Basics of Finite Element Analysis - Part II
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Lecture -41
Numerical Integration scheme for 2-D problems

Hello, welcome to Basics of Finite Element Analysis Part two. This is the seventh week of this course. And what we will be discussing today as well as tomorrow, is how do we go around doing numerical integration for problems which are to dimensional in nature. So, we had discuss this concept for one dimensional situations and we will extend the same thinking to two dimensional problems, but before we do start discussing the two dimensional context very quickly, we will recap what we did for one dimensional systems.

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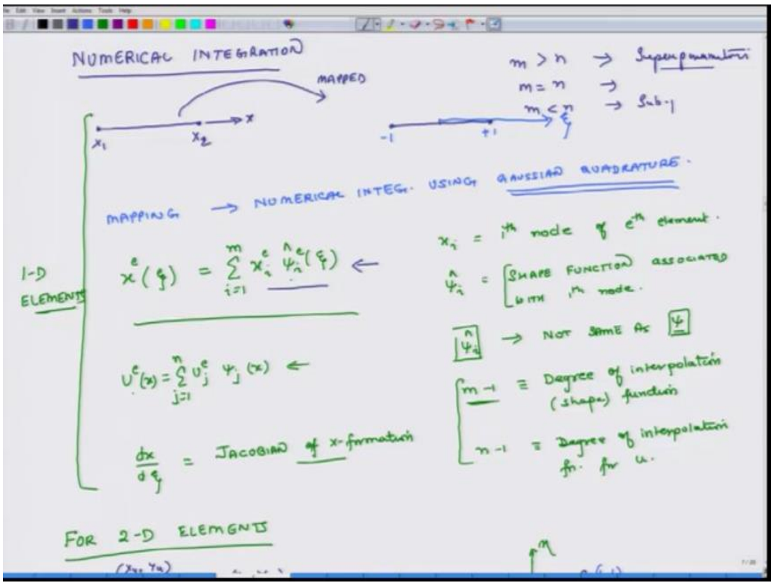

So, our theme is going to numerical integration. And when we were doing the numerical integration in one degree one dimensional situation, then the first thing which we had done was that if we had any element. So, it is in x coordinate and the coordinates of the nodes are x 1 to x 2. So, this is a two nodded element, but the same thinking we had applied for quadratic and cubic elements then we had mapped this, element to another element. And here the coordinate was not x, but it was zeta right and what the dimensions of the element. This was plus 1 2 minus 1. So, we were, we had mapped this element to zeta coordinate system. And then we did numerical integration in the zeta coordinate system using Gaussian Quadrature.

So, the first thing we did was mapping. And then we did the numerical integration. Using actually we discussed two methods one was Gaussian Quadrature. The reason we have to map this in we said that we have to map it in the zeta domain is because, Gaussian Quadrature. Works the numerical integration scheme of Gaussian Quadrature works in zeta domain, where the size of the element is minus 1 to 1. So, that is why we have to map it, now when we did this mapping. What did we do? We had written expressions something like this, that x. So, in the x domain the coordinate is x. When I am mapping it is nothing, but a function of zeta. And this is equal to x i psy i head which is a function of zeta. i is equal to 1 to m and here, x i is the i-th node of. So, I will also put a superscript e. So, it is the i-th node of e-th element.

So, if it is a linear element then it will have two nodes. If it is the quadratic element then the node will have on each element three nodes. So, on and. So, forth and psi i hat is the shape function associated with i-th node. And please note that psi i hat. So, this the shape function associated, with the i-th node which helps us map from x domain to the zeta domain, this is not same as psi, because these psi is where what, these are approximation functions which are used to approximate u or v or w right. So, this helps us approximate the primary variable, but this hat psi i hat it helps us map from x domain to zeta domain.

So, that is why we have put a hat here, to mark the distinction and then the last parameter and this m. And what is m? m is minus 1 corresponds to degree of interpolation function. Or in this case, because it helps us map the shape of the element. So, I can also call it shape degree of shape function, m minus1.

So, for a linear element m is 2, because there will be two shape functions and m minus 1 is tells us that the degree of polynomial for each shape function will be 1. It will be linear element. If there are three nodes then the degree of interpolation function or shape function in this case, will be two.

So, forth now again have put m minus 1. For use right we had n, because we had also approximated u at element level. And that we had written as u e j psi j x j is equal to 1 to n right. So, this is an approximation. Here, psi j is the approximation function for u here, psi i hat is the shape function. Which helps us map between x and zeta domains. So, it is important to understand this, and here n is what n minus 1 represents degree of interpolation function for u.

So, this is distinctions here, important to remember this distinction. And finally, we had said that d x by d zeta is what we given it a name. If we remember and we said that it is a Jacobean of transformation. So, this is this entire thing was for 1-d elements. Now we will extend this thinking to 2-d elements.

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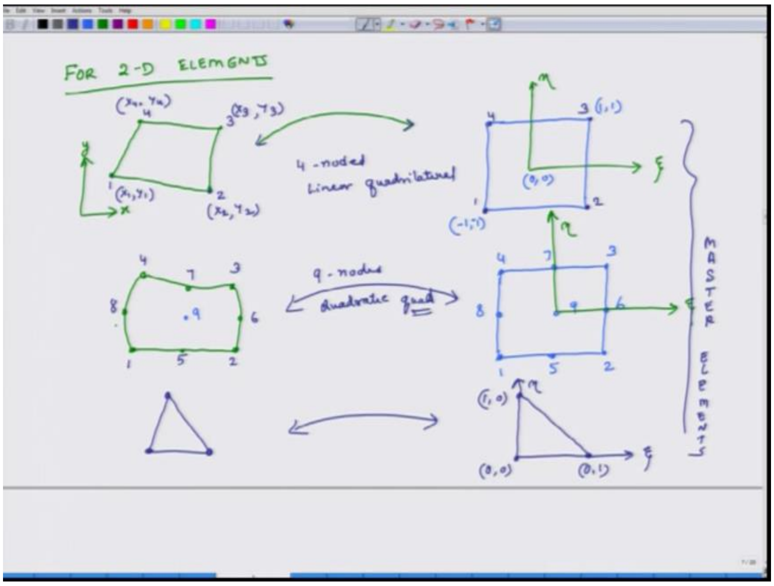

So, for 2-d elements if I have any element, it could be a quadrilateral and this in x y coordinate system. If I have to do numerical integration because, I do a lot fem using computers.

So, I have to use numerical approaches, if I have to do numerical integration. Then I have to map this, into because, and when I do numerical integration, I use Gaussian Quadrature method of integration. Then I have to integrate in the zeta eta domain. And here the size of the element is 2 by 2. So, this is my 0 0, this is 1 1, this is minus 1 1. And let us look at the node numbers. So, these are local node numbers 2, 3, 4 and in the zeta domain. This is node 1, node 2, node 3, and node 4.

So, this is the map. Now here, the mapping is for a four nodded linear quadrilateral. The coordinates of this four nodded quadrilateral are x 1 y 1, x 2 y 2, x 3 y 3, x 4 y 4. So, node one in this original actual coordinates system gets mapped to minus 1 minus 1 in this zeta eta domain. So, any quadrilateral I can map into a standard square element in the zeta neta domain. And what is a relation will see that in a while, but this for a four nodded quadrilateral. For an eight nodded quadrilateral. Suppose this is an element, and I am having the edges may be curved, there is no rule which says that edges can be curved. Actually I will change this something like this. So, here 1, 2,3,4,5, I missed one node here, 6, 7, 8.

So, again here, it is the x coordinate system. It gets mapped into the zeta neta domain again as and there is a ninth node here 9. So, this 1,2,3,4,5,6,7,8,9 and my coordinate system is still same. So, here a eight nodded quadrilateral with curved edges it maps into rectangular not rectangular a square in the zeta neta domain, and if I have triangular elements. So, I can have any triangle then it gets mapped as a right angle triangle.

So, this is 0 0, this is 0 1 and this is 1 0. Similarly if I have a triangle with curved edges, I can map it as a quadratic you know second order. So, this is nine nodded quadratic quad elements. Here, this is a three nodded linear triangle; I can map a three, six nodded quadratic triangle into a standard triangle in the zeta neta domain also. If this is three nodded, but I can also have six nodded which is quadratic triangle.

So, this is the master element, these are master elements. We do numerical integration on master elements. Gaussian integration Quadrature is performed master of elements. So, what we will do is we will discuss only this transformation. In this course, this will give you an idea, but if you want to learn about other transformations you can refer the books and the theory is very similar. So, we will only cover the transformation of a four nodded quadrilateral to a four nodded master element in the zeta neta domain.

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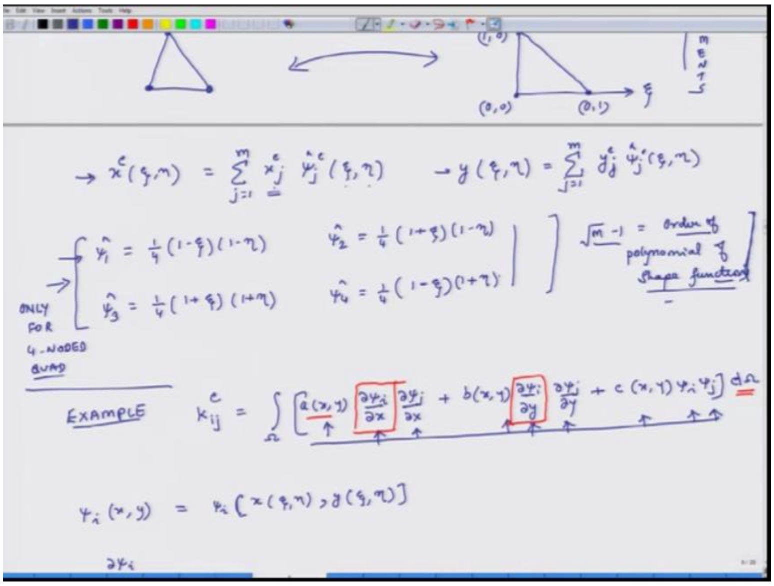

So, what are the transformation relations? So, x is a function of these two parameters zeta and neta. So, in our 1-d we had this as the transformation x of each element was a function of zeta and it was this functions right. And extending the same thinking here at element x and this is equal to. And because in two dimensions we have two coordinates, we have a relationship not only for x, but also for y. So, their relation to map y, this one we had here is this. So, this psi or shape functions are in, what domains there are in the neta zeta domain?

So, these shape functions do not change from one element to other element. The only thing changes from if I change from one move from one element to other elements these coordinates. So, x j and y j are the coordinates in x y coordinate system for any specific element. So, these are specific to each element and this is hat. So, the psi function does not change, because the psi function is always in zeta eta coordinate system with and the location of this square is at the is standard. So, these shape functions do not change. When I move from one element to other, but what changes is the coordinates of each element. So, the coordin transformation relation it changes forms one element to other.

So, what are these shapes functions, and they change their standard for any four nodded quadrilateral. So, it is psi 1 equals 1 over 4 1 minus zeta 1 minus eta. So, i 2 hat equals 1 over 4 1 plus zeta 1 minus eta, sides we had equals 1 over 4 1 plus zeta 1 minus 1 plus eta psi for hat is 1 over 4. So, these are the standard relations. If I use this transformation, I can map any four nodded quadrilateral in a regular real coordinate system to square element of dimension to n zeta neta coordinates system. And then neta zeta coordinates system, I can integrate it using Gaussian Quadrature. And get my value s for different elements of k and f matrix or etcetera k matrix and f vector.

So, this psi these are valid only for four nodded quad. This psi is only valid for four nodded quadrilateral as we have seen several times earlier. So, I will like re instate restate the value of psi 1 at node one is 1 and at node two, node three, node four, will be 0 the value of psi 1 at node one. So, this is my node one this is node one, two, three, four. So, these psis have been such crafted that the value of psi 1 is 1 at node one. What is the value of zeta and eta at node one this is node one minus 1 and minus 1. If I put minus 1 and minus 1, here I get minus 2 times minus 2, no 2 times 2 divided by 4, is 1. At all other nodes this will work out to be 0 the value of psi 2 will be 1 at node two and at all other nodes it will be 0 and so on and so forth.

So, it is important to understand this. So, m minus 1 is the order of polynomial of shape function. So, if I have an eight nodded quadratic element or an nine nodded quadratic element. Then it will get mapped in a nine nodded square. In that case, m will be 3. So, here it will be m. So, in that case, it will be 9. So, m will go from m 1 to right. So, what is m? So, in case, of one dimensional element we had said that m minus 1 is the order of polynomial of the shape function. In this case, it is square root of m minus 1 that is the order of the polynomial of the shape function.

So, as another similarity here, is that in this case, when in for 1-d situation, when m was greater than n then what was the formulation, super parametric. That is when the degree of shape function was higher than the degree of interpolation function for u. Then we said that it is super parametric formulation, when it was same as n then it was iso parametric formulation right. When m was less then n then it was sub parametric formulation.

So, the same, classification of formulation of fe problems exists here, that we have super parametric, iso parametric and sub parametric formulation. So, what have done till. So, far is we have developed relations, which transform x coordinate into zeta neta coordinate and y also into zeta neta. So, using this coordinate transformation, I can transform a four nodded quadrilateral into a square element of size two in zeta neta coordinate system, but what is our original? Our original goal is to do numerical integration; that we can easily compute the values of k i j and f i for element matrixes. So, let us look at an example.

So, k i j for e-th element, we have developed earlier this relations several times, we can expert it as e x y some constant times del psi i over del x del psi j over del y or in this case, I will still keep it del x plus b of xy del psi over del x, I am sorry, del y del psi j over del y plus c x y psi i psi j d. So, suppose I have to calculate this integral.

So, that I can assign the values of k i j for a specific element, then how do I do that. I have what I do is, I have to compute each of these terms. In terms of zeta and eta because, then if I reassign these terms in terms of zeta and neta. Then I can do the numerical integration using Gaussian Quadrature. So, what we will, for now what we will next 5 to 7 minutes is we will learn is, how do, how this term and this term does. How I can express it in terms of zeta and neta. Then later we will know how to handle these terms and also how to handle these terms. We will individually look at each of these terms and we will change the scope of the problem.

So, we know that zeta psi is a function of x and y. Now what is psi in this case, this is not psi hat. So, it represents the approximation function associated with u. psi hat represents approximation, or shape function associated with this transformation psi represents approximation function associated with u. So, this we can write it as psi i an x an x itself is a function of these and y also is a function of these parameters.

So, del psi i over del x. So, suppose I have to compute del psi i over del x. Then I can write it as del psi i over times del x, here psi is a function of x, but x itself is a function of zeta and neta. So, if I have to differentiate with respect to x, I differentiate with respect to zeta with x. and then I differentiate it with respect to eta times x. This is I will actually modify this, and I can express this whole expression in a different way. So, I am interested in finding the partial derivative of psi i with respect to x right, but I know that these relations psi i with respect to zeta and neta, I know these relation right. What I will do is first I will write an expression. This I can calculate psi i with respect to zeta using these approximation functions I can very easily calculate these.

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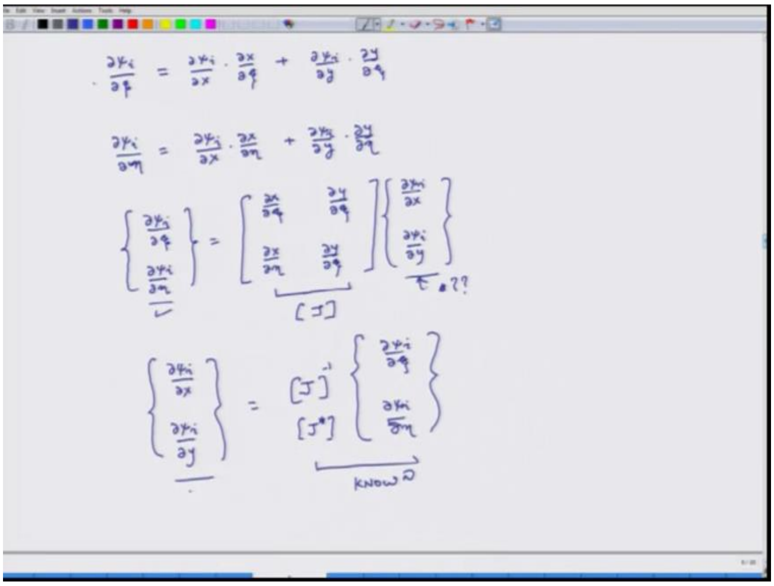

So, I know this thing, and this I can also write it in terms of x. So, this is psi i over del x times del x over del zeta plus del psi i over del y times del y over del zeta. So, I know what is a value of left side and on the right side, I know something's I may not know something's agreed. Similarly, del psi over del y del eta equals del psi i over del x times del x over del eta plus del psi i over del y times del y over del eta. Or I can write this in the matrix form, del psi i over del neta del psi i over del eta equals 2 by 2 matrix. And in this 2 by 2 matrix I have del x over del zeta del y over del zeta del x over del eta and del y over del zeta times del psi i over del x del psi i over del y in this equitation. For k i j I am interested in finding, del psi i over del x and del psi i over del y, from these relations for approximations functions. I can calculate this side left side this is what I am in interested in right.

So, this is what we want find out. So, if I know this matrix. I already know this left side, if know this matrix, then I can find out this unknown vector del psi i over del x and del psi over del y, how do I find out this matrix. The way we find out this matrix is that. These are my transformation equations. So, this is equation one this is equation two right. So, from equation one, I can find out del psi i over del zeta I am sorry, del x over del zeta, why because these expressions are given here this.

So, equation one helps us compute this matrix. Equation two helps us compute the left vector. So, then I can calculate this unknown column. Understood, this matrix is called the Jacobean matrix. So, in one dimensional system the Jacobean was a number in a two dimensional system, the Jacobean is a matrix, in a three dimensional system Jacobean will be symmetric matrix. There we will have del psi i del x over del zeta del x over del del y over del zeta and del z over del zeta so, one and. so, forth.

So, the movement from 2 d to 3 d is still same, we still have to deal with matrices it is from 1-d to 2-d, we go from now numbers to matrixes. So, if that is the case. If this is a Jacobean, then I can write del psi over del x del psi i over del y equals j inverse del psi i over del zeta del psi i over del eta. Lot of times I also call this j star. So, that I do not have to write it several times.

So, I compute the partial derivative of psi i with respect to x and y, by first computing with j star vector, which is the inverse or j vector a j matrix, and then multiplying this j star matrix with partial derivative of the size with respect to zeta and eta. And this site is known. So, I can compute del psi i over del x and del psi i over del y.

So, in this way, if I have to compute this k i j, I now that I have will be calculating this integral in what. Using which method Gaussian Quadrature method, because I have to do numerical integration for Gaussian Quadrature method, I have to go a zeta neta coordinate system. And if I have to do in zeta neta coordinate system, I have to express these partial derivatives in terms zeta and neta, and that is how I express them in terms of neta and zeta. So, we will continue this discussion in the class.

What we will do in the next class is that we will actually do an example. So, that you learn a little bit more about how to do these operations.

Thank you very much.