Basics of Finite Elements Analysis - Part II
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Lecture - 45
Plane Elasticity Problems

Hello, welcome to Basics of Finite Element Analysis. This is the second part of this course. The first part was done a couple of months earlier and today we are in the last week of this course. This is third day of this particular week and for the remaining portion of this week we will essentially cover plane elasticity problem and the reason we are covering this plane elasticity problem and trying to understand it in terms of its FEA formulation is because it involves two differential equations, simultaneously. In each differential equation we have two variables. So, and this is in two dimensions. So, it is a multi dimensional problem with multiple variables and once we get a good handle on this type of a problem then, we can understand this; we can extend our understanding to three dimensions relatively easily.

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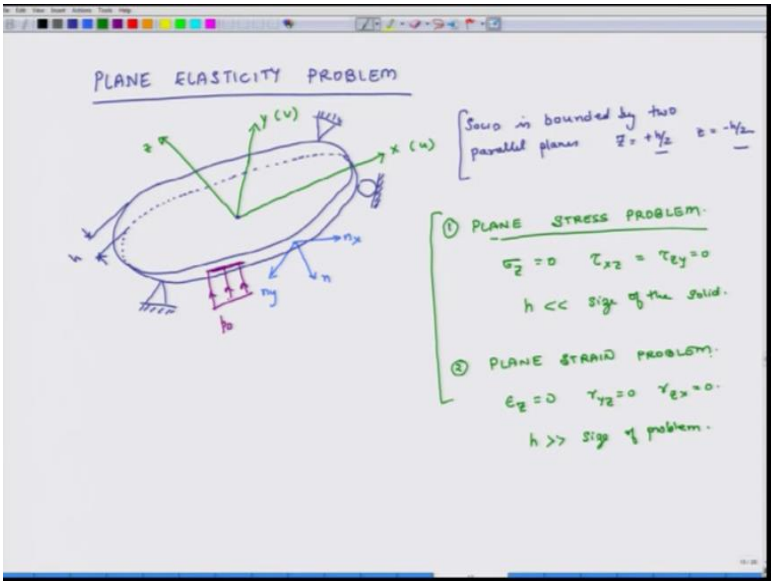

So, we will discuss plane elasticity problem. So, consider a solid object such that it has uniform thickness. The thickness can be extremely small or it can be extremely large, but the thickness has to be uniform across the domain. So, and I have my axis. So, one axis is in the plane of the solid, other axis is in the z direction out of the plane and the third axis is also in the plane of the solid. So, z is in the thickness direction and other two directions are in the plane of the solid. So, the displacement in x direction is u, displacement in y direction is v and as I said the thickness of this solid is uniform across the entire surface. So, what I can say is that the solid is bounded by two parallel planes.

Z is equal to plus h over 2 and Z is equal to minus h over 2. So, this origin is such that it is at the center of the solid and the top surface is at a height h over 2 and the bottom surface is a height minus h over 2. So, this height is how much? This is h; I can have some constraints on this solid. So, I can have a fixed height. I can also have a roller contact. I can also have again this thing fixed at third point and so on and so forth. So, I have at some number of points, I fixed it. The other thing is that at any point on the edge of the domain, I have a normal it is normal to the edge and this normal can have two components. One is n x and the other one is n y and also I can have let us said along part of the edge I can have some force or external pressure. So, I call it p 0.

So, what we want to understand is how does this solid behave? When it is constrained at a finite number of locations and with seeing forces at on its boundary, there are two classes of problems. So, this is called a plane elasticity problem. It is an elasticity problem because we are assuming that the material is elastic, it is elastic. So, as I keep on increasing my stress or I get a strain and the relationship between stress and the strain is a straight line and if I remove the external force, the system comes back to its original position. That is the nature of elasticity and we are assuming that it is linear elasticity because the curve is a straight line.

It is plane elasticity because we do not worry about the third dimension. So, we are only worried about two dimensions. So, there are two types of elasticity problems plane elasticity problems the first one is.

Plane stress problem, this is called a plane stress problem because the stress is only exists in the plane. They do not exist out of plane, if the stress is does not exist out of plane then what stresses are 0, in this case? What are the different stresses? Sigma Z is 0, is there anything else, shear stresses in x z and z y is 0. So, it is called a plane stress problem because the stresses out of plane direction, they are 0. Now this type of problem develops then h is very small compared to size of the solid.

If the thickness this h is extremely thin, if it is extremely thin then we can make an approximation that sigma z and the shear the stresses x z and z y they will be pretty close to 0. They will not be exactly 0, but they will freely close to so we can make that approximation. So, this is one plane elasticity problem. The other plane elasticity problem is plane strain problem and it is plane strain because, the strain only exists in the plane. There are no out of plane strains. In plane stress problem you can have out of plane strains and plane strain problem you can have out of plane stresses. So, here Epsilon Z is equal to 0 Gamma y z is equal to 0 and Gamma z x is equal to 0.

And this happens when h is extremely large compared to the size of the problem. It is extremely large. So, if it is a very fat thing, so it is a very fat thing the material provides sufficient restraint for motion to happening in the z direction and if there is no strain happening in the z direction then it is a plane strain problem. So, the thickness provides by itself by virtue of its material, a significant amount of constraint. So, that there is no strain in the z direction. So, we will analyze both these problems, but in a unified way.

Now we will not go into the mechanics of this problem, but such a plane elasticity problem is captured by an equation of motion right. So, this equations; of motion we get some Newton's laws. So, how many directions we are concerned with the motion of this object? 2. It can move in x direction it can move in y direction. We are not bothered about the z direction motion because they have no forces in the z direction and it is a plane elasticity problem. So, if motion can happen in x axis in the x direction and y direction how many equations of motion we will have, 2 equations; one to govern the x direction motion, the other to govern y direction motion.

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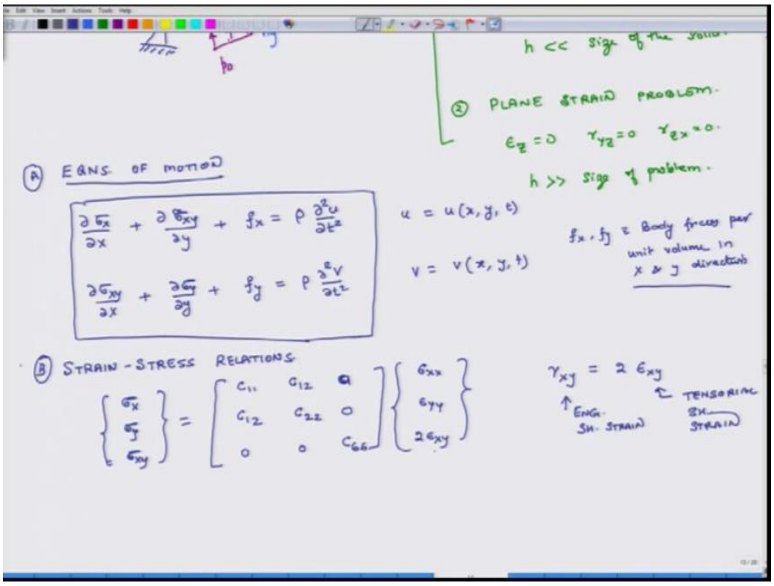

So, first we will write down, equations of motion and these we essentially get from Newton's second law, force is equal to mass times acceleration, but if we do all the mathematics this is the equation we get.

So, here I am using sigma x y which is same as tau x y which is the shear stress and then the second equation is here, you is the displacement in x direction of what of any point in the, of any material in the body. So, u changes with respect to x y and what else time. It can change the time also. If it is a static problem then the time is not important, but if it is a dynamic problem then time is important. So, this the time is important. So, u is a function of x y and t. Similarly v is a function of x y and t and then f x f y and body forces, but unit volume in x and y direction. What would be the example of f x? Suppose this is the body and suppose my x axis is like this, my y axis is like this and z axis is in this normal direction. This is my x axis, this is my y axis, this is the z axis and the body is oriented like this.

There are no forces in the z direction right and the second thing is that the thickness of this is uniform. So, it is a plane elasticity problem I can put some forces here, some force here, some force here and I am constraining it like this. So, it is a plane elasticity problem, what would be the body force? Now this full body is also seeing gravity, the same gravity right and what is the gravitational force? Density time's volume times G right. The gravity is acting in this direction, but my x axis is like this, y axis is like this.

So, there will be one component of gravity force in the x direction and other component in the y direction. So, those are body forces.

So, these are the equations of motion. The second thing is we have strain stress relations. So, what stresses we have in the system? Sigma x sigma y and I will actually call it sigma x y and that equals 0. So, these are the strains Epsilon x x, Epsilon y y and twice Epsilon x y. So, a lot of times this shear strain, this engineering shear strain this equals twice of Epsilon x y which is. So, this is engineering shear strain and this is Tensorial shear strain. So, here we are using Tensorial shear strain, but I can replace this by Gamma x y, it does not matter. So, this is the second set of equations. So, we have equation of motion, strain stress relations. The third equations are strain displacement relations.

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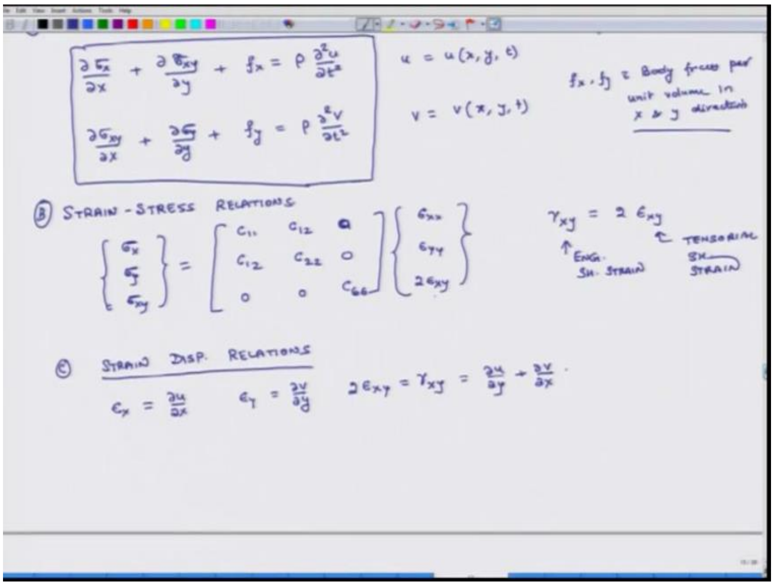

Strain displacement relations. So, Epsilon x is equal to Del u over Del x Epsilon y equals Del v over Del y Epsilon x y twice which is same as engineering shear strain, this is equal to Del u over Del x plus Del v over Del x and finally. So, all these equations A, B, C are valid for both set of problems; plane stress and plane strain problems. So, if we solve this set of equations we can solve for both. If I use these equations, equations of motion, stress strain relations and strain displacement relations, we do not distinguish between plane stress and plane strain problem. The only difference comes in our definition of C 1 1, C 1 2, C 2 1 and C 6 6 because the values of these Cs are different for plane stress and plane strain problems.

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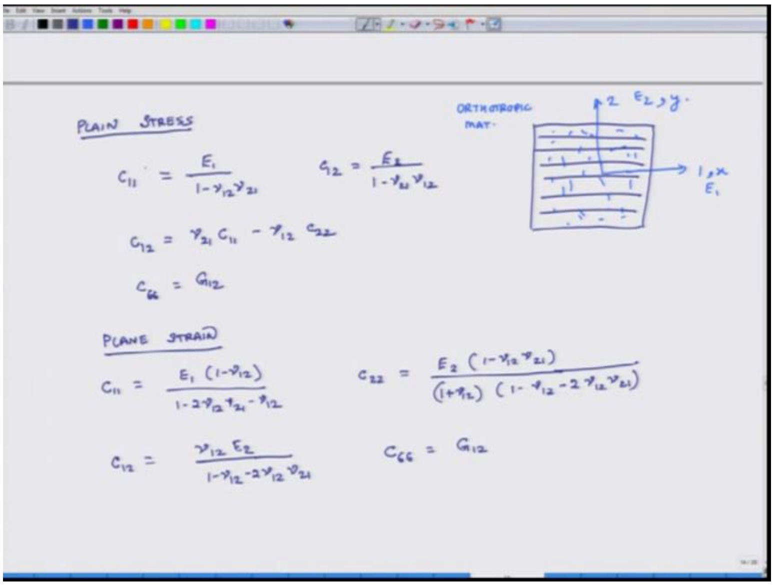

So, what are these values? For clean stress problem C 1 1 equals E 1 divided by 1 minus mu 1 2, mu 2 1. So, you may be wondering what is E 1 mu 1 2 mu 2 1. So, what we are assuming is a material and let us says this material has lot of fibers and then all these fibers are bound together by some glue by some matrix. So, this is direction 1, this is direction 2. So, if I pull it in 1 direction, it will have some modules right. For instance; wood along the length of the fiber wood is stiffer. So, is E is higher. So, in the direction 1 its modulus E 1, in direction 2 its modulus is E 2. These types of materials are called orthotropic materials.

Orthotropic materials and what we are also assuming is that 1 direction is same as x direction and 2 directions is same as y direction. So, C 1 1 is this, C 1 2 equals E 2 divided by 1 minus mu 2 1 mu 1 2, C 1 2 equals mu 2 1 C 1 1 minus mu 1 2 C 2 2 and C 6 6 is equal to G 1 2. This is the shear modulus. And for plane strain problem these constants are different C 1 1 equals E 1 minus 1 minus mu 1 2 divided by 1 minus 2 mu 1 2 mu 2 1 minus mu 1 2 C 2 2 equals E 2 into 1 minus mu 1 2 mu 2 1 divided by 1 plus mu 1 2, 1 minus mu 1 2 minus 2 mu 1 2 mu 2 1 and C 1 2 equals mu 1 2 E 2 by 1 minus mu 1 2 minus 2 mu 1 2 mu 2 1 and C 6 6 is equal to G 1 2. So, these are the constants. So, if you have to solve a plane stress problem, you use these constants. If you have to solve plane strain problem use these constraints, but the solution procedure for solving the procedure these equations is the same.

Finally, if the material is isotropic so, here we have assumed that the material is orthotropic. If material is isotropic, for instance steal our regular materials, rubber steel not rubber steel, aluminum, brass, bronze things like that.

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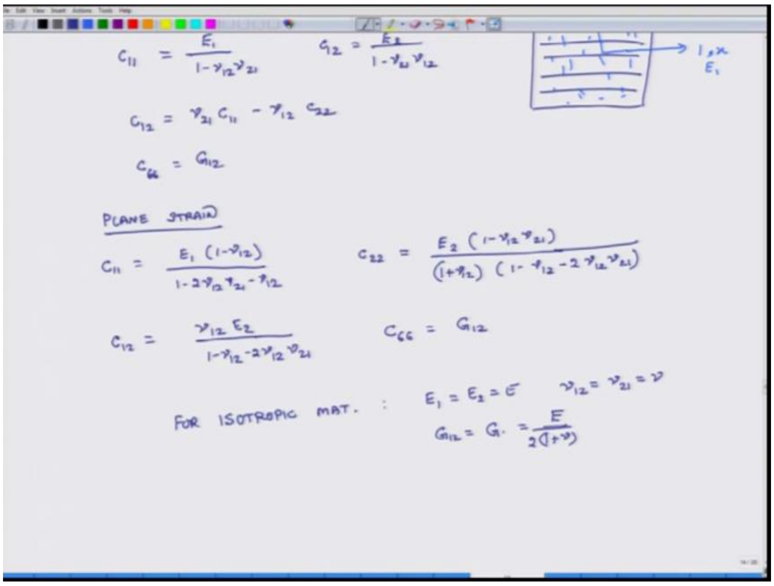

In that case for isotropic material E 1 is same as E 2 is same as E. So, you can simplify these relations, mu 1 2 12 is equal to mu 2 1 is equal to Poisson’s Ratio and G 1 2 is same as shear modulus, which is equal to E divided by 1 plus mu times 2.

So, this is the overall plane elasticity problem definition. So, what we will do is we will close our discussion today here and going forward we will now start developing the formulation for this problem and actually solve this problem over the next couple of lectures, several lectures thank you and look forward to seeing you tomorrow. Bye.