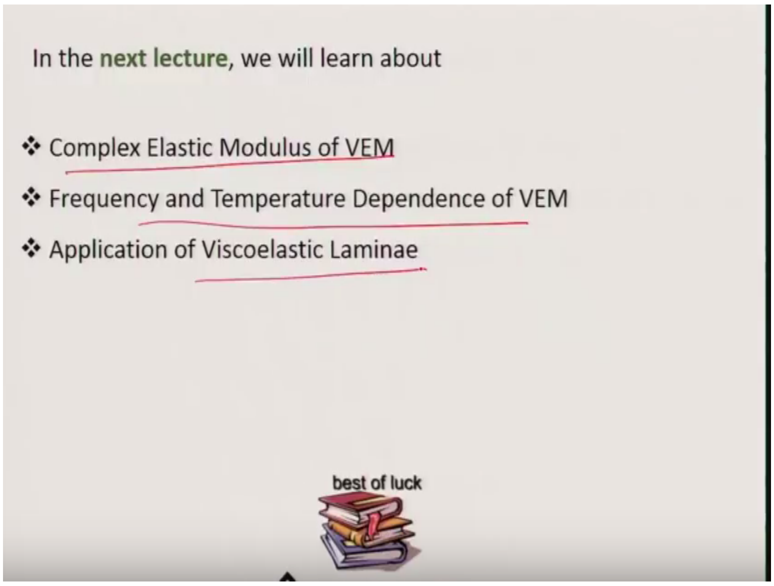

So with these hopefully you will be able to model viscoelastic materials properly with the help of any of these material models, so this is where we will come to an end. In the next lecture, we will talk about complex elastic modulus viscoelastic material, frequency, and temperature dependence of viscoelastic material and application of viscoelastic laminae. Thank you.