Basics of Finite Element Analysis - Part II
Prof. Nachiketa Tiwari
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Lecture - 42
Jacobian and transformation matrix for 2-D problems

Hello, welcome back to Basics of Finite Element Analysis Part II, today is the last day of this week which is the seventh week and what we will do today is we will actually calculate this Jacobian matrix and also the transformations for three specific elements and we will learn a little bit more about the conditions required for having valid transformations.

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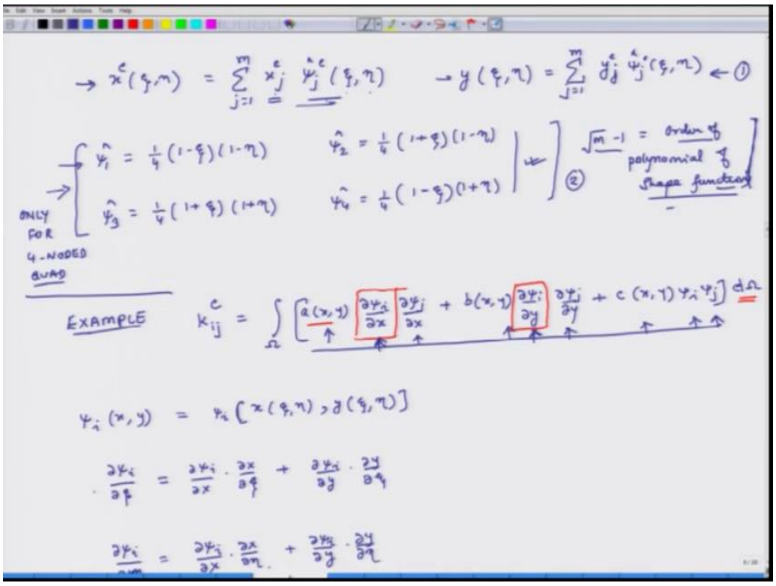

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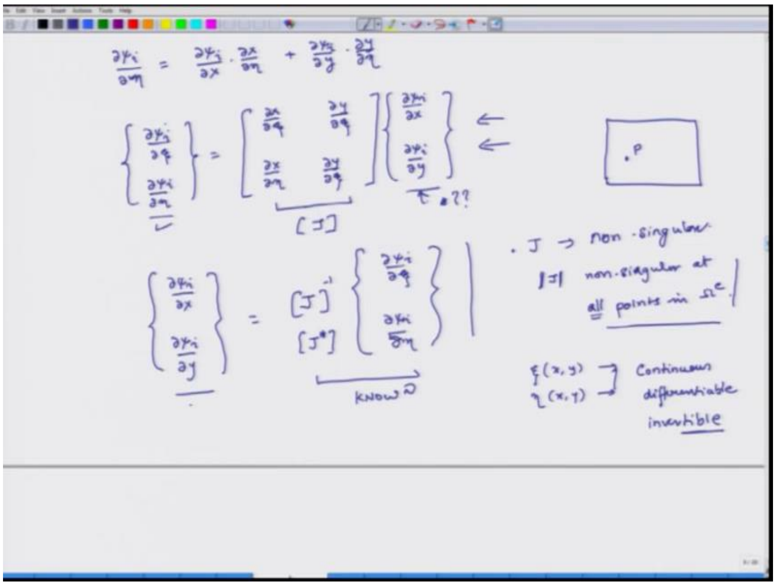

So, what we have shown in the last class was that we have to express, if we have to express partial derivatives of psi functions, which are interpolation functions for primary variable in terms of their partial derivative in terms of x and y then we have to compute the inverse of Jacobian matrix which is j star and multiply that matrix with partial derivatives of psi with respect to zeta and eta this one. Now for this inversion to happen; we have to; if I have to invert a matrix then what should be the requirement for that matrix.

So, if j has to be inverted then j should be non-singular, this is very important, what does that mean? What; that means, is that its determinant should be non 0, its determinant should be non 0 and another thing, this j matrix exists at which point in the element. So, this is all happening at element level, in this entire discussion we have not put superscript e, but all this description is in context of calculating k i j for e-th element. So, all this is happening for e-th element, now the elements of j matrix del x over del zeta, del y over del zeta, del x over del zeta and so on so forth.

They are at what locations are we calculating these derivatives, we are calculating them at all the points in a specific element, because we are trying to find out k i j for e-th element. So, at each point in the element, so suppose I have a rectangular element or a quadrilateral there is some point p, at each point there will be a differential of del x over del zeta, del y over del zeta and so on and so forth.

So, all these terms in this equation they exist at every point in the thing, so if j has to be nonsingular; at which point should it be non-singular. It should be non-singular at all points in the domain. So, it should be non-singulars at all points in e-th element and a condition for that to happen is that zeta and eta which are functions of x and y, they should be continuous differentiable and invertible, it should be continuous differentiable and invertible. So, it has to be non-singular because if it is not non-singular at all points then we cannot create, have this inversion and then we cannot compute partial derivatives of psi with respect to x and y, in terms of their partial derivative, in terms of the zeta and eta. So, this is important, so what we will do is we will do an example; are actually three quick examples.

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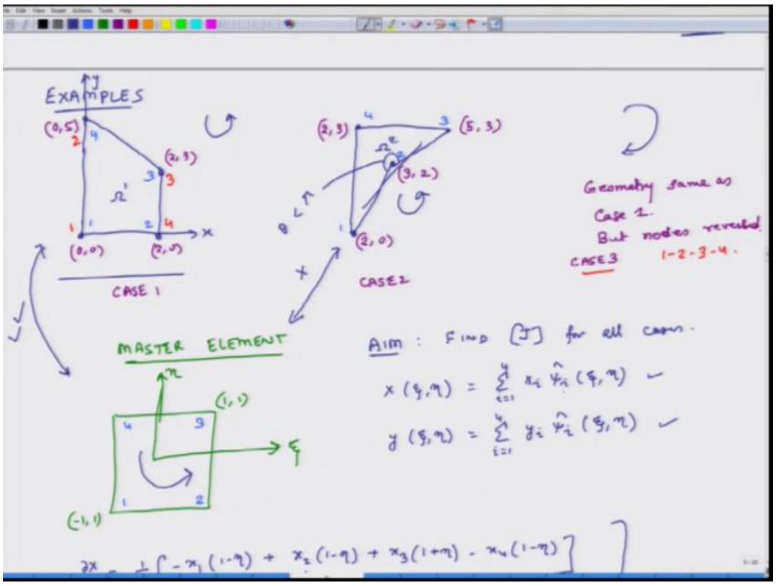

So, the first example is an element which looks like this, it is a quadrilateral I should be able to map it to a square, in the zeta eta domain. So, here my x this is my x axis and this is my y axis and the local coordinates are 1, 2, 3, 4, so this is my first domain or first element. We will see whether we can compute for this or not what are the; to map this; what do I have to know, I have to know the coordinates of this thing. So, the coordinates are 0; 0, 2;0, 2;3 and this is 0;5.

Then I will take another element, it looks like this, so this is also forwarded element. So, this is my second element omega 2 and here also I will first; so whenever we draw an element we have to put its node numbers very clearly, so this is not 1, 2, 3 and 4 and the coordinates are 2;0, 3;2, 5;3 and 2;3. So, what I will suggest is when you revise this, you actually make this element with in actual dimensions with these coordinates and you will see that the shape looks something like this, this is important to understand.

The third case, so this is omega 2 so this is case 1, this is case 2 and then we will also have case 3 and what is case 3; geometry same as case 1, but nodes reversed; nodes are reversed. So, what are the reversed nodes so here I in the origin case 1, I was going 1, 2, 3, 4 anti clockwise here I will do 1, 2, 3 and 4. So, here my node numbers are in red and in all these 3 cases, I have to map these into zeta eta coordinate system where the shape of the element is square size is 2.

So in all these 3 cases, the master element will be the same, so what is the master element; this is there and I will also mark the node numbers 1, 2, 3, 4. So, the node numbers do not change, size of the element does not change, the coordinate system does not change for the master element, it is fixed; does not change. So our aim what do you want, we want to see whether we can invert these things correctly or not. So our aim is, if I have to invert it, if I have to compute this, I have to make sure that the Jacobian matrix is non-singular. So, we will see whether the Jacobian matrix is non-singular for all these three cases or not, so find j for all cases.

So to find j, what do I have to do? We have to find these parameters del x over del zeta, del y over del zeta right. So, what does that mean I have x, this is the expression we had developed earlier right. If I know this expression then I can compute all those derivatives in the j matrix; in this matrix I can compute all those derivatives and psi i hat represents what? What are psi i hats? They are the shape function, there was an interpolation function, but in this case they actually specify the shape, so they are the shape functions which map x and y into the zeta eta domain and they help us transform arbitrary quadrilateral into a square, so these are the shape functions and what are these shape functions? These shape functions will have a value of 1; psi 1 will have a value of one at node 1 and it will be 0 at all other locations.

So, we had developed these functions earlier, we had written expressions these are the four shape functions, so I do not have to write these four shape functions again, so what I will do is del x over del zeta is equal to 1 by 4 minus x 1. So, what is x 1, it is the coordinate of the first node whatever element we are dealing with, if we are dealing with this first case 1 then it will be 0, if we are dealing with case 2; it will be 2.

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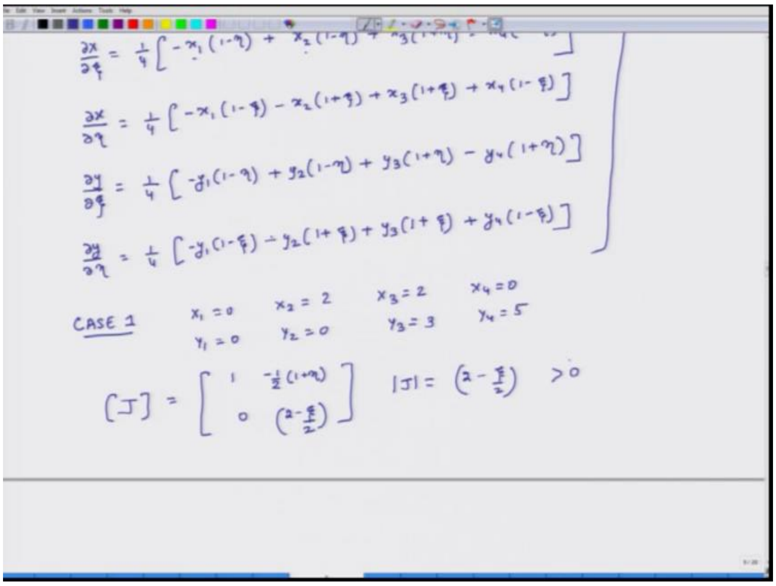

So, it is the coordinate of first node, so del x over del zeta is minus x 1 into 1 minus eta plus x 2 plus x 3 minus x 4, del x over del eta is equal to 1 by 4 minus x 1 minus x 2 plus x 3 plus x 4. So, these are the four expressions and how did we get these four expressions, we essentially differentiated this relation; these two relations and the expressions for psi as well given earlier these are the expressions for psi. So, we plugged in these expressions and x 1, x 2, x 3, x 4, y 1, y 2, y 3 y 4 correspond to the coordinates of local nodes 1, 2, 3, 4 for any element which we consider. So, these expressions do not change from element to element; the only thing which changes is values has x 1, x 2, x 3, x 4, y 1, y 2, y 3, y 4. So now what we will do is using these formulas, we will find out the value of Jacobian for each of the 3 cases.

So, case 1; x 1 equals 0, x 2 equals 2, x 3 equals 0, x 4 equals 0, y 1 equals 0, y 2 equals 0, y 3 equals 3, y 4 equals 5; these are the values, how did we get these values? We got it from this picture. So, what we will do is, we will plug in these values in these expressions and I will write the expression for j matrix and it comes as 1 minus half into 1 plus eta 0; 2 minus zeta by 2. So, the determinant of j matrix is 2 minus zeta by 2, is this 2 minus zeta by 2, always more than 0; yes because what is the maximum value of zeta possible 1, zeta varies from minus 1 to 1. So even if it is 1, then the value of this j matrix becomes 1.5, so this is always 0 which means that I can validly transform this quadrilateral in case 1 to a master element back and forth and that transformation is valid and I do not have a problem, now let us look at case 2.

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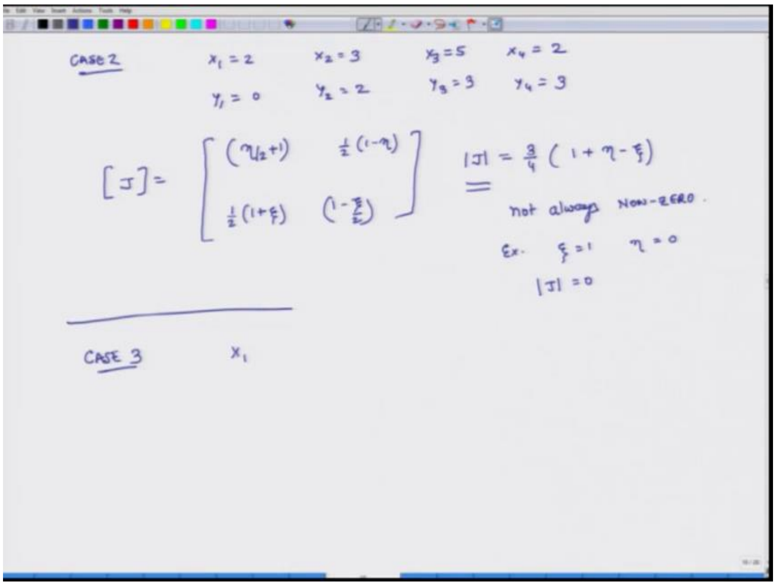

Here x 1 was 2, x 2 is 3, x 4 equals 5, x 4 equals 2, actually this is x 3 and y 1 equals 0, y 2 equals 2, y 3 equals 3 and y 4 equals 3. So, the Jacobian matrix becomes eta by 2 plus 1, half 1 minus eta half 1 plus zeta and 1 minus zeta by 2. So, the value of determinant is 3 over 4 into 1 plus eta minus zeta. Is this Jacobian always positive, can we find at least one single point where it is not positive or not 0, it is not always positive.

Actually it is not always non 0; we have not talked about positive and negative till so far. For example, even if we find one single point when it is non 0 then the transformation will not work, example when zeta equals 1, eta equal 0; Jacobian comes to be 0 and we can find several such points actually. So, what that means, is that I cannot transform this to this and back and forth, this is not valid. Here to here, it will work; no problem but case 2 to case 3 will not work and then the last case; case 3 again you can put in these different values of x 1, x 2. So, here the element was same, but the numbering changed, so x 1; node number 2 changed, node number 3 remain same, but node number 4 changed, so the numbering changed.

So, in the first case the numbering was anti clock wise, see here it was anti clock wise and in first case it was also anti clock wise and here also it was anti clock wise, but in third case it was clock wise; numbering was clock wise. So, in your master element it is anti clock wise, in your actual element it is clock wise; what does that mean? Let us see, x 1 is 0, x 2 is 0, x 3 is equal to 2, x 4 is equal to 2, y 1 equals 0, y 2 equals 5, y 3 equals 3, y 4 equals 0. So, the Jacobian is eta 2 minus eta by 2, 1 minus 1 plus zeta by 2.

So, it is determinant is minus 2 plus eta, zeta by 2; can it be ever 0, what is the maximum value of or minimum value of eta times zeta minus 1, zeta eta they vary between minus 1 and plus 1. So this term, add the minimum it will be 2 minus half which will be 1.5, but there's a negative sign here. So, it will always be negative and when it is negative, this type of transformation is not desirable you will get your answer, but what will happen is at an element level there will not be any problem, but when you do that assembly level, some terms will become negative, some terms will become positive and the assembly will not be valid.

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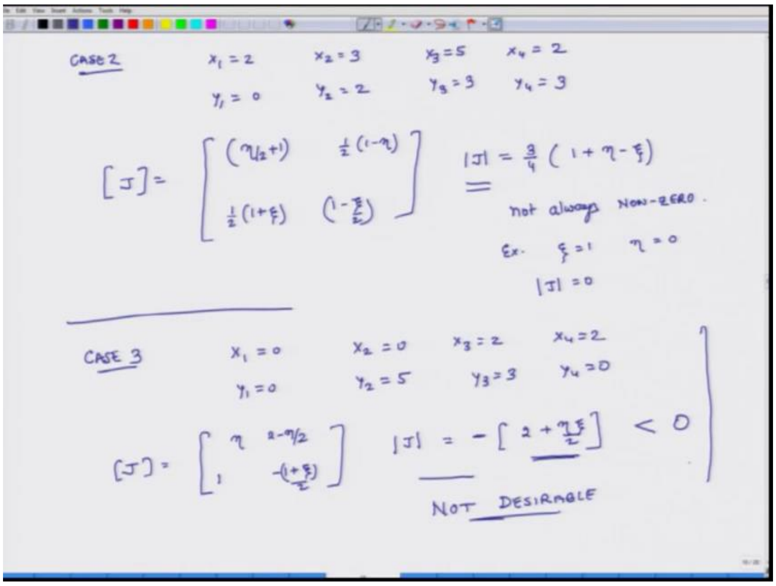

So when you are doing these transformations, you have to make sure that the node numbering in each element is consistent, if you are using this anti clock wise in your master element, in every element it has to be anti-clock wise, so this is one; couple of things. So, case 3 we do not like it because when we have assembly that would be problems, case 2 is the situation where it is the matrix is not; j matrix is not invertible, so we cannot compute k matrix and why it is not invertible. In first f a course, we had said and with this we will close the discussion for today that the domain should be convex.

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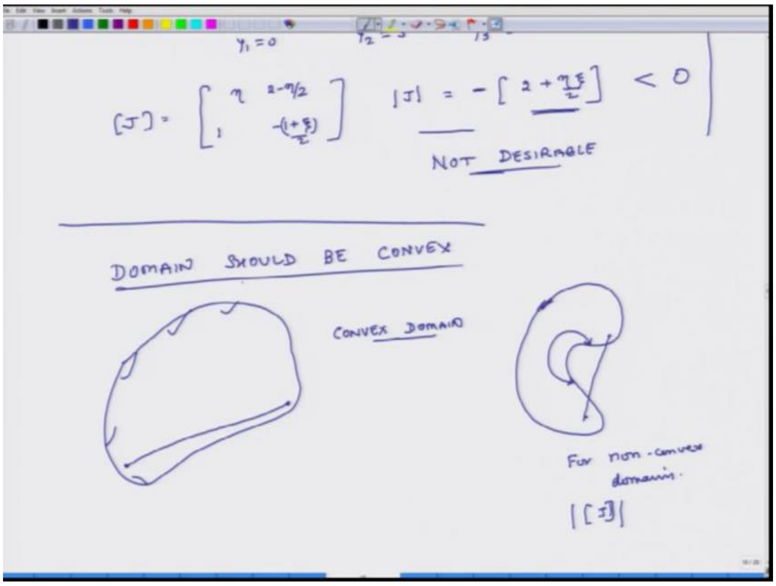

And what was the definition of convex, that if I have any point, so this is my domain if I have any point in the system, I should be able to connect these 2 points with a straight line and the straight line should not go out of the boundary of the element. Between any 2 points, so in this shape this is a convex domain, now if I have; this is not a convex domain because if I have to connect these 2 points, I go out of the system. So far non convex domains, my Jacobian will not necessarily be non 0 at all points; it will not necessarily be non 0.

Now in this case this element is it convex or non-convex, it is non-convex because if I connect these 2 points, I have to go out here. So, essentially what that means is that this angle should never be all interior angles should never be; so theta should always be less than pi. If all interior angles in the element are less than pi then I will have a convex domain, see here also the angle is more than pi. Here at every point the angle is less than pi, so j is invertible only for non-convex domains.

So, two important things we have learnt that we have to make sure that our elements should remain convex and the second thing is that our numbering is important otherwise at assembly level we will have a problem. So, with this we conclude the discussion for this week and in the next week, we will start developing details on some other problems and with this we conclude the discussion and thank you very much and have a great day bye.