Basics of Finite Element Analysis – Part II

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Lecture – 36
Boundary integrals for Triangular element

Hello. Welcome to Basics of Finite Element Analysis Part II. Today is the last day of the six week of lectures, and what we plan to do today is continue the discussion which we were having in the last lecture and specifically what we are trying to do complex is develop methods for finding out the boundary integrals and that two in particular context of a triangular element. So, in context of the triangular element what we said was that.

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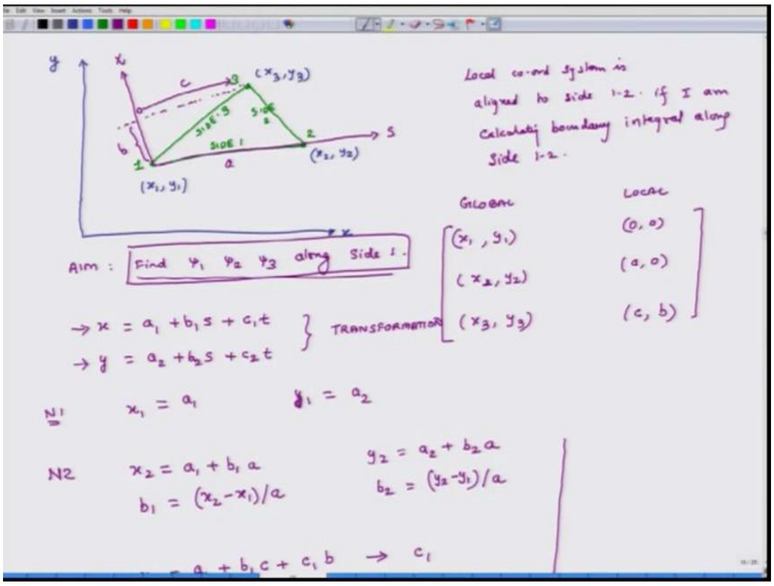

We are interested right now only in finding the integral alongside one. So, for that purpose we will use different coordinate system local coordinated system with coordinates s and t. And the transformation between x and y coordinate system and s and t coordinate system is given by these relations and we had figured out the values of a 1 b 1 c 1 a 2 b 2 c 2 and so on so forth.

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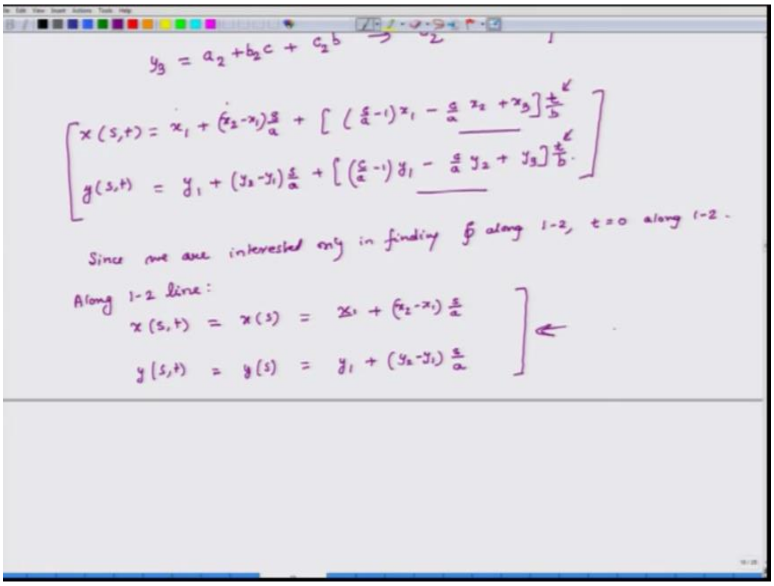

So, with these values are relation for X now x could be expressed as a function of s and t, it could be written as x 1 plus x 2 minus x 1 times s divided a plus c over a minus 1 x 1 minus c over a x 2 plus x 3 times t divided by d and expression for y. So, this is by a 1 this is by b 1 and so on and so forth right. So, expression for y is equal to y 1 plus y two minus y 1 s over a plus c over a minus 1 y 1 minus c over a y 2 plus y 3 t over b. So, these are the transformation equations and here for every point in local coordinate system s and t I can calculated x and y coordinates using these formulas, because everything in this equation is known x 1 x 2 x 3 y 1 y 2 y 3.

So, for every point s t in the local coordinate system I can compute its x and y values, now what we are interested in is we are not interested in finding any properties at all points in the domain, we are only interested in finding the points along the line 1 2 now along the line 1 2 the values of t is along line 1 2 or along x axis. What is the value of t 0? So, and because we are interested only in finding out the values of boundary integral along that line, this term will be 0 because t is 0 here right. So, since we are interested only in finding boundary integral along 1 2 t is 0 along 1 2 line along 1 2 line x s t.

So, in a general case x is a function of s and t, but now because t is 0 x is only a function of s along 1 2 line and that is equal to x 1 plus x 1 minus x 1 times s by a and y which in general case is a function of s and t is now along 1 2 line it only depends on s. So, y is only a function of s and that is equal to y 1 plus y 2 minus y 1 times s by a. So, what does this mean? What this means is that the boundary integral is only and integral for a 1 d element because this is here the element if I have to make elements along this line it will be only a one d elements. So, it is only a one dimensional integral, this is important to understand.

So, let us call this equation 1, now are original goal is to find psi 1 psi 2 psi 3 and use these size to compute the boundary integral along side 1.

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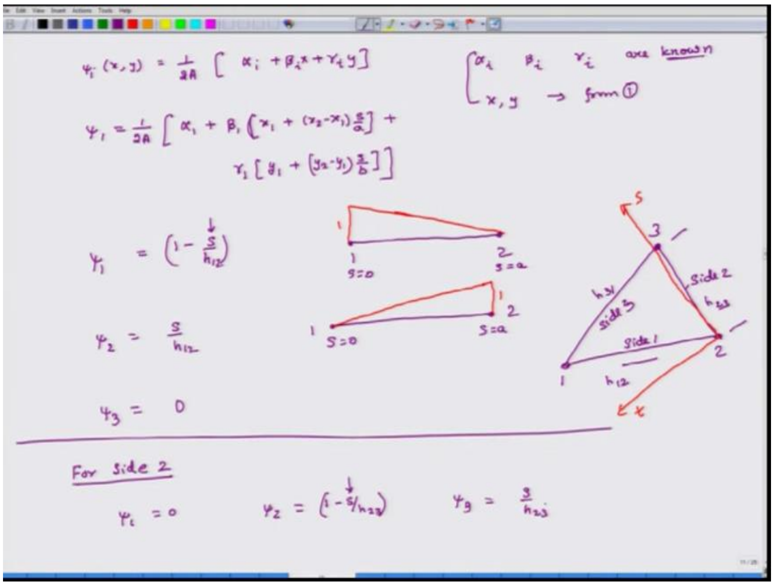

So, psi 1 or psi i is equal to 1 over 2 and I will not use that super script e for purposes of simplicity and that equals beta i x plus gamma i y. So, we can put this equation 1 in this equation for psi and what we get is. So, for different values of i, I get different values of psi i right, if I put i equals 1 then I will get psi 1, if I get i equals 2 then I get psi 2 and so on and so forth.

So, alpha i beta i gamma i we know we have calculated earlier several lectures back we know these values right. So, alpha i beta i gamma i are known, they are known and x and y we calculate from 1. So, using these 2 I can calculate psi 1 psi 2 and psi 3. So, what are the relations first psi; 1 psi 2 psi 3? So, I will right down the relation for psi 1. So, if I use these substitutions the relation for psi 1 is 1 over 2 a alpha 1 plus beta 1 x 1 plus x 2 minus x 1 times s by a plus gamma 1 times y 1 plus y 2 minus y 1 times s by b. This is the expression for this thing and if i simplify and if i substitute the values of alpha 1 beta

1 gamma 1 and if i simplify it essentially what this boils down to is 1 minus s by a to psi 1 becomes 1 minus s by a.

What does this mean? So, this psi 1 is valued for what where is it valid this psi function, this psi function is valid for or line 1 2. So, what does this mean along line 1 2 what is the value of psi 1 at. So, here s is equal to 0 here s is equal to a. So, at s is equal to 0 the value of psi is 1 and at s is equal to a the value of psi is 0. So, this is how and approximation function is in 1 d element right, psi 2 you we use the same method we will find that it equals s by a. So, for psi 2 again s is equal to 0 this is node 1 this is node 2 here s is equal to a how does psi 2 behave it goes like this. So, this is 1. What you thing psi 3 will be because i goes from 1 2 3 for a triangular element. So, what will be psi 3, psi 3 will be 0 because 1 a 1 dimensional linear element you can have only 2 approximation function. So, psi 3 will be 0. Now so this is by triangle this is psi. So, this is 1 2 3 this is side 1 this is side 2 now this is side 3 and this is side 2.

These 3 sides are value for side 1. So, for side 2 what do I do I reconstruct my local coordinate system. So, this is my s axis this is my t axis, if I have to compute share functions for side 2 then in that case psi 1 will be psi 1 is share function associated with node 1. So, that will be 0 because that does not figure out in this psi 2 will be 1 minus s by a, but this s will be different n then this s and then psi 3 will be s by a and here a is this length here a is the length of psi 2 3. So, to be a little bit more accurate and precise we will not use this term a and we will call it h 1 2 that make said more explicit h 1 2 is this length this is h 2 3 this is h 3 1 ok.

So, here we replace it by h 1 2 and here a will be replace by h 2 3 and same thing for this thing.

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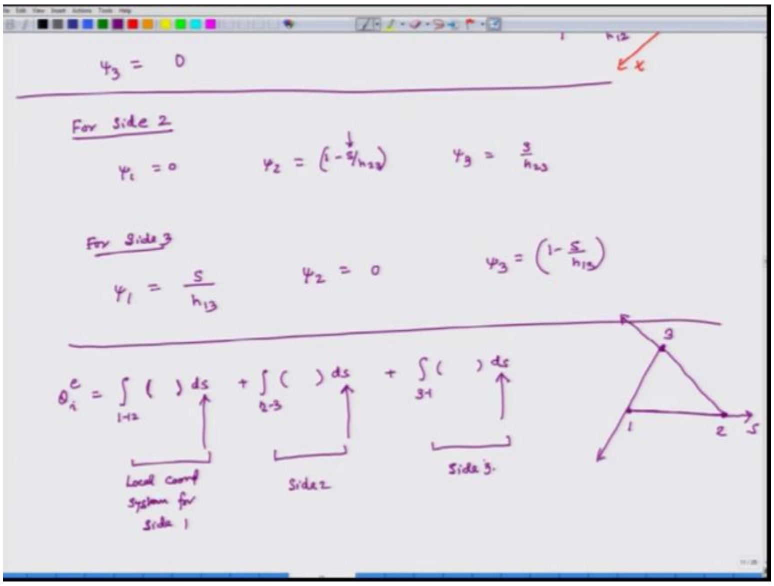

And for side 3 psi 1 will be s divided by h 1 3 what will be the value of psi 2 for psi 3 0 and psi 3 will be 1 minus s by h 1 3. So, if there is a triangle suppose this is a triangle and this is having 3 sides 1 2 3 and suppose I am interested even though I had earlier side that I should calculate the q vector only for which sides. Which yes, but in general we should calculate q for only the side which is not shared suppose it just happens that I have just 1 element in the whole system. So, I have to calculate q for which sides, if I have just one element in the whole mesh then for which sides I have to calculate q for all the sides.

So, suppose I am interested in calculating q for all the sides then q i e will be integral from 1 to 2 d s plus integral from 2 to 3 d s plus integral from 3 to 1, but there is an important thing to note that the definition of this s and the definition of this s and the definition of this s are the same or their different, they are different this s is s along this line.

So, when I am calculating this as and all the parameters in the bracket I should use the transformation based on the coordinates of nodes 1 and 2. This as has a different expression and this has a different expression understood. So, even though this s s are same, but in reality this is this uses local coordinate system for side 1, here the local coordinate system uses what has the x axis or the s axis side 2 and here it uses side 3.

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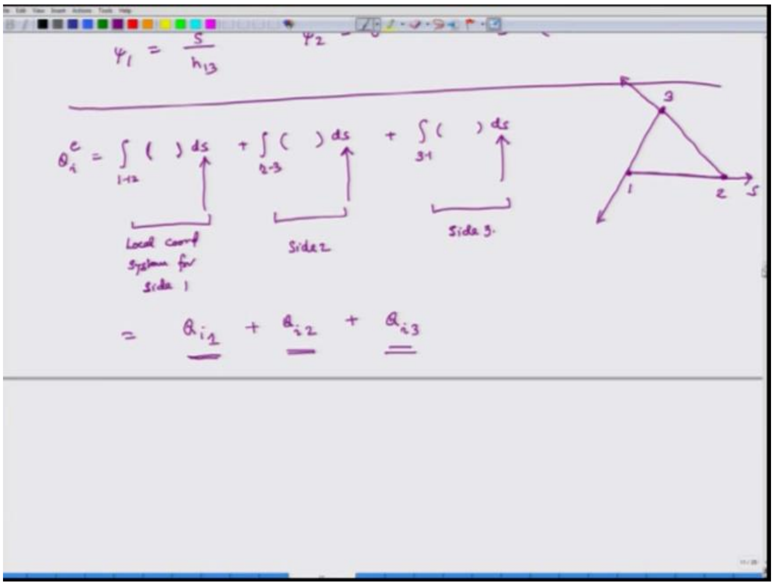

And when we do this we will be able to calculate 3 components of this q. So, what will be the 3 components q i 1 means side 1 plus q i 2. So, q i 2 is the component due to side 2 plus q i 3 is the component due to side 3 understood. So, this is very important to understand. So, this is the contribution to q from side 1, this is the contribution from side 2, this is the contribution from side 3. Next we will do an example and we will close this discussion.

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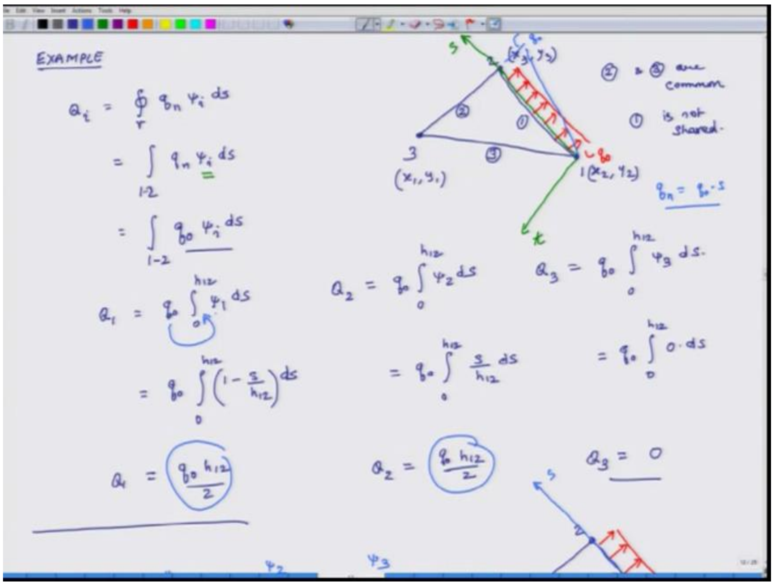

So, suppose I have a triangle, it has three sides 1 2 and 3 and when I look at the overall mesh I see that, sides 2 and 3 are common and side 1 is not shared is this is given side 1 is not shared.

In this case I will compute Q for which component which contribution of q will i calculate, first side I will calculate first side. So, q i equals boundary integrals of q n psi i d s this is the general formula this formula applies to all triangles and squares whatever. Now I am interested in finding out the value of q corresponding to side 1 because 2 and 3 and 3 and 1 are share sides. So, when I do the assembly they will cancels I do not have to worry about it. So, this in this case will be equal to. So, I am interested in only calculating for side 1. So, I will calculated from 1 to 2 q n psi i d s agreed, in that case what will be my s axis the s axis goes from 1 to 2. So, this is my s axis right it goes from 1 to 2 and the t axis is normal to it. So, this is my t axis, I just shifted the node. So, node 1 i is this node.

To calculate this formula I know how to calculate psi i, I have learned that i learnt how to calculates psi i for this I know. So, what is this x 1 y one x 2 y 2 x 3 y 3? So, I know all these coordinates. So, psi i we can calculated to just learnt it in last two lectures we can calculate psi i, but i also should know q n if I do not know q n I cannot calculate this integral. So I have to know, what is the value of q n? So, in this case I say that q n is the flux normal to this h, such that it is having a constant and its values q 0. So, this is equal to 1 to 2 q 0 psi i d s.

So, now what will be q 1 q one will be integral of q 0 time’s psi 1 times d s. So, q 1 and q 0 is constant. So, it is equal to q 0 1 to 0 psi 1 times d s q 1 is equal to q 0 integral 1 to 2 psi 2 times d s. So, this should be 1 and q 3 will be q 0 1 to 2 psi 3 times d s right and what is psi 1? Psi 1 is we had just calculated it is 1 minus s divided by h 1 2 element length right. So, it is 1 minus s divided by 1 2 d s and this is equal to q naught 1 to 2 1 to 2. So, I should replace these 1 to 2 by what see this 1 to 2 represents global coordinate system, now I am in the local coordinate system. So, I just integrate from 0 to h 1 2, here also 0 to h 1 2 h 1 2 is the length of side 1 2 this is also from 0 to h 1 2.

So, this is 0 to h 1 2 and then the second integral is q naught and what is psi 2 second integral is q naught integral from 0 to h 1 2, what is the value of psi 2 for side for the side 1 s by a and a is h 1 2. So, this is this and what is the value of psi 3 for psi 1 0 yes. So, I get Q naught what is the integral of this a 2 and the integral of this is 0 agreed. So, this is Q 1 this is Q 2 and this is Q 3 and physically what does it mean if the flux is constant across the whole length what will be the total flux, total flux will be q naught times the length that will be q naught times h 1 2 right and because it is constant across the whole length then what that means, is that the nodal value at node 1 will be half of that q naught h 1 to divided by 2 and the nodal entity for node 2 will be also half of that and because this flux does not influence 3.

So, its value is 0 agreed one minus you integrate this we will do one more case. So, if I do. So, here I have one exposed h I could have a situation where I have two exposed adjust 1 2 3.

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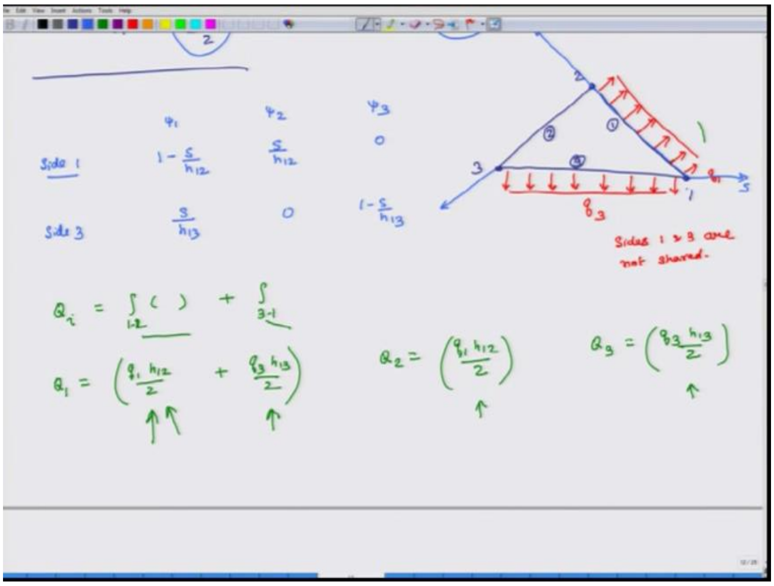

So, this is side 1 side 2 side 3 and 2 exposed adjust let us say the exposed 2 unshared ages are this. So, here the value of q is let us say q 1 and here the value of q I will call it q 3. So, for which sides these are sides 1 and there are not shared. So, I have to calculate q i for these sides right for the third side which is side 2 I do not have to worry about it, sides 1 and three are not shared to is shares. So, shares I do not have to worry because when do a simply it will cancel out. So, for side 1 psi 1 psi 2 psi 3 actually I will just write down the relations, side 1 this is 1 minus s divided by h 1 2.

What is psi 2 s by h 1 2 and psi 3 is 0, for psi to I do not have to worry for psi 3, it is what s by h 1 3 see it is cyclic. So, you have to make sure that this is not 1 minus h by it is cyclic because why is it cyclic. So, your coordinate system here is goes from 1 to 2. So, this is s axis for side 1, if I have to calculate s axis for psi 2 then this would have been the s axis for side 2 and for side 3 this is the s axis. So, for side 3 psi 1 will be s by h 1 3, it will not be 1 minus s by h 1 3 understood and then psi 2 is 0 and this is 1 minus s by h 1 3.

So, now if I am interested in calculating Q I what will do? I will integrate from 1 to 2 and I will also integrate from 3 to 1. So, Q 1 will be what Q 1 will be from 1 to 2 I will get q 1 h 1 2 by 2 and from this 3 to 1 I will get q 3 h 1 3 by 2. I have omitted a lot of details, but if you do the math you will get this number. Q 2 will be q 1 h 1 2 by 2 and q 3 will be q 3 h 1 3 by 2, it will be h 1 3 by 2. Once again we please work out the details, because I have omitted few steps you have to calculate these integrals then you will see how these numbers come.

But what does this mean? See node 1 is common node 1 is common to side 1 and side 3. So, there is a component coming from side 1 and there is component coming from side 3 right and the component from side 1 is what q 1 times the length of the element divided by 2 and the component from side 3 is q three times length divided by 2. So, that is what it is for q 2 there is only 1 component and same thing is for q 3. Now these formulas are good if q is constant if q was varying like this. So, I will go back. So, here we had assumed that q is constant if q was constant suppose q was varying like this.

So, let us say this is q naught, then what will be q n will be nothing, but q naught times s in that case right, in that case this q will go inside the integral then you have to integrated and accordingly update it these numbers understood. So, if q is not constant you just do not take q outside and integrated and you will again get the correct answers. So, this is how we calculate the boundary terms and what we have done today is in these 6 lectures we have learnt how to calculate different terms of stiffness matrices fours vector and also the boundary vector for triangular as well as rectangular elements.

So, with this I things we have a form understanding of how to develop element level equations for two dimensional problem having a single variable. In the next week will learn how to assemble these equations and solved these equations. So, with that I wanted to close the discussion for today and I look forward to seeing you tomorrow once again.

Thank you very much, have a great day.