Basics of Finite Element Analysis – Part 2
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Lecture – 04
Weak Formulation

Hello, welcome to basic of finite element analysis part 2. Today is the first week 4th day of this particular course and what we will do is, we will continue the review of some of the important concepts we covered in FEA 1. What we are going to talk about today is weak formulation because; this is (Refer Time: 00:38) formulation will be using again and again throughout this course.

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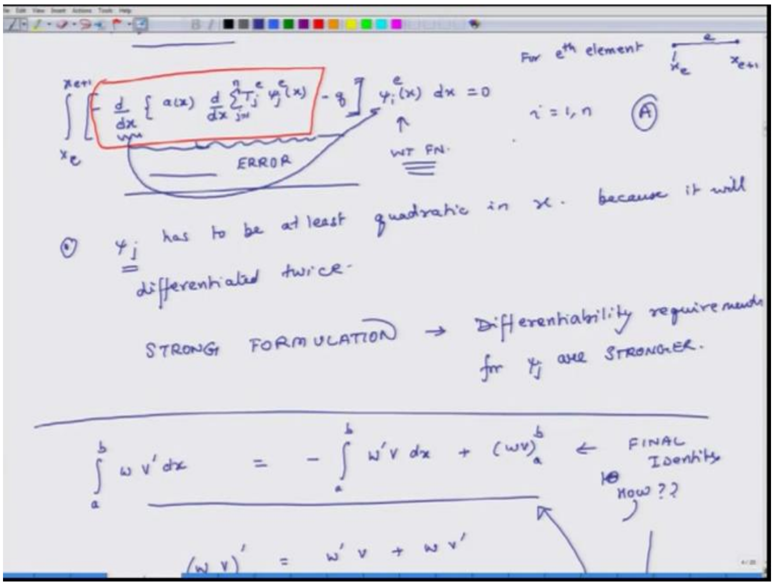

So, this is weak formulation, now let us consider a differential equation. Suppose you want develop a solution for this equation over a domain. Now let us say the domain is x is equal to 0 to l, so we want to develop the solution. Here x is equal to 0 and this is x is equal to l.

So, the first step in the finite element processes, that we break the domain into smaller elements. So, that is what we have done, so this is first element, second element, third element, forth element and so and so forth. You can develop break it up in to as many elements as we want. The second step is that we find the error know we assume an interpolation function, using that interpolation function we find the error in the equation, multiply that error with the weight function and integrate the product of weight function in the error over the domain of the element and equate it to 0, so that is what we will develop. So, consider e th element and for eth element, this is the e th element its first coordinate is x e and second coordinates x e plus 1. So, I am going to integrate the weighted error or weighted reside well to 0 in this domain x c 2, x c plus 1.

First we will write the expression for error minus d by dx ax and u will be replaced by an interpolation function. So, it is d over dx, T j, psi j which is the function of x and this is for the eth element; minus q and this dx equals 0. So, this is the error, I made a mistake I have to multiply by a weight function. So, psi i which is the function of x dx equals to 0. So, this is the error, this eth item and this is the weight function, so error times weight function I integrated to over the entire domain and make it 0. Once again or and this T j psi j has to be added up, this is the summation and here it is j equals 1 to n and this represents how many equations, this represents n equations because psi i can take n values i can be 1 to n.

So this is the thing; now when you look at it and I will ask you to look at this term carefully the one in the red box and we make some observations. So, when I differentiate T j psi j, T j is a constant. So when I differentiate basically I will differentiating psi j, once and then when I differentiated again because there is another d by dx operator so, psi j has to be differentiated twice if we have to calculate the error which means, that for this equation psi j has to be at least quadratic in x because, it will be differentiated twice. If it was a linear function let us say a plus bx then when I differentiated once I get a constant of when I differentiated again what do I get 0. So, then it will not make sense right?

So, it has to be at least quadratic to get a non zero entity it has to be at least quadratic in nature, so this is the thing. When we look at the weight function, the weight function is not differentiated at all. So, I can take a weight function of any order, but psi j which represents in approximation of the variable that has to be at least contract in nature, so this is important to understand. So, this kind of a formulation has a different requirement for the weight function. Here the weight function could even be a constant, but psi j has to be at least quadratic. This kind of formulation where the requirement of psi j to be having a higher order of polynomial nature compare to weight function is called as strong formulation. It is a strong formulation because differentiability requirements for psi j are stronger that is why it called as strong formulation.

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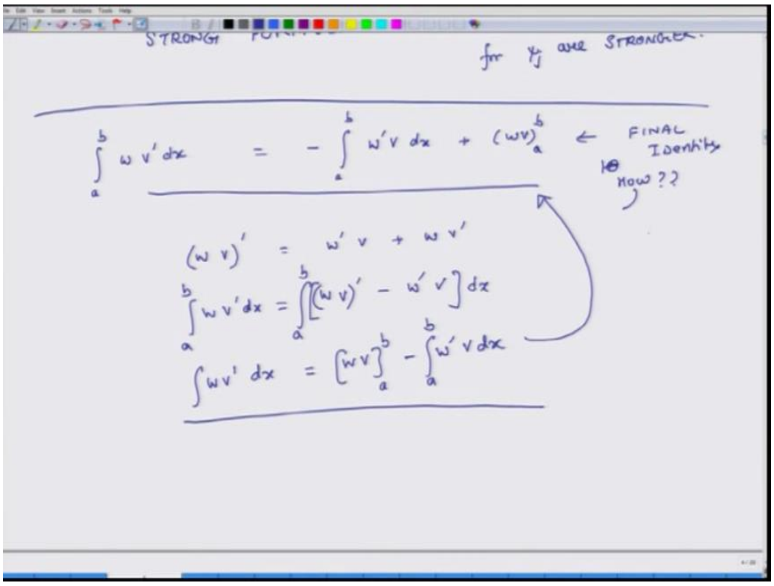

Now, we will look at what is called as week formulation. Before we talk about the week formulation we will look at one mathematical rule. So, we know that if I integrate w v prime dx so, this is something different in the limits a to b. This is same as minus of integral of a to b, w prime v dx plus w v, a to b. How do I get this? I know so this is the final identity and how do I get this? Consider w v, if I differentiate it, this will be w prime v plus w v prime or w v prime equals w v the whole thing differentiated minus w prime v. When I integrate both sides, what do I get? This is what I get, so this is same as this earlier relation. Now I use this rule to shift the differentiability operator on this expression, so let us call this equation A.

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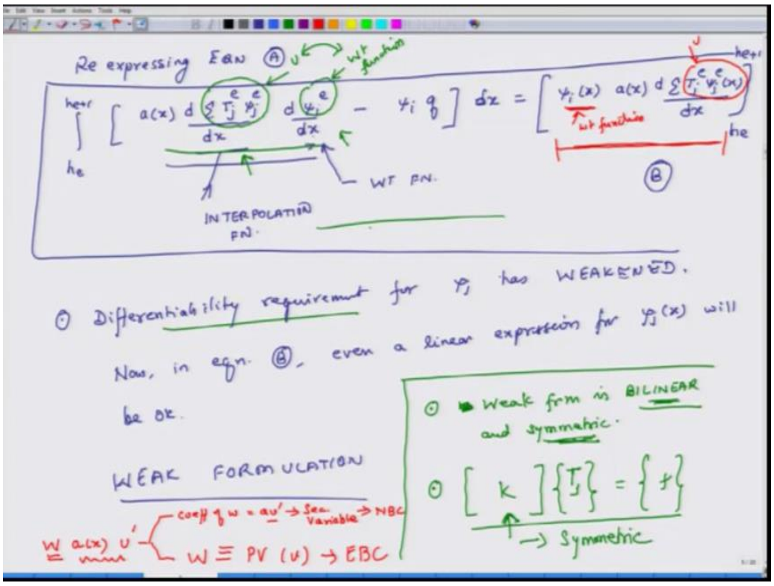

So, using this rule I can re express equation A in the following form. I can re express it as h e to h e plus 1, a x d T j psi, j psi j the function of x time dx, times d psi e over dx. So, what I am done is this here, this differentiability operator has shifted to this term, using this relation; using this method. So, this is why I am getting d psi i over d x minus psi i times q and then I get a boundary term see even when I do this integration I get a boundary term here w v evaluated at a and b. So, this whole thing is integrated with respect to x and that equals psi i, x, a x d T j psi j x differentiated with respect to x.

So, this is the boundary term and it is evaluated at h e and h e plus 1. So, this term is related weight function, this term is related to interpolation functions and what we see in this expressions so, let us call this expression B. So, what we see in this expression B is that psi j is been differentiated only once. In the expression A it was getting differentiated 2 times, here it is getting differentiated only once and psi 1 is also getting differentiated in only once which means, differentiability requirement for psi j has weakened. So, earlier we needed at least the quadratic expression now we can even handle with manage with a linear variation.

So, now in equation B, even a linear expression for psi j which is the function of x will be ok. So, mathematically equation A and equation B are same, but we have a more flexible choice in terms of selecting psi j. So, that is why this type of formulation is called a week formulation because the requirements for differentiability are not as a strong, they are weaker; weak formulation. So, this is a very important distinction between weak formulation and strong formulation. We will make some other observations.

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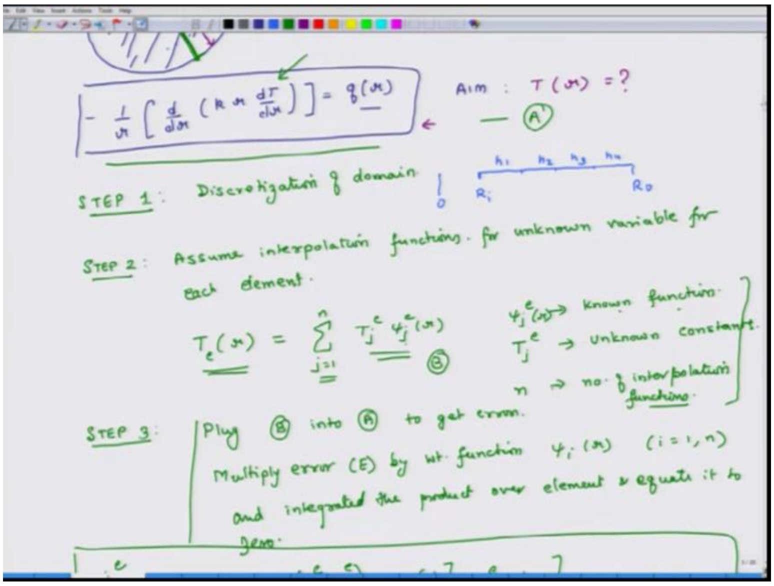

Now, it just happens that at least in this equation see our original equation was this. This is the original differential equation and this is an equation which is a linear in T because, the power of T is 1. It is a second order differential equation; second order ordinarily differentiated equation because I am having 2 times differentiation, but the order of T power of T is 1. So, it linear in T, when we do week formulation is especially for linear differential equations, in a lot of cases we end up this situations where the week formulation is bilinear and symmetric.

This is true only for linear systems. What does this bilinear and symmetric mean? Bilinear means that we have d psi over dx and we have d psi j over dx. So, both of these are linear terms. These psi i over dx and when we do differentiation here we will get d psi j over dx. So, this is one thing and the other thing is if I replace psi j with psi i and psi i with psi j; equation does not change. So, this is bilinear as well as symmetric, this is a functional and this functional is bilinear and symmetric, it is bilinear and symmetric. See this is weight function, this is the interpolation function and this is bilinear weight function and interpolation on in the primary variable.

Second thing is just to clarify; this entire expression represents what? Actually I will let me erase this; this entire expression it represents u and this entire expression, it represents weight function. Either you wanted to make it more clear what is bilinear and symmetric.

Now, you saw this equation is linear in u because the power of u is only 1 and this equation is also linear in weight function. I can see like I can replace this T j psi j by u and I can replace psi i by someway another number letter w; weight function. So, this equation is linear in u because its power is only 1. This equation is also linear in weight function, power is only 1, so that is why it a bilinear in u and w. The second thing is it is symmetric. Why it is symmetric? If I replace u by w and w by u the equation does not change. If I replace u by w and w by u equation this entire expression it does not change, so that it is symmetric. So, because of this property, when I express at an element level in matrix form I will get some matrices form equations. This matrix k matrix will be symmetric because the week form is symmetric in nature.

So, week form not only in reduces the differentiability requirement, but it also ensures that the k matrix is symmetric for linear systems, for linear conservative systems. We are discussed all this terms and in our previous lecture. The third thing is it helps us identify what kind of boundary conditions we should worry about. So, let us look at this term, which is on the right hand side. Here T j psi j it is what u, is it right? This is u we are approximating u as T j psi j. So, T j psi j is u and this is the weight function, so if I look at it a little more closely, what I have is this is weight function psi i I can call rewrite it as w and then I have a x and then I have T j psi j summation of T j psi j is nothing, but u. So, this is u prime right T j, psi j and I can break this expression in to 2 parts.

One is coefficient of w, this is w this is the coefficient. So, this is a u prime and this is known as secondary variable, if I have to specify this value I say that I am going to specify natural boundary condition. What does a u prime represents, suppose I have a bar and I am pulling it, u prime is essentially strain du over dx. du over dx is strain and a is a multiple of e and cross section area.

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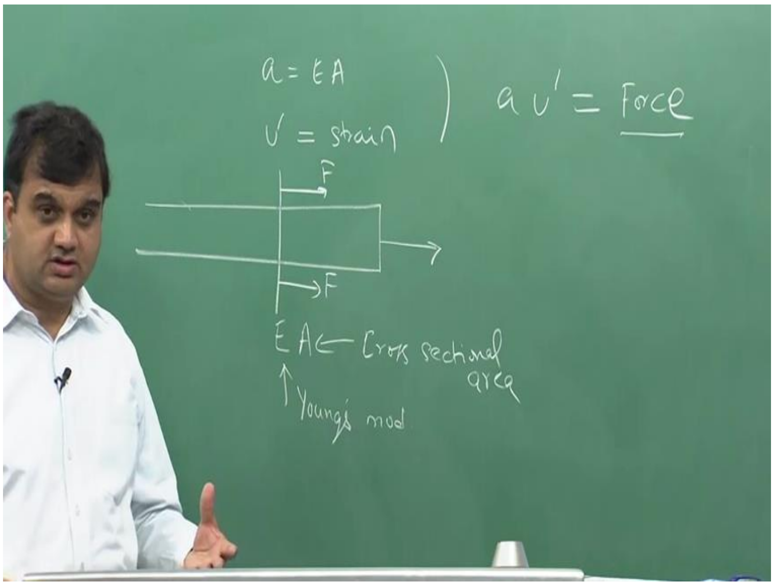

Suppose I have a bar, I am pulling it, at this cross section e times a. e is Young's modules, a is cross sectional area and so a is equal to e a at least in context of the bar problem and you prime is strain right so, a times u prime equals force.

So if I have a bar and I am at this point somehow gripping this bar and pulling it in that directions then I know what is the force being applied here and that force will be a time u prime. So, this force if I am applying from externally, then this force is called a natural boundary condition. So, in constant of the bar problem this a u prime is known as the secondary variable and it is and if I specify a I say that I have a specified natural boundary condition and then the other thing is w itself. So, w is the weight function and earlier in FEA1 we had discussed that it represents variation in the primary variable. So, w corresponds to, so it is not same as primary variable but it corresponds to primary variable and the primar4y variable is u.

So, if I specify u it is the primary variable and then when I specify it I call it that I specified essential boundary condition. So, this boundary term, it helps us figure out what boundary conditions do we need to know in a system. So when we do this week formulation, it not only reduces the differentiability requirement for the interpolation function but it also generates a boundary term and look at the boundary term we can figure out what kind of essential boundary conditions and are natural boundary conditions are required to solve the problem. So, this is the second very important benefit. So, with this we close the discussion for today and we will continue this discussion tomorrow by actually illustrating this in detail.

Thank you.