Basics of Finite Element Analysis - Part II
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Lecture - 43
Numerical Integration Schemes for 2-D Problems: Closure

Hello, welcome to Basics of Finite Element Analysis Part II. This is the start of the last week of this course, and this week we will complete our discussion on single variable two dimensional problems, especially in context of numerical integration and also how we go around doing post processing of the results and then once we're done with this, we will be moving on in the later part of this week to two dimensional, two variable problems because what that will help us understand is that, in two dimensions how do we deal with multiple variables. So, once we understand that then it is pretty straight forward to graduate to 3 d problems with multiple variables. So, that is more or less what we plan to cover in this course and hopefully with this understanding, you will be able to expand your knowledge in area finite element analysis for much more complex problems because whatever we have been covering and we will cover this week in this course as well as if you want, which happened a few months back that I feel should be sufficient to get you going in f a and also using a lot of finite element analysis tools which are commercially available effectively and smartly.

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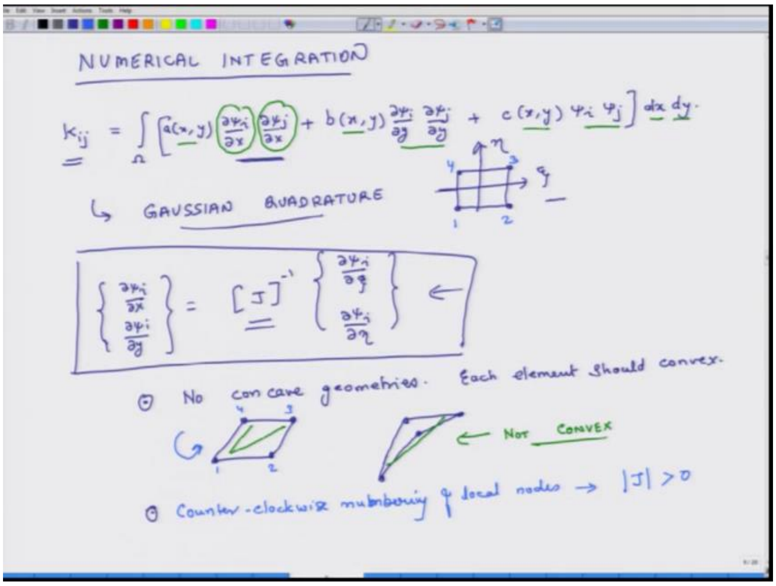

So, what we will do in this today's lecture is, we will continue our discussion on numerical integration. And to recap the context, the purposes or the aim is that suppose, we have a stiffness matrix whose elements are defined by k i j in such a way, then if we have to evaluate different elements of this k matrix. We have to perform this integration on two dimensions and to do this we usually in f a use Gaussian Quadrature method and this Gaussian Quadrature method it works in zeta eta space and typically we integrate it over a standard element in this case it is a four nodded square of side two. So, to use Gaussian Quadrature we have to work in this space zeta eta and if we have to work in this space, we have to calculate these parameters all these parameters in zeta eta space and also we have to transform this into the d zeta d eta ok.

So, what we have learnt was in the last class that these derivatives of approximation functions, they can be expressed as inverse of the Jacobian matrix and this should be partial derivative psi y psi with respect to x. So, what we had learned in the last week was that we can calculate partial derivative of i with respect to x in terms of partials of psi in with respect to zeta and eta. If we multiply that by inverse of the Jacobian matrix and we had actually calculated this Jacobian matrix for several situations. So, that helps us handle these two components and before we move on, to move to other parameters we would like to recap what we understood, that this relation is valid only if there is no concave geometries. So, each element should be convex element. What does that mean? that if there is an element because I am my domain of integration is the element not the entire geometry, then any point in the element, if I pick up any two arbitrary points then they should be I should be able to connect them, with the line such that the line does not cross the boundary of the element. If the element was not convex, for instance, this is an element. Now if I have to connect these points, then my line crosses. It goes out of the boundary. So, then this is not a convex element and because and if, the element is not convex then j matrix may not be non singular at some points and if, that is the case then I will not be able to invert it.

So, that is one thing we learnt. Second thing we learned was, that whenever I number the nodes local numbers, then I should go in a counter clockwise way because that is what we, how we, number the master element as well. So, counter clockwise numbering of local nodes, this ensures that the determinant of j is greater than 0, as long as the element is convex.

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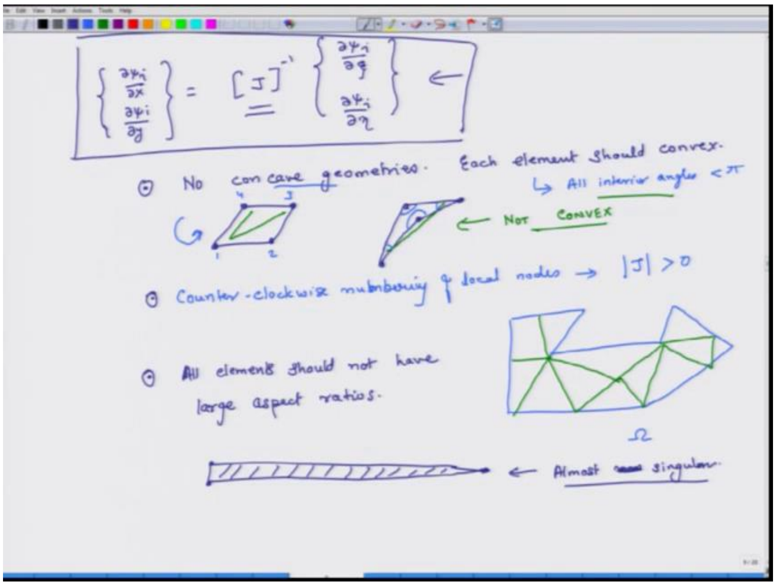

So, if there is no concave geometry, it essentially implies that all interior angles should be less than pi gradients. So, this is less than pi, this is less than pi, this is less than pi, but this angle is more than pi and because of that the geometry is not convex. So, what that means, is that suppose, I have a complicated geometry suppose, my geometry is like this. So, this is the overall domain, but that does not mean that this domain cannot be understood using f a. As long I can still analyze this domain using f a as long as my each individual element is convex. So, what I can do is, I can break this up into convex elements and so on and so forth. So, now, even though I have a complicated non con convex domain I can break it up into smaller elements which are concave in nature. So, that is there.

So, interior angles we have said that that should be less than pi and then there are more restrictions for higher order elements, but at least in the context of this course, we will not touch upon those and, then the last important consideration is that the element, all elements should not have large aspect ratios, because we had seen in the last week that if, suppose there is an element which is having extremely large aspect ratio. For instance, this then, when we try to develop its shape functions, not state function approximation functions.

We again have to go to, we again have to do some inversion of some matrices and when we are trying to do the inversion of this matrix, the area of this element comes into picture and if the aspect ratio is very large then the element becomes almost non singular. Almost singular, this is what we had learnt in the last week. So, it becomes almost, not last week in the sixth week, we learnt this, that becomes almost similar. So, we should avoid elements with very large aspect ratios. This is important, otherwise our computationally accuracy can really degrade very fast.

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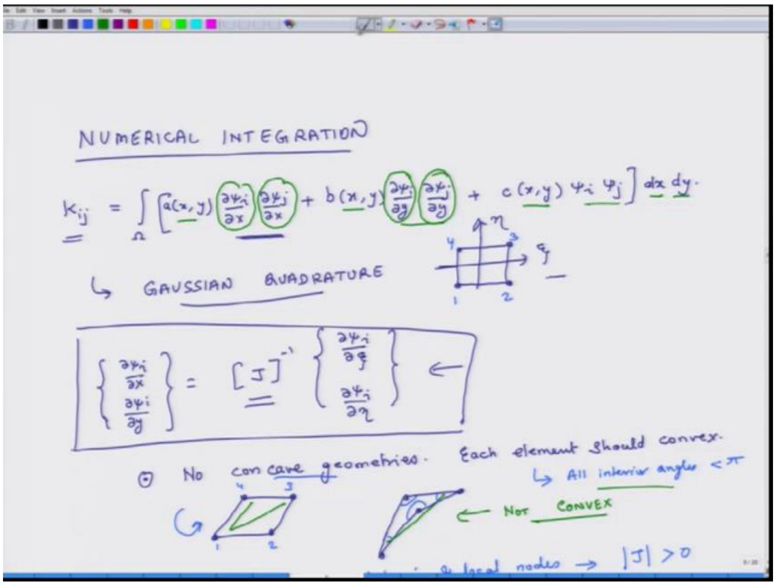

So, with this understanding we know how to calculate parameters these parameters which are in green circles. So, the next thing, we have to learn is what to do about a x y and b x y. So, now what we will learn is how to handle a x y b x y c x y because they are functions of x and y and we have to express the inverse in terms of zeta and eta. So, that is not difficult.

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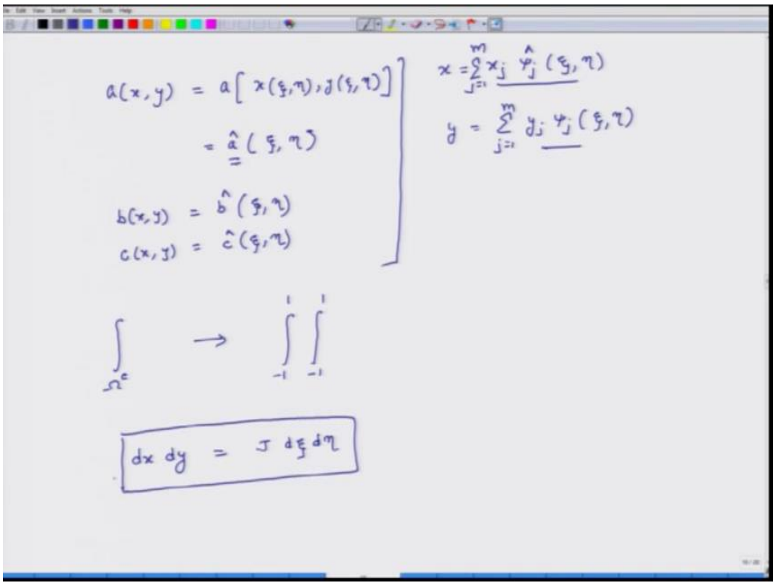

So, a is a function of let us say x and y. Then we know that x is what? x j psi j hat zeta eta j is equal to 1 2 m and y is equal to y j times psi j zeta eta j is equal to 1 to m. So, what I can do is whatever, is the relation for x, I express that x in terms of the zeta and eta. So, my equation becomes my term becomes x which is a function of zeta and eta y which is a function of zeta and eta. So, wherever I see x I replace it with x j psi j summation of that, wherever I see y, I replace it with this relation.

So, essentially what I get is a new function a hat zeta and eta. So, the nature of this function is different, because it depends now on zeta and eta, but I have, using this I can transform which is a function of x and y into a hat which is a function of zeta and eta. Similarly b x y can become v hat and c x y can become c hat. So, wherever I have functions of x and y, I can very easily in a straightforward way convert them. So, these terms can also be handled and the same is true for these also, psi I psi j, we know how to express them in terms of zeta and eta.

So, the next thing is this domain. So, when I am integrating, Let us say if I am talking about a quadrilateral. So, I am integrating it over this area of the quadrilateral, but if I have to do Gaussian Quadrature way of numerical integration then I have to integrate in this space. So, the integral the limits of integral they become minus 1 to 1 and minus 1 to 1 because now I am going in the zeta and eta domain and the last thing is d x times d y d x times d y, that also has to be transformed. So, that becomes this d x and d y also has to be changed to d zeta and d of eta. So, this becomes Jacobian times d zeta d eta and this is a transformation which we had developed also in context of one dimensional Gaussian Quadrature scheme.

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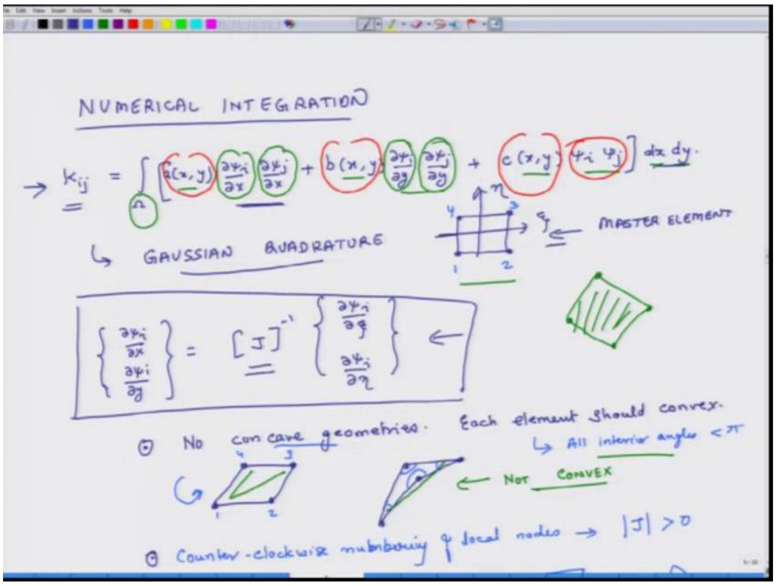

So, once I am able to do all these, these transformations, once I am able to do all these transformations, then I can instead of integrating over an arbitrary quadrilateral, I am going to integrate it over to this master element. This is the master element. So, all integration for all elements I perform over master elements but when I move from one element to other element, it is the, these, this Jacobian comes into picture. That comes into picture.

So, that is what I get. So, essentially my new relationship becomes k i j for the eth element equals my domain of integration changes from the physical area to minus 1 to 1 minus to 1.

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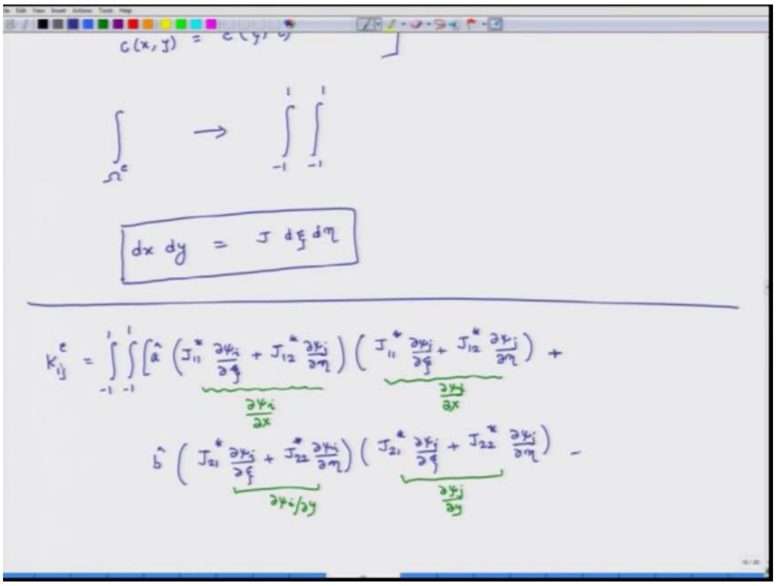

And then a hat times j11 star del psi i over del zeta plus. So, this is j12 star del psi j over del eta. So, what is this? This is effectively del psi i over del x because of this relation, the inverse elements are stars. So, del psi i over del x is equal to j11 star plus j12 star. j11 star times this plus j12 star times this. So, what I have done is I have replaced a y s a hat. I replace psi i with a partial with respect to x in terms of zeta and eta. And then psi j partial of psi j with respect to x is j. So, this term what is this term? Is still del psi i over del x. So, this is a times del psi del x times del psi. This is del psi 0 del x plus. Then I have b hat j21 star del psi i over del zeta plus j22 del psi i over del eta eta. This is again a star times j21 del psi j over del zeta plus j22 star partial of psi j with respect to eta.

So, these are del psi i over del y this is also del psi j over del y plus. So, what was the third term? c of x and y times psi i psi j. So, I write it. c star c hat and c hat is a now a function of zeta and eta [FL] times psi i psi j.

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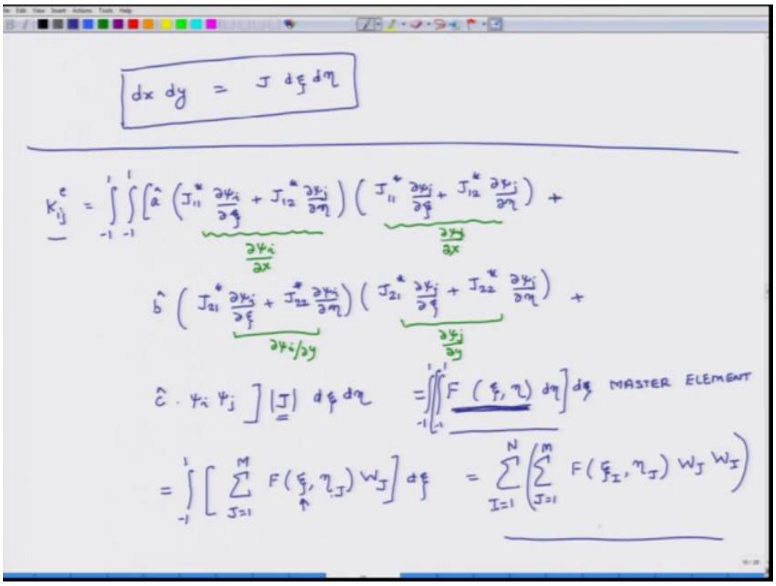

So, I can express psi I and psi j in terms of zeta eta. That is, we have already given you the functions. Bracket closed. j d x d y no excuse me, it is a number. I should have written. So, number. So, this entire expression here it is being integrated over zeta and eta. So, I can end it, only involves zeta and eta, all everything else is a constant. So, I can express it as a function of this thing.

So, I have to evaluate this function. Now, this function is being evaluated over which element? This function, originally, it was being evaluated over the actual quadrilateral. Now we are evaluating this function over the master element and when I have moved on to master element, I can use Gaussian Quadrature straightaway. Actually, so, I will still hold on to my integral sign and. So, I am integrating this function big F now earlier it was there but now integrating this function big F, which is a function of zeta and eta, over the master element area.

Now, when we go back to our understanding of Gaussian Quadrature, we know that in context of 1 d elements and integral over the master element of any function was nothing, but the value of that element, at specific points, which are known as Gaussian points, times what? Weight functions. So, this integral is nothing, but. So, first what I will do is I will integrate it. Actually, first I will integrate it with respect to eta. Does not matter, I can do it in the other way. So, first I am going to do the integral on eta. Then I am going to do the integral on zeta.

So, when I am going to integrate it on eta, then this is nothing, but the value of that function, evaluated at j-th Gaussian point times a weight which is actually a constant number and that weight w j. So, I have removed this integral sign. And how am I have performed that integration. By evaluating that function F, at this point, zeta coma eta j and we know the values of eta j. It depends on what Quadrature we are using. So, we know the values and then, we multiply this f zeta coma eta j with some weight. Weight, this is again a constant. It is a number w j. So, but now this is still a function because this is still variable and it is still getting integrated.

So, to remove this integration sign, I will again apply one more times Gaussian Quadrature. So, the first summation is j equals 1 to m. And please see things a little carefully. Here, just to differentiate j and m from earlier js and m, I have put capital letters. So, capital letters means these are indices associated with Gaussian Quadrature scheme and m is the number of Gaussian Quadrature points. It is the number of Gaussian Quadrature points. So, now, I am going to integrated it again over the dimension zeta. Then using this Gaussian Quadrature method I am able to integrate and get the value of k i j using this method.

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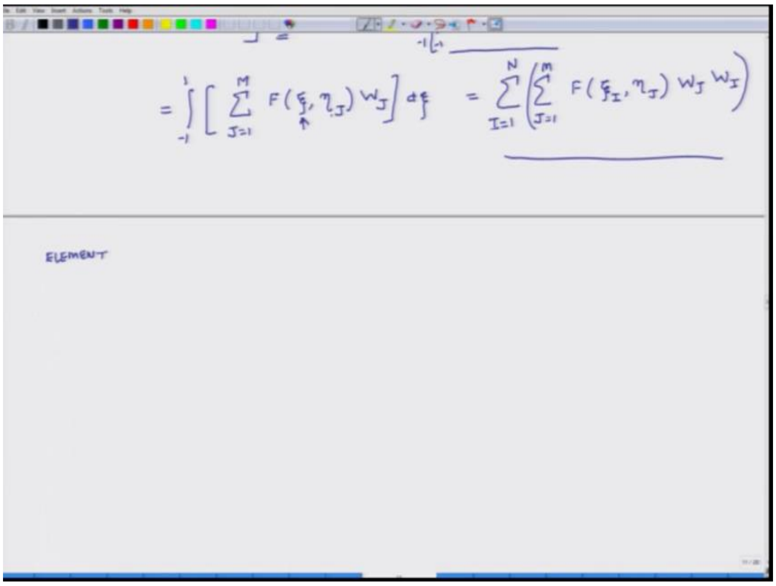

So, what are these Gaussian Quadrature points? We will very quickly look at some of these points element. So, what we will do is, we will conclude our discussion for today and we will continue this discussion on numerical integration tomorrow once again, for a brief period and once we are done with that we will move on to a somewhat different topic.

Thanks a lot and have a very great day Bye.