Basics of Finite Element Analysis – Part II

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Lecture - 08
Measures of Errors in FEA Solution

Hello. Welcome to Basic of Finite Element Analysis Part II. This is the second week for this particular course and today is the second day of this particular week. Yesterday we discussed different types of errors which are there in FE solution related to the exact solution.

What we had discussed and explained where that there are principally 3 different types of errors which exist in any finite element analysis solution. The first type of error which is there in FE solution is, domain approximation error. That particular error is not present in 1D problem, but it does exist in 2D and 3D problems because, if we have to use straight edged or straight phased elements, we cannot exactly replicate the geometry of a curved 2D or 3D object. So, because of this there is a domain approximation error which keeps it in to the finite element solution;

The second error which is there is known as approximation error. This keeps in because we assume a particular function to represent the variation of unknown variables over the domain. If this approximation function is not exact replica of the actual function, then there will be this approximation error and this error can be reduced by increasing the order of approximation function. Especially, if we are using polynomial functions to represent this variation of variables over the domain.

And the third error is computational in nature. It rise inherent in any algorithm, where we are adding, subtracting, dividing taking square routes and doing several mathematical operations. So, it depends on the accuracy of our calculations and also the accuracy of the quadrature method which is used to integrate to perform computational or numerical integrations. So, 3 errors domain approximation error, then polynomial approximation error and third one are quadrature and finite arithmetic error.

What we will do today is discuss how can we measure and quantify these errors in their several ways to measure these errors.

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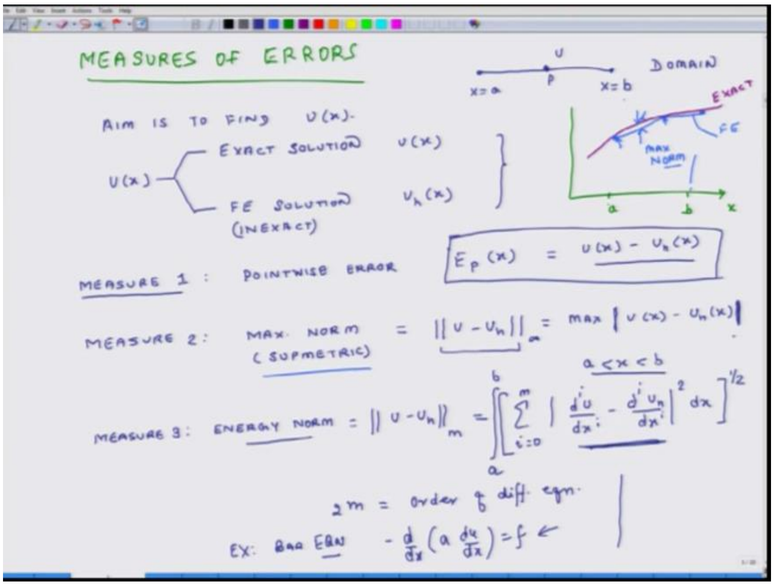

That is what we are going to discuss today; measures of errors. So, we will again explain these measures in context of a problem. So, suppose this is the domain and it is a one dimensional domain and here x is equal to a and here x is equal to b along the X axis and there is some variable at any point P and this variable is U. So you want to know, what is the value of U for different locations of probability? So, basically our aim is to find U as a function of x. Now this U of x could have 2 solutions. One is exact solution and I call this exact solution as U of x. Then there is another solution and this is inexact solution FE solution and by nature finite element solution is inexact. I call this finite element solution as U h this is also function of x and x varies from x is equal to a to x is equal to b. So, we have two solutions one is the exact solution which is u of x and the other solution is inexact solution which is U h as a function of x.

So, the first measure 1, it could be pointwise error and I call this Ep and this is a function x it itself. So, what is a pointwise error? It is just the difference between U the exact solution minus the FE solution. So, that is my pointwise error. So, what does it mean that from point to point the value of this error will change?

Now, an important thing to note about this is that using this function Ep x I do not have a single number. See Ep is a function. So, I cannot say that error is 1 percent or 2 percent because, this is a function. So, this number will change from point to point. So, I cannot assign a single value. So, I cannot compare 2 solutions because, this is not one single number. So, in that contest we will discuss another measure, measure 2 and this is called max norm, maximum norm and some people also call it "Supmetric", and this is expressed as this is format in which is written. So, this is U minus Uh So, I am just omitting the x, but U is function of x and Uh is a function of x. Just for purposes of gravity and dropping out the x. This is the max norm and what is it? It is equal to maximum value of absolute value of U of x minus Uh of x. So, this is the max norm. Here x ranges between b and a. So, first thing is that this is a positive number because it is an absolute value and it is the maximum possible value of this difference.

So, for instance if I have so, this is X axis this is a this is b. Let us say this is the exact solution. So, this is exact and I am interest only between a and b domains and this is the inexact solution. Suppose I am having two elements and I am using linear interpolation. So, this is the inexact this blue is FE solution. So, the difference between the exact and the inexact solution is maximum at this location. So, this is max norm at this location difference is 0 at this location difference is 0 at this location difference is 0. Difference keeps on increase and it becomes maximum at this location right and not all other location this difference is less compare to this maximum value. So, this is defined as a max norm or supmetric. So, this is a second number for quantifying the type of or the magnitude of error maximum.

Then there is a third measure and this is called energy norm and oh I forgot to mention that here there is a subscript which is infinity. So, this is how it is designated. Two parallel bars then you write U minus Uh and then another two parallel bars and then you put a subscript infinity. Now this energy norm is written as U minus Uh and here the subscript is m. So, m designates energy and this if I have to calculate it, it is a little complicated formula and the relation for this is

So, this is the expression. So, it is a little complicated, but will explain this. So, first thing is that it is a summation of is it is some first thing is o. So, this is integrated over x is equal to a two x is equal to b. So, first thing is this energy norm is an integral and because it is integrated over domain a to b and because it is integrated it is still a single number and then what do we integrate? We integrate a series of terms where this series is an addictive series and this i changes from 0 to m. What is m will discuss that so, this is i changes from 0 to m and then you take absolute magnitude of derivatives of U and derivatives of Uh. It is not just first derivative it could be second derivative third derivative whatever. So, it is derivatives of u and derivatives of h, you take the difference square it integrated and then take the square root that is what gives you energy norm.

Now what is m? M two m corresponds to order of differential equation. Two m represents order of differential equation. For instance, the example bars equation where we have a bar under tension. The equation for this is minus d over dx a, d over dx equals f. So, this is a second order differential equation. So, this second order differential equation means that m is one. In beams especially Euler-Bernoulli beam it is a fourth order differential equation. So, in that case m will be 2. So, if I have to do solve find the max norm for the second order differential equation m will be one. So, I will range from zero to one. So, first we will have d 0 u over dx 0 which means you itself right zero third derivative of any function is the function it itself. Then it will also include first order derivatives. So, it will include U and U prime. If I go to beams it will include U, U prime, U double prime it will go up to that so on and so forth.

So, this is how we generate energy norm. The reason with is known as energy norm is that, here we have squares of derivatives involved in the calculation of the norm and when you look at especially elasticity problems when you calculate the energy of the system, that also is involves is squares of derivatives. For instance strain energy is E which is Young’s modulus times is strain square divided by 2. So, that is the strain energy density. So, here computation of norm involves derivatives of use and squares of derivatives of use and that is why it is called energy norm. So, this is the third measure.

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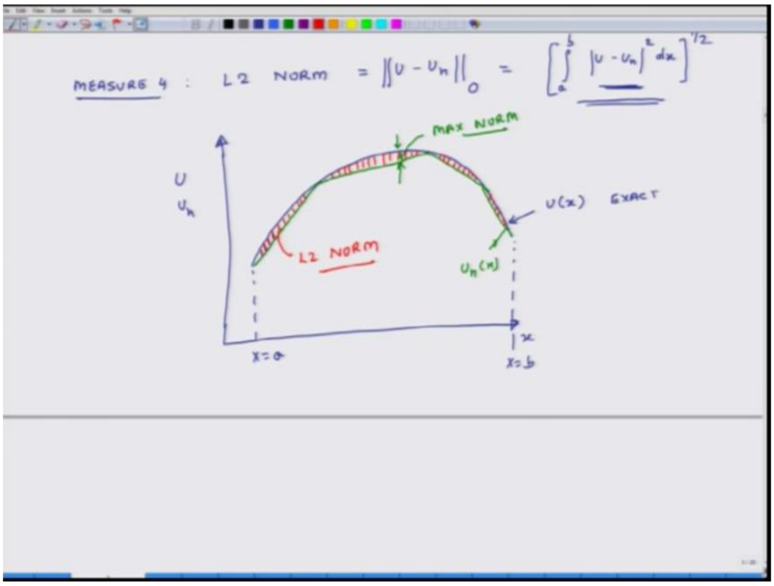

And the fourth measure is called L2 norm and this is written as U minus Uh again on both sides we put 2 parallel bars and here we put a subscript o points. So, that is an energy norm. You take the whole thing and then take the square rote of that o i excuse me I put this bracket here. So, this bracket should be here so this is important here. So, the L2 norm this is defined as integral of a to b U minus Uh and then you take the absolute magnitude square it and then integrate it and you take the square route of the integral. So, that is the L2 Norm. Now this L2 norm if you look at it represents the area the difference in areas right. So, what is this u time's dx is what the area under the curve U. If you integrate U times dx if integrate U with respect to x then the integral with represent the area under the curve for U. So, will integral of Uh times dx will be area under the curve for Uh right. So, this norm it represents the difference in the areas. This is somewhat tells us about energy this tells somewhat about difference in areas to give you an illustration.

So, this is x and here I am plotting U. Let us say this is my U and plotting it lets say this is x is equal to b and this is x equals a. So, this is U of x this is exact. Next what I am going to plot is. So this green line is the FE solution which is Uh. So, what you see is that I have broken the domain for FE into how many parts, four parts and I am using interpolation function which is linear right. So, this is Uh, h of x right. So, my max norm is this difference. This is max norm because the deference is maximum at this point. My L2 norm is this red area it is this red area. So, if I add up all there red areas and add them up that is my L2 norm and max norm is the maximum difference which is shown here. Energy norm I cannot so, easily depict it on this graph. In that case I have to plot energies and then someone normalize it and then I will be plot the energy norm. So, these are 4 different measures to quantify how accurate are solution is.

The first measure is pointwise error right and it is actually functions so, it is not measure so it is not a single number, so we do not call it a measure. Even though in this discussion we have called it a measure, but it is a function it is not a single number and the second measure is max norm or supmetric and here the way we find it out is that over the domain we find the difference between the exact solution and the FE solution and wherever that the maximum value of that difference absolute value of the maximum value of absolute value of difference is defined as maximum. In energy norm we compute the differences and derivatives and then square them, differences take the absolute value square them, integrate this over the domain and take the square route and that is how calculate the energy norm.

And finally, there this L2 norm which represents that difference in areas under the exact and inexact solution curves. Here it is basically U minus Un the absolute value of that square it and then integrate it over the domain and take square root. Now all these norms are we have discussed are in context of single variable 1D situation, but we can extend the same logic for multi variable problems and also multi dimensions. So, in case of if the domain is 2D, then we just do not integrate it over x, but we integrate to over x and y. In the domain is 3D, then we integrated to on all the 3 domains and then when we are calculating norms we are taking differences so, again if there 2 variables u and V then we take U minus Uh and then also we do V minus Vh. so, like that we calculate these norms, but we still end of with one single number.

So, specifically 3 important metrics of measures are there max norm, energy norm and L2 norm and then of course, there is a pointwise error function also and that is basically the difference between exact solution and the inexact solution.

So, this completes our discussion for today and we will continue this discussion tomorrow. What we are going to do tomorrow is, we will start talking about convergence and how finite element solutions converge and on what parameters they depend on and to what extent.

Thanks a lot and look forward to seeing you tomorrow. Bye.