Basics of Finite Element Analysis – Part II
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Lecture – 09
Convergence and Accuracy of Solution (Part -I)

Hello. Welcome to Basics of Finite Element Analysis Part II. Yesterday and day before yesterday we have been discussing different types of errors which exist in finite element solutions and also how these errors can be measured and quantified in terms of a single number. Today what we are going to discuss is that how do finite elements solutions converge that is they become closer to the exact solution and on what parameters they depend and to what extent they depend on these parameters. So, our theme for today is convergence.

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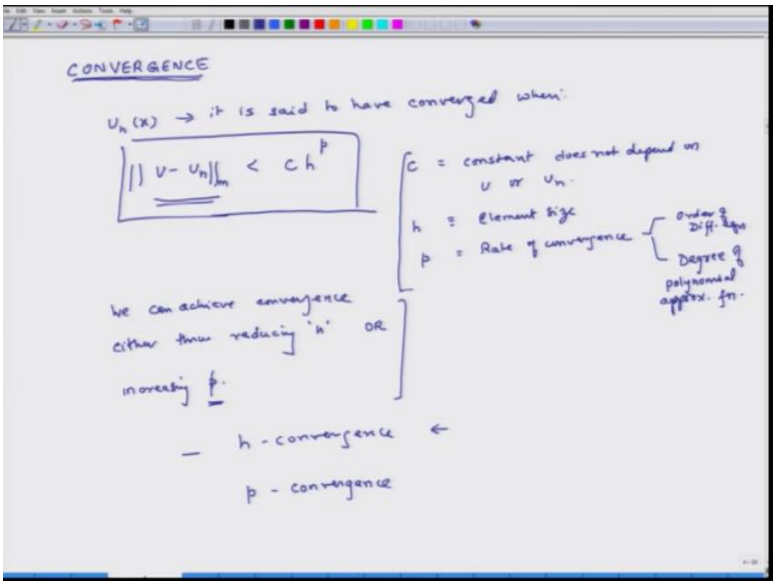

So, first I will give you some brief idea what is convergence; if I have a finite element solution, and with five elements it will have some difference with respect to the exact solution. If I use the same approximation functions, but double the number of elements, the error will go down. If I make the number of elements four times, error will go down further and as I keep on making my finite element solution. So, as I keep on increasing the number of elements and keep on making my solution more and more accurate the error reduces and this process of reduction of errors is designated as convergence because our finite element solutions is slowly or fastly depends on how good our algorithm and approximations are, but essentially the finite element solution is converging towards the actual solution. So, this process of convergence of finite element solution, towards the exact solution is known as convergence.

Whenever we do finite analysis for any problem we have to make sure that our solution has converged. It is important to understand that be and what that means is; that once the solution has converged that we can be reasonably sure that our solution is fairly close to the exact solution. If we are not sure whether the solution has converged, then we cannot rely on those finite element solutions because we do not know whether those finite element solutions will reflect the reality or not.

So, that is why convergence is important. Now in a mathematical sense, the solution u h of x is said to have converged. So, it is said to have converged it said to have converged, when, the norm u minus u h m. So, this is the max norm and we have already defined what this max norm is. So, the definition of max norm is given here. I am sorry, this is the energy norm, sorry; so this definition of energy norm is given here.

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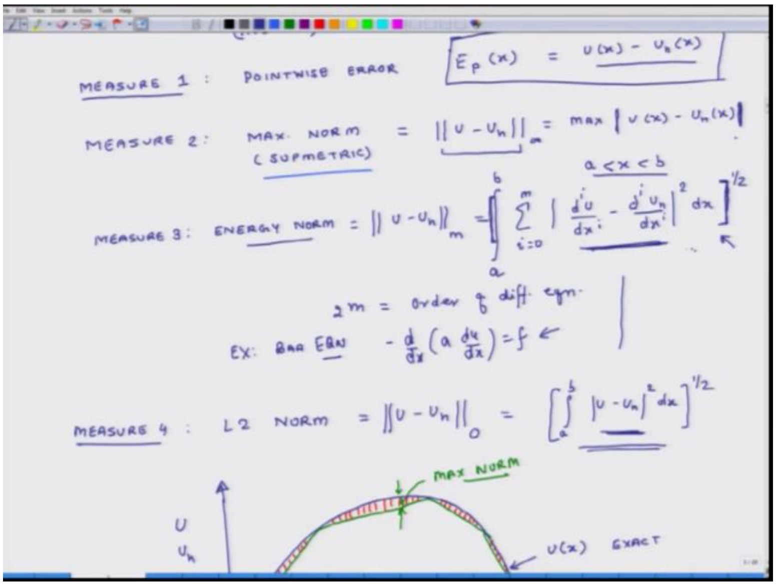

So, when this norm is less than a number c h to the power of p.

And here c is a constant and it does not depend, depend on u or u h. h it depends on element size. So, it may need not be the exact element size because you may have a mesh with all sorts of element sizes, but it depends on the element size. So, overall if your element size is small, then h is small and so on and so forth and p is known as rate of convergence, and it depends on two things it depends on order of differential equation and it also depends on power of or degree of polynomial function. So, it depends on degree of polynomial approximation functions. If we use a linear function, it may some value use quadratic function. It may have, but so it depends on these two orders of differential equation and degree of polynomial approximation functions.

And we will see these things a little bit more in detail may be later tomorrow or day after tomorrow, but at this stage it is important to understand that convergence is said to have happened when the energy norm is less than equal the c times h to the power of p. Now what this shows is that I can ensure convergence either by playing with h, right or y by playing with p. So, we can achieve convergence, either through reducing h that is making our element sizes smaller or; increasing p, or increasing p.

When I reduce h what happens? The elements size becomes smaller. So, my domain approximation error start shrinking, and also, if a function is if a variable is changing in a very complicated way over the domain, if h has gone down then it will achieve accurate more accurately capture that variation even if it is a linear function, it will more accurately capture that variation. So, that is how h by reducing h, we can make the convergence faster by having more number of elements.

When I increase p it means that the degree of an approximation function has gone up which means complicated variation of variables over the domain are accurately captured. So, that also in makes our convergence faster. So, the process of convergence, process of convergence of the solution is of two types - one is h convergence. So, in lot of literature, they say, this is the h converge solution which means that convergence was obtained by reducing the element size progressively.

So, we start with ten elements, then you do twenty elements and you find that the difference in solutions, because you do not know the exact solution, that is why you are solving the problem. So, how do you achieve h convergence? Let us say you have a domain, you break it up into ten elements and you find the solution. Then you break it up into twenty elements, you again find the solution, and find the difference between those two solutions. Find the difference, may be, by finding out this energy norm. So, for 20 element solution and 10 element solution, let us say the difference is ten percent then you go to 40 elements. Let us say the difference becomes half a percent. Then you go to 50 elements, may be the difference goes down to 0.1 percent.

And if you thing now that the differences are not changing increasing, difference is very small, then you say that then you say that your solution is converged. So, in this approach, you have got convergence by reducing the element size progressively. So, this is called h convergence. And then you have p convergence, where the number of elements is not changing, but what you are doing is that you are gradually increasing the degree of the polynomial approximation function. So, you start with elements which may be linear in nature, then you go to quart quadratic elements, then you go to cubic elements, may be then you go to quadratic elements, fifth order elements, sixth order elements and so on and so forth.

So, here the mesh is not changing. The discretization of the domain does not change as, but what is changing is the, this, the value of p by increasing the degree of polynomial approximation functions. So, that is there in reality we tend to do sometimes both, we increase h and we also increase p, we can have a mixer, but these are two fundamental approaches of convergence, h convergence and p convergence. In as we discuss this further, I would like to say that, there is one inherent advantage, if we go for p convergence, compared to h convergence.

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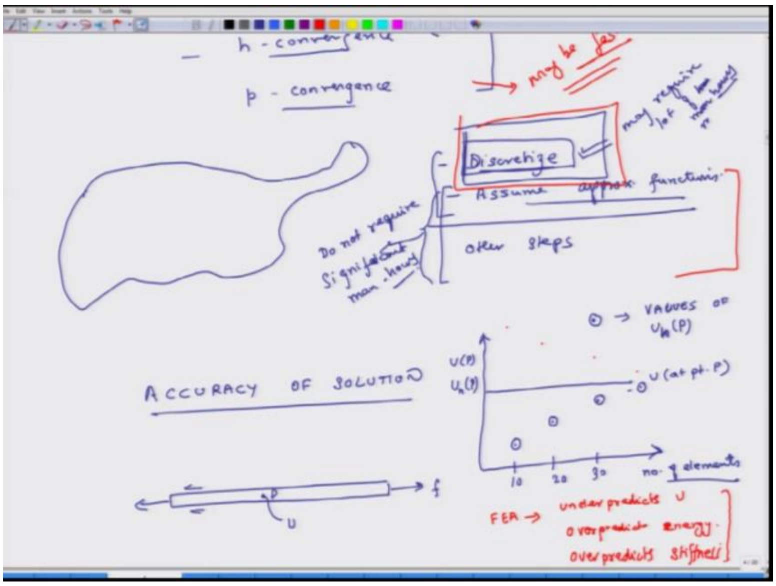

And the reason for that is I will explain that. Suppose, I have a complicated shape and I discretize it, so the first step is discretize. And then the second step is, assume approximation functions, right? When we actually conduct finite analysis, and if you are already doing that, you would already know, we spend a lot of time in discretizing the domain. We spend a lot of time in generating the mesh, because you have to make sure that the mesh is accurate, there are no elements which have extreme aspect ratios and there are several other criteria, we will probably discus that in subsequent weeks.

So, a lot of time, and by time, I mean man hours, it is not computer hours because once you submit to the computer, computer keeps on running. Then the human labor is not invested, but lot of labor is invested in meshing the geometry and for some times complex assemblies, it can days or even 3 4 5 6 weeks just to mesh the geometry and to get a very good nice mesh. So, lot of this may require lot of man hours, and then once the thing has been discritized then we can assume approximation functions and assign them and start the solution process.

So, all the other subsequent others steps these do not require many man hours, do not require significant man hours. So, if we have to ensure that the system is to converge we initially discritize it into hundred thousand elements then we discrite into 2 lakh elements then 3 lakh elements and each time we are discritizing it. We are investing a lot of man hours. We are investing lot of man hours this is. In other steps, the computer takes care of it. So, we do not bother too much now only thing which concerns us that we should not be just waiting for the results for weeks or months.

But the computer is doing our job. So, labor is not lost. So, if I have to do h convergence, each time I have to mesh it again and that requires a lot of man hours if I have to do p convergence I can mesh it once, make sure that the mesh is working correctly and then all what I am do is, I am increasing the order of elements. Oder of polynomial functions and that takes cares of the convergence.

So, because of that reason, a lot of some several cases, h convergence approach requires much more human time compared to p convergence approach and because of the same reason p convergence approach can be automated because once you have a mesh all you have to do just keep on increasing the thing, but in h convergence human approach, human intervention is required repeatedly. So, that is why this sometimes may be faster, but we will see that a lot of times it becomes impractical to go to elements which are more than fourth order, or fifth order, right. It sometimes it becomes impractical to go to sixth order element, seventh order element, eight order element.

So, but from our conceptually stand point this may be faster, but from practically stand point, you cannot go for p converge may be not beyond fourth or fifth order elements. So, there is something important to understand. So, the next thing in this context, we will discuss is accuracy of the solution, accuracy of the solution. So, what we are specifically interested in is, that when we say that how accurate are solution is and we can measure the accuracy in terms of all these norms right, max norm, energy norm, l two norm, but we are also interested in figuring out that when I have a finite element analysis, hm.

So, let us say this is number of elements and suppose I have a bar and I am applying some force here. I am also applying some fraction here and I am interested in finding u at this point, at point p. So, the exact solution is this u and this u at point p. So, in this case we are not computing u at different point, but we are interested in finding the value of u at point p.

So, the exact solution does not depend on number of elements, it is a constant line, correct. So, here we are potting u at point p and u h at point p. And then what we will find is that when I compute u h, let us say this is number of elements is ten, this is twenty, this is thirty. I will find that, these are the values of u p. So, whatever points are there they represent values of u p. Of u h p, that is they represent the FEA solutions when number of elements was ten, when number of elements was twenty, when number of elements was thirty, when number of elements was 40 and so on and so forth.

Now, you will see that I have drawn all these values; they are below this u line, u line below the theoretical line. People can think that the solution can come from top also and it can converge from this direction also, right. But in reality, in very large number of cases, you will see that the solution converges from below for displacement, and for energy it converges from above. If I am plotting displacement then it converges from below if I am plotting displacement that in convergence from above. So, we will try to understand why this is happening.

We will try to understand why this is happening. Essentially, what we are saying, here is that FEA solutions it under predicts displacement. So, when most of the cases, lot of the cases when we you are trying to predict displacement, the theoretical value will be more than the FEA value. It will be more then the FEA value, but as you keep on increasing the number of elements, the FEA value will approach that theoretical value and the FEA solution over predicts energy. So, if we are trying to compute the overall energy in the system, FEA will give you higher energy, little bit higher energy and it also over predicts stiffness. It over predicts stiffness. So, what we will try to understand in next several lectures is what are the mathematical reason and some basis for understanding this. So, that is what we are going to explore [FL].

So, that is the going that is what going to be the focus of our lectures in next two three lectures, that why does FEA predict under predict u or displacement. Displacements, why does over predict energy and why does it over predict stiffness. This is what we are going to discus. So, with this we will close the discussion for today and we will look forward to meeting you tomorrow.

Thanks.