Basics of Finite Element Analysis – Part II
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Lecture - 05
Weak Formulation : Example Problem

Hello, welcome to Basic of Finite Element Analysis Part II. In last module or last lecture we had discussed a weak formulation. What we are and what we are going today as well as in the next class is develop weak formulation based finite element equations for an actual problem. In this problem we have a bar which is being subjected to external forces. Some of these forces are tensile in nature and others are tractional in nature. We will try to develop solutions for this problem using the finite element method. In finite element method we will use the weak formulation to develop the solution.

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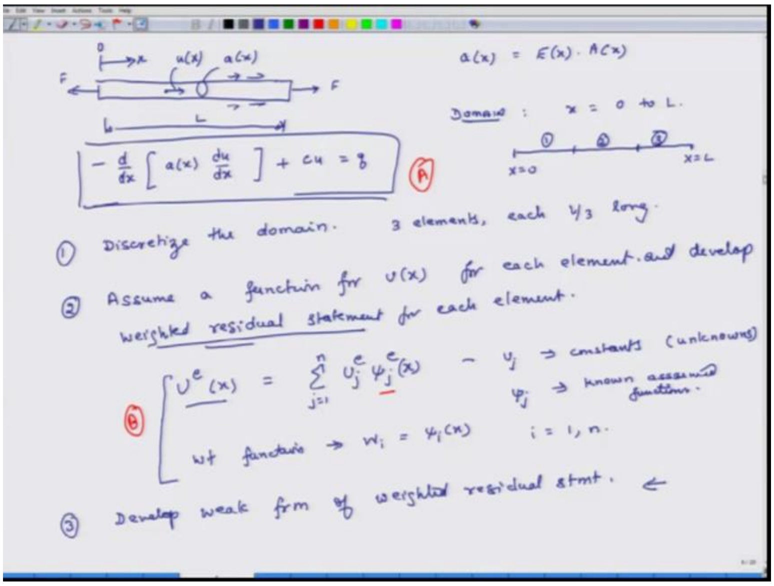

So, first let us look at the bar problem. So, you can have a bar and it is cross sectional area is a x actually it is not cross sectional area. So, a x is Young's modulus and that could also vary with respect to x times cross sectional area, which is this thing. This problem I could apply some external force F and I could also apply some traction force like this. So, the governing equation for this kind of a problem in general can be written as d over dx, ax du over dx. So, the displacement at any point is u. So, u is a function of x plus cu equals q. Let say the length of the bar is l. So, here the domain is x is equal to 0 to L. So, this is the length of the bar. This is my coordinate origin. So, x is equal to 0 at this location n x is equal to 0 at x is equal to l at the other extreme end.

Now, if I have to develop a formulation for this problem, the first step finite element formulation is, the first step is that, this is my domain. So, the first step is discretize the domain so, I break this up. In this case for purposes of simplicity I will break this entire domain into 3 elements. So, this is element number 1, element 2, element 3 and will just assume for purposes of simplicity that each element is of equal length. So, we have three elements each L by 3 long.

The next step is that we assume a function for u x for each element, assume a function for u x for each element and then develop a waited residual function and develop weighted residual statement for each element. For each element U is varying over the length of the element. So, this is nothing but some interpolation functions times uj for each element. So, uj are constants and they are unknowns and psi js are known assumed functions. There are n such functions. So, I am going to add this up over n terms and also we assume a weight function. So, weight function because to develop this weighted residual statement we have to assume u and we also have to assume weight function.

So, weight function is wi is equal to psi ix and i is equal to one to n. So, will take different weight functions in this thing, with these things now we develop so, we have developed the weighted residual statement, we can plug all this. So, let us call this equation A let us call this equation B. So, we plug B into A to get weighted residual statement, but we have seen that if we have a regular weighted residual statement the differentiability requirement for psi j is higher and it has to be at least in this case it has to be at least quadratic. So, we will weaken the differentiability requirement by going for the weak formulation. So, we will say now develop weak form of weighted residual statement. So, this is an additional step because, now we are using a weak formulation.

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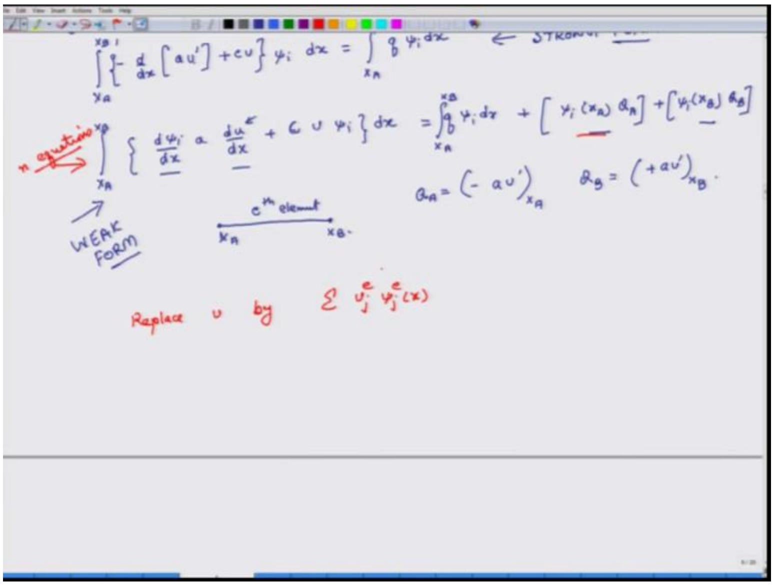

So, the weak form of the weighted of, so first we will write the strong form then we will write the weak form. So, my domain of integration is he h plus one, minus d by dx au prime, plus cu I mean in reality I have to replace this u by this entire expression, but for purposes of gravity I am not doing that and now I am multiplying it by weight function psi I, dx is equal to he, he plus one, q psi I, dx, this is the strong form. Now I will mathematically manipulate the strong form to convert into a weak form by integrating by parts. What I get is he, he plus one, d psi i over dx, a du over dx plus cu, psi I, dx, is equal to q, psi I, dx, integrated over the domain of the element and plus I get some extra boundary terms. So I get psi i evaluated xa, qa and I will define these entities later plus psi I, xb evaluate times qb. These are the 2 extra terms.

These are the terms boundary terms when I do integration by parts, I get those boundary terms we had seen that. So, these are these things. Here Qa equals minus au prime evaluated at xA right. So, what is xA? This is actually to be consistent. What I will do is I will erase these limit is and I will call it XA, XB XA XB XA XB XA XB. So, this is eth element. It is first coordinate is XA and second coordinate is XB so, QA is minus a times u prime evaluated at x is equal to XA QB is equal to plus au prime evaluated at XB.

So, this one is the weak form and here the differentiability requirement on the weight function and the primary variable is same right. So, that is why it is lesser compare to the strong form so, this is the weak form. We know that u, what is u? We had assumed that u is this function right? Weight function is this. So, once again in the weak form i can take different values i is equal to 1 to n. So, this represents n equations. So, now what we will do is we will replace u by uj psi j x and because it is for the each element I have this superscripts E.

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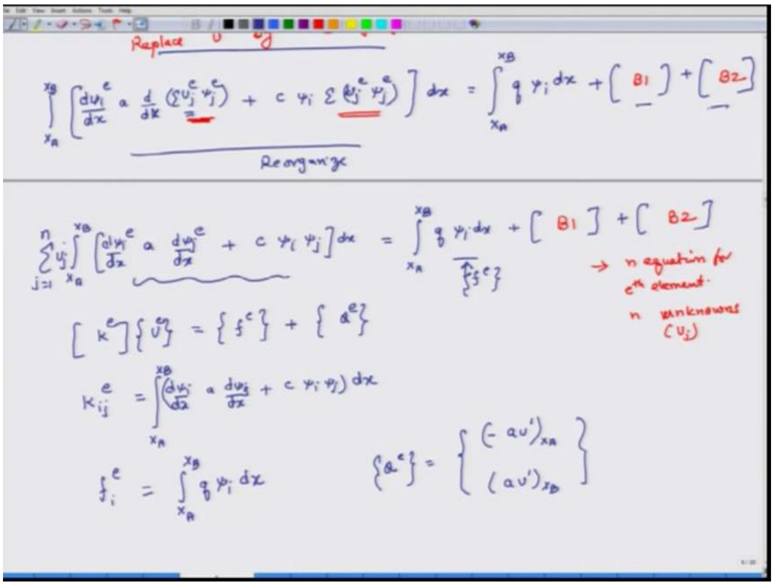

So, we do this and what we get is XA to XB and now because I am having we are taking about e th elements. So, I am you know just psi i superscript e times a times d over dx, u j psi j plus c psi i and actually this has to be summed up, u j psi j integrated with respect to x equals q psi i dx, xa to xb plus I get this boundary term plus I get the other boundary term. So, I will just call this boundary term as B1 and I will call it B2.

Now uj is are unknowns, but they are constants psi j are known functions psi j are known functions. So, I will reorganize this thing. Once again these are n equations and I will reorganize this thing. So, what I will do is I will bring this summation sign outside the integral. So, I integrated from XA to XB d psi i e over dx a s psi j over dx e plus c psi i psi j. So, this is the expression which I get. Just note that u j is now outside the integral sign and the reason I could take it out is because it is a constant. So, when I differentiate it becomes 0. So, the only thing which maters while when I am differentiating this term i am sorry this term. This entire thing is psi j. I can take out u j outside because it is constant right? So, again this represents n equations for e th element. How many variables are there? n variables. It has having n variables not variables n unknowns. These are u js. So, for the e th element there are n unknowns which are u js and we have n equations.

So, now I can express this in a matrix form for the e th element. I can call this entire thing is K e it is being multiplied by a vector u e which is u j and this is equal to f e. So, this gives me force wave f plus I get another vector Qe, from B1 and B2. From B1 and B2 I get Q e. So, here k i j for the e th element is d psi i over dx a d psi i over d x excuse me j. So, this is the definition of k matrix. The definition of f i e is q times psi i times dx and by Q matrix it will be essentially minus au prime evaluated at XA and au prime evaluated XB from B1 and B2. So, this is QA and this is QB, oh i am sorry this has to be multiplied by psi i times xa psi i.

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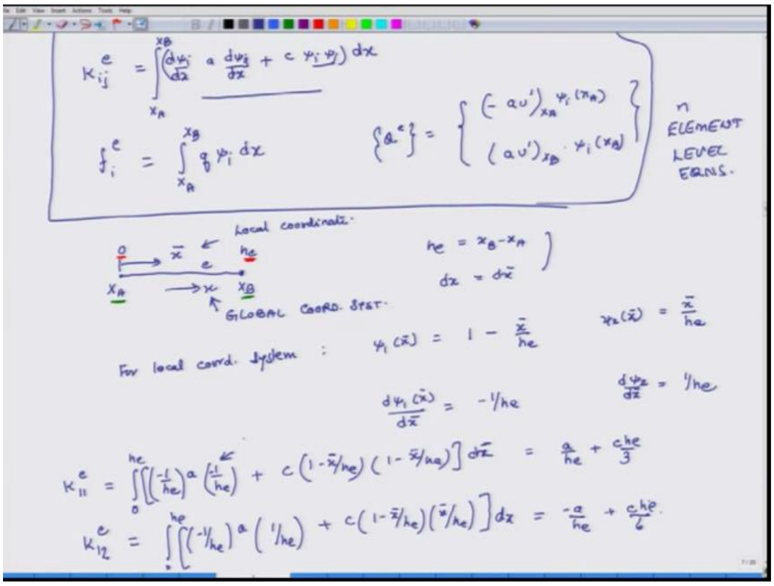

Times evaluated at xa times psi i evaluated xb. So, this the element level equations and they are n n element level equations and they evolve n unknowns which are used. Now what we will do is we will actually compute these values. So, we will actually compute actual values of these K matrix and f matrix and Q matrix. For this we have to actually know the definition of size. If you know the definition of size and we can differentiate it and integrate it and we can get the values.

Now, in part 1 we had discussed that for a 2 nodded element for a bar, we can have linear approximation functions right. We had developed the relations for those functions. So, we will use the same linear approximation functions in this case, but before we do that what we will do is, see here the integration is happening from XA to XB and this is the global coordinate system. So, here x is equal to 0 here x is equal to l and for instance in this case XA will be l by three and XB for second element XA will be l by 3 and XB will be two l by 3, but what we will do is we will go to a local coordinate system because it makes our computation faster.

So, before we actually calculate we will develop local coordinate system. Suppose you have an element and it its global coordinate are XA and XB. So, this is coordinate x then. So, this is global coordinate system. Then what I do is I develop another coordinate system and I call it x bar and this is local coordinate. So, this local coordinate system means that it is valid only for e th element this is e th element. It is only valid for e th element beyond e th element it does not it is purposeless. So, in this local coordinate system at XA the value of x bar is 0 and at XB the value of x bar is h e.

So, these are the local coordinates and these are global coordinates and h e is equal to XB minus XA. Now given this we see that dx is same as dx bar. So, with this transformation we will calculate the values for elements of k matrix f matrix and stuff like this. So, for local coordinate system my interpolation functions are psi one x bar now psi is a function of x bar not x and this is equal to one minus x bar over h e and psi 2 which is the function of x bar is equal to x bar over h e.

Then d psi 1 x bar over d x is equal to minus one by h e and d psi 2 over d x bar is equal to one by h e. So, let us calculate k 11 for the e the element and I will use local coordinate system to calculate it. So, my limit is will be when it is XA it becomes 0 and when it is XB x bar is h e and here in this re relation I replace i with 1 and j with 1 to calculate k 11. So, d psi 1 over dx is minus 1 minus by h e and I assume that a is constant and d psi j, j is what 1. So, it is 1 by h e plus c times psi 1 x because psi is psi 1 psi j right. So, this is one minus x bar over h e and psi j, j is one. So, it is again one by x bar over h e dx. So, what I get is a over h e plus c h e over 3 and if I do the math similarly k 12 for the e th element if I do the same thing 0 to h e minus 1 by h e a oh there should be a negative sign here I am sorry there should be negative sign here because, j is 1. So, for while calculating k 12 it is 1 over h e times a times 1 over h e plus c times 1 minus x bar over h e times x bar over h e dx and this we get as minus a by h e plus c h e over 6.

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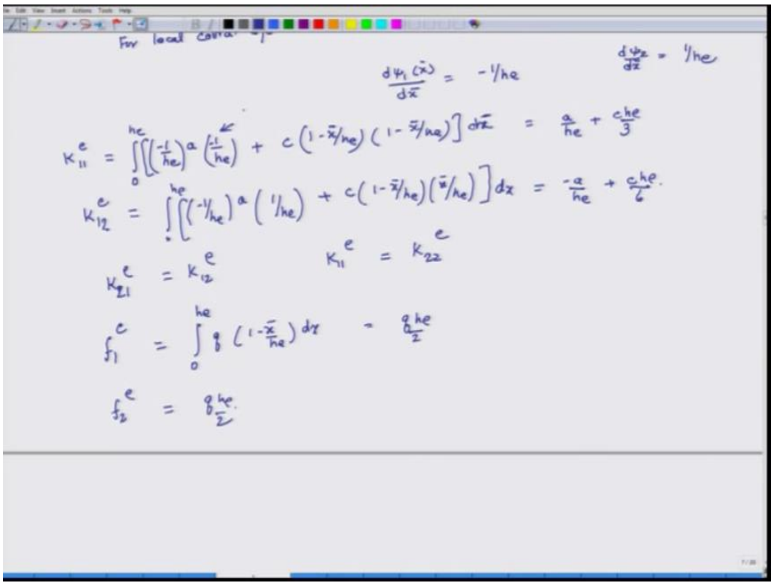

And we find that K 21 e is same as K 12 e and K 11 e is same as K 22 e. If we do the calculation this is what we find out. Likewise we can also calculate the force vector. So, f 1 e what is force? Force vector is q times psi i d x. So, f 1 is q times 1 minus x bar over h e dx integrated between 0 to h e and that comes to be q h e over 2 and f 2 e is equal to q h e over 2.

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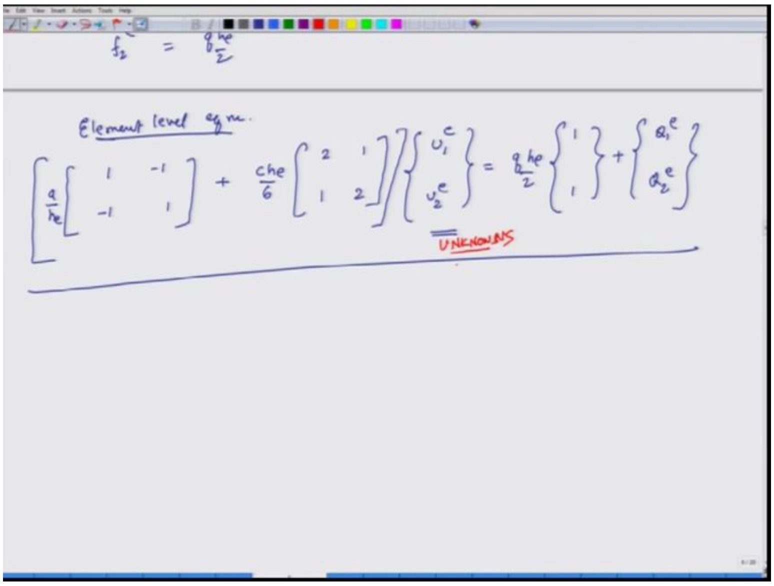

So, ultimately I get a element level equation as this 1 minus 1 minus 1 1, a over h e plus c h e over 6, 2 1 1 2 and this entire thing is added up and multiplied by U 1 e, U2 e and this equals q h e over 2 11 plus Q 1 e then Q 2 e. So, this is the element level equation. Then this element level equation again we have 2 unknowns these are the 2 unknowns right c a q all these parameters are known. So, we have 2 unknowns and 2 equations. 2 unknown constants and 2 equations, in the next class what we will do is which is the last week last class of this week is, we will now assemble these equations and also apply boundary conditions. So, then that will give us a comprehensive over view of whatever we did in the last course. So, thanks a lot and look forward to see you in the next class.