Basics of Finite Element Analysis – Part II
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Lecture - 35
Boundary Elements for Finite element equations

Hello. Welcome to Basics of Finite Element Analysis Part II. Today is the fifth day of sixth week of this course and what we will start discussing today and also probably tomorrow is how do we actually evaluate the boundary integral which appears as Q term in the element level finite element equation.

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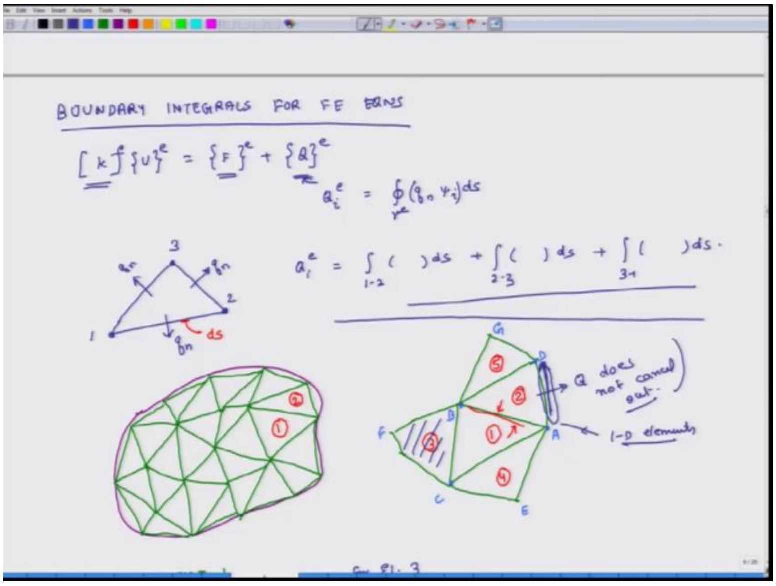

So, that is what our focus is going to be that is we will learn how to evaluate boundary integrals for finite element equations. So, our finite element equation for the e th element looks something like this here elements of k matrix and elements of f vector. They are calculated using surface integral because, we are integrating over the domain and if the domain is the 2 dimensional domain then, they are evaluated using surface integral, but to calculate this we have to use a boundary integral because Q I for the e th element is defined as Q n psi I d s evaluated on the boundary.

What does that mean? So, suppose I have a triangular element. So, this is node 1, 2, 3 then Q n represents the flux for instance if it is a heat conduction problem it is the flow of heat in the normal direction in the direction normal to line. 1, 2 similarly over line 2, 3 this is the direction of Q n similarly for line 1, 3 this is the direction of Q n. So, Q n represents flux in case of heat transfer it could remain it, will tell us how much heat is getting transferred across this boundary 1 2 2 3 and so on so forth. So, for a triangle, for a triangle element what do I have to integrate from 1 to 2 this whole term? So and DSS, what is d s? D s is a small line element. So, Q n could be flux per unit length and if I multiplied by d s it is the total amount of flux which is crossing the d s element.

So, when I integrate it over line 1 2 it is the total amount of flux which is getting out over line 1 2. So, going back Q I will be integral of this term Q n psi I d s over line 1 2 plus same thing over line 2 3 plus. So, this is 2 3 and plus the same thing in 3 to 1 of d s now consider a domain. So, let us consider this domain and I can break these domains, and so this is the physical domain which is the purple boundary the o actual domain. I have discredited this domain in to a large number of elements.

Now, we will consider 2 elements this is element one and this is element 2. So, I will draw these 2 elements separately. So, this is element one and this is element 2, element 2 is having 2 neighbors element, one is having neighbors on all the four sides on all the 3 sides I am sorry. So, element one is surrounded by elements all around it element 2 is surrounded by elements only on 2 sides. So, third is exposed to the outside, if I have to calculate Q for element 1.

So, let us level these. So, let us level these nodes a, b, c, d these are nodes. So, the boundary terms boundary terms for element one for element 1.

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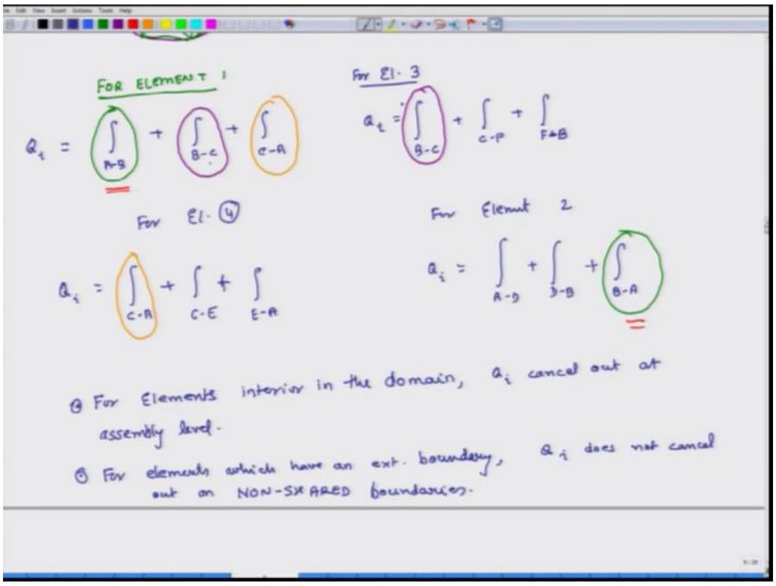

The boundary terms will be A to B plus B to C plus C to A; these will be the 3 boundary terms using this expression here. So, these will be for element one these will be the 3 boundary terms Q I will be this let us all are also actually make another 2 more elements 3 4 and 5 now. So, this is for element 1, what will be the terms for, so I will put 2 more nodes. So, D E F G, for element 2 element 3 what will be there for element 3 Q I will be will have 3 components B to C plus C to F plus F to B right and then, for element four Q vector will be integral of C to A plus A to E plus integral of C to E plus E to A for element four and for element 2 what will be the thing? Q I will be A to D plus D to B plus B to A. So, what I have writing down is boundary terms for element one and it is neighbours element 3 is a neighbor, element 2 is a neighbor, element 4 is a neighbor. So, I have written boundary term for it is neighbour.

Now, what we see is that you have. So, these are at the element level. So, these terms will appear at the element level right, but ultimately what we do? We assemble these elements together we assemble these elements together when, we do the assembly what happens this term a to b also appears for element 2 right, but they are the sign will be reversed they are the sign will be reversed what does that mean that A to B is this line for element one the flux will be happening in this direction outwards for element 2 the flux will be happening in this direction again outwards.

So, as a consequence when, I do the assembly these 2 terms will add up and their total will be 0. Similarly the terms corresponding to B to C; So B to C is the inter face of element 1 and 3. So, this term will cancel with this term and C to A, C to A is the common boundary between 1 and 4. So, this term will cancel out this term. So, what I see is that because an element is surrounded on all other side all the 3 sides by other elements then I do the assembly the boundary terms will add up and their total will be 0 that will be the case for an element if, it is boundary is surrounded on all the 3 sides or on all the sides by other elements that is for an element which is in the interior of the domain.

So, for elements interior in the domain Q I is cancel out at assembly level they cancel out at the assembly level. So, if they are going to cancel out at the assembly level then, what is the point of calculating those boundary terms there is no point in calculating those boundary terms or interior elements. But for elements which have an external boundary Q I do not cancel out on non shared boundaries this is important to understand.

So, let us look at the picture again. So, for element one all the 3 boundaries a, b, b, c, c, a; they are shared with other elements. So, the at assembly level all the Qs will cancel out, but for element 2 Q is cancel out for boundary a b they, will cancel out for boundary b d, but they will not cancel out for boundary d a because, that is not a common boundary to more than one element. So, here Q does not cancel out. So, we have to calculate Q only for this boundary. So, we have to in the overall element I mean mesh we have to look at those boundaries which are exposed and which are not shared and we have calculate Q's only for those boundaries this is important to understand.

So, that is what we will do. So, that is what we planned to do starting today the other thing which we should note is this is the boundary integral. So, so Q is a boundary integral right we are calculating this Q on the boundary it is a line integral or a boundary integral for 2 dimensional surfaces this is the boundary integral becomes a line integral for 3 dimensional surface a 3 dimensional systems the boundary integral becomes, a surface integral. So, here it is a line integral and we will see that essentially what; that means, is that for calculating this boundary integral we only have to vary about one d elements because, it just a line it is not a surface to calculate integral over this entire domain it is a surface. So, it is a 2 d element, but here we are only calculating the integral n, one dimension which is in this particular dimension we only calculating the integral and this dimension which is dimension along a d.

So, this is important to understand. So, that is what we planned to do. So, what is our aim our aim is 2 figures out, how do we calculate line integrals around a specific boundary of a element and in this case, we will start with a triangular element and figure out how to calculate the line integral.

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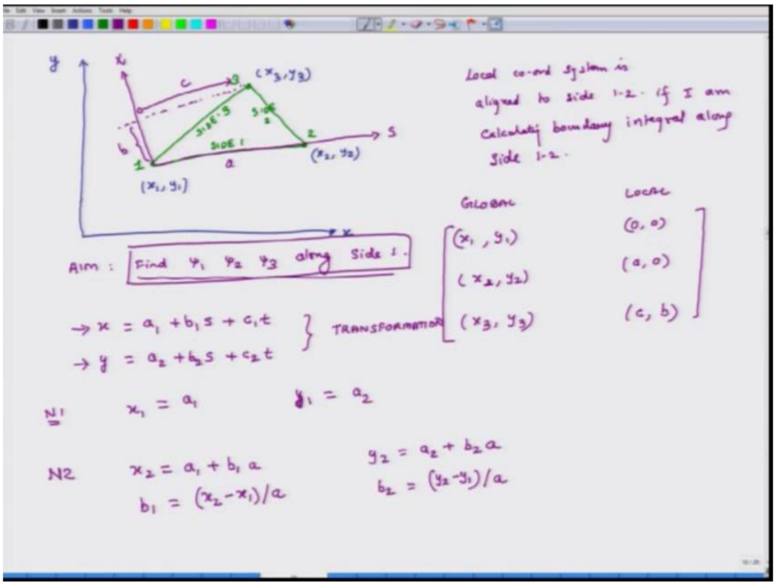

So, this is my global coordinate system x y and this is my triangle. So, we have side one side 2 and we have side 3 we have node one node 2 node 3 the global coordinates of this are x1, y1, x2, y2 and x3, y3, x1, y1, x2, y2, x3, y3 they are the global coordinates.

Now, we will also develop a local coordinate system. So, the local coordinate system this is one x axis s and there is another axis normal to it this is x t. So, this local coordinate system important to remember is aligned to side 1, 2. So, it is aligned to side 1, 2 here my interest is in calculating the boundary integral. If I am calculating boundary integral along side 1 2, if I was interested in calculating boundary integral alongside 2 3 then my x axis would be a long line 2, 3 and my t axis would be normal to that understood and the origins would lie at node 2.

If I was interested in calculating the boundary integral along line 3, 1 because I am interested in calculating about specific boundaries then, in that case my x axis would be in the direction of 3, 2, 1 origin would be at node 3 and the second axis would be normal to that s axis understood. So, this is not exactly in normal I will make it up better normal. So, this is t axis. So, let say this side a side 1 is a long. So, my global it is a long and this distance this distance is b which is parallel to side a and this distance is c global and local global is x1y1 and local is 0, 0 global is x2 y2 and local is a 0 and third one is x3 y3 and what is the local coordinate in that case the value of s is c and this is b agreed.

So, this is the mapping for each of these nodes now what is our aim our aim is to find psi one psi 2 psi 3 alongside 1 we are interested in finding psi 1, psi 2, psi 3 alongside one because, I am interested in calculating the boundary integral alongside 1. If I was interested in finding boundary in alongside 2, then my local coordinate system will get reoriented and reposition.

So, this is our aim we want to find psi 1, psi 2, psi 3 alongside one why do we want to do that because, in my boundary integral I have psi I and if I wanting to be use local coordinate system then, I have to find the formula for this understood and what about Q n Q n would be known right Q n would be known because only then I can calculate. If I do not know Q n then, how can I calculate the boundary integral? So, so Q n has to be known then I can calculate the boundary integral. So, this x1 y1 maps to 0 0 x2 y2 maps to a0 x3 y3 maps to c b.

So, I have to first thing is I have to find out some formula which transforms x in to x y in to s t and this is a linear this is a linear transformation. So, I can write x is equal to a1 plus b1 s plus c1 t and y equals a 2 plus b 2 s plus c 2 t. So, this is the transformation this these equation to transform local coordinate in to global coordinate and vice versa, but we do not know the values of a1 b1 c1 a2 b2 c2. So, now we have to figure out their values. So, we have 3 points using these 3 points we calculate these a1 b1 c1 a2 b2 c2 and c4. So, now they are this is our ultimate aim to find size, but before we want to find psi we have to know what is the nature of this transformation, right? So, that is what we plan to do.

So, let proceed. So, when s is equal to 0 t is equal to 0 that is for node one x is equal to x1and y equals y1 right actually this we have already written here. So, I do not have to write it. So, let consider the first point for node one let consider node one and apply equation one what is the value of x for node one x1and that equals using this relation a1 what is the value of s 0 and. So, so b1 time 0 is 0 plus c1 time's t is 0. So, a1 is equal to x1and similarly for y y1. Now we use the second equation for y. So, y1 for node one what is the value of I y1 that equals a2 because for node one x and t are 0. So, in the second equation which is this y1 equal's a2 so, we have calculated figured out what is the value of a1 and a2.

Now, let us look at node 2 node 2 what is the value of x at node 2 x2 is equal to a1 plus b1 times what is the value of s a and what is the value of t 0. So, this gives me b1 equals x2 and what is a1, a1 is x1. So, minus x1divided by a similarly y2 is equal to a 2 plus b2. So, I get b2 is equal to y2 minus y1 divided by a understood.

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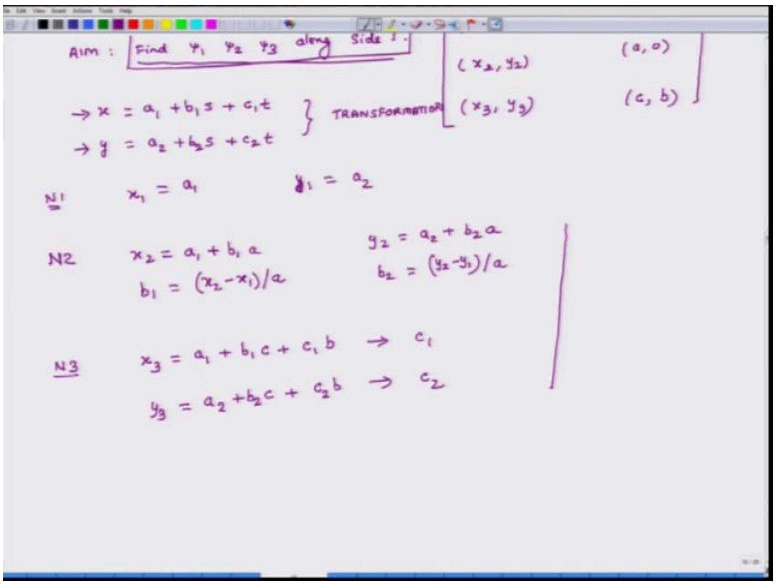

So, this is the thing for node 2 and for node 3 the relation is x coordinate of x3 this is equal to a1 plus b1 times c plus c1 times b and this will give me the value for c1 because I know a1 and b1 y3 equals a 2 plus b 2 c plus c 2 b this will give me the value of c2. So, what we have done is we have calculated the values of a1 b1 c1 a2 b2 c2. And we will use this information to calculate first the psi functions, and use those psi functions for calculating the boundary integral.

So, that is what we will do and that is what we will accomplish tomorrow.

Thank you.