Basics of Finite Element Analysis – Part II
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Lecture -32
Evaluation of Stiffness and Force matrices

Hello. Welcome to Basics of Finite Element Analysis Part II. This is the sixth week of this course today is the second day of this week and yesterday what we were discussing is how do we go around computing psi functions or approximation functions for a phone need nodded linier rectangular element and in that context what we had shown was that we can develop that expression in a step by step way.

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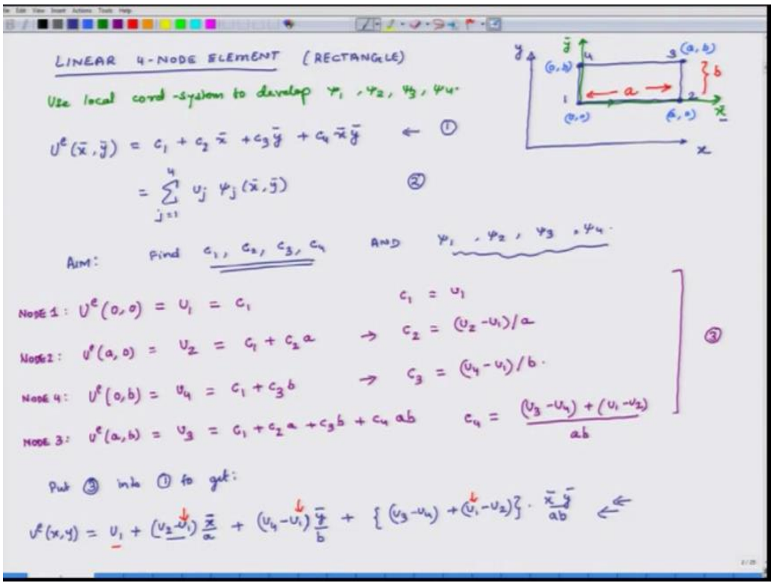

The first step is we had undertaken was we had developed the local coordinate system x bar and y bar and in those local coordinate system the expression for u could be written as C1 plus C2 x bar plus C3 y bar plus C4 x bar y bar. And then we had calculated the values of C1 C2 C3 and C4 and using so from that calculation we had arrived at this expression. So, now, what we will do is we will use this expression to find out the approximation functions which are associated with this rectangular element.

So, there will be four approximation functions psi 1 psi 2 psi 3 and psi 4. So, that is why in this equation 2 we are having addition over an Indies up to 4.

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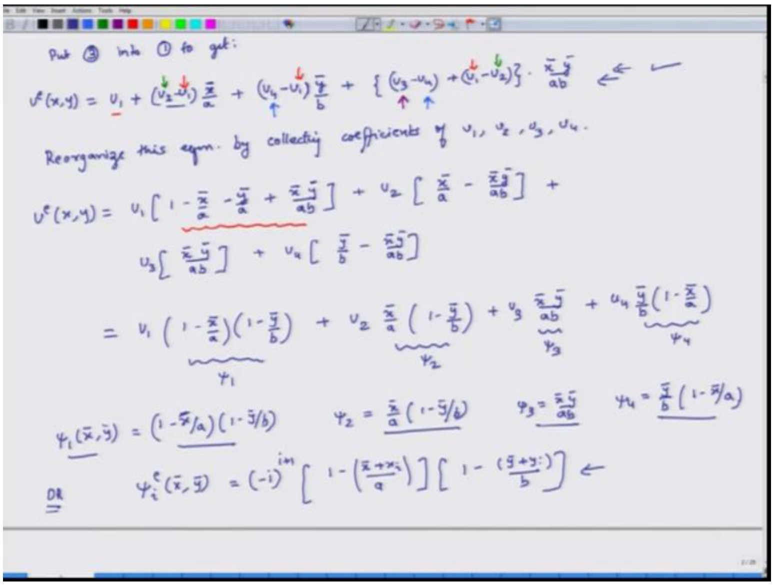

So, our aim will be to use this expression to compute those 4 approximation functions which are associated with the 4 nodded linier rectangular elements. So, the way we are going to do is that we will reorganise this equation by collecting coefficients of u 1 u 2 u 3 u 4. So, u e of x and y equals u 1 and what I have already marked is this the first term involving u1 is u1. So, its coefficient is 1 the second term is minus u1 times x by a it is coefficient is x bar divided by a for the third term the coefficient is minus y bar divided by a and for the fourth term the coefficient is x bar y bar divided by a b.

Next look at coefficient for U2; so where do we see U2, U2 is appearing here and it is also appearing in the last term. So, it appears 2 times, so because of that the first coefficient of U2 is x bar by a and the second coefficient of U2 is minus x bar y bar by a b plus there are U3 terms. So, u three terms appears only at 1 place. So, the expression for that is U3 times x bar y bar by a b and finally, we have U4 term and that U4 term appears once actually it appears at 2 locations. So, U4 times the first coefficient of U4 term is minus y bar by b and the second coefficient is minus x bar y bar by a b.

So, this should be plus y bar by b. So, this now I further organise it in a better way. So, U1 and this entire thing I can express it as multiple of 1 minus x bar by a times 1 minus y bar by b, if I factorise this term in the red bracket. Plus U2 x bar by a in to 1 minus y bar by b plus u 3 into x bar y bar by a b and finally, we have U4 y bar by b times 1 minus x bar by a. So, this is by psi 1, this is by psi 2, this is psi 3 and this is psi 4. So, to formally write this we can write it as psi 1 x bar which is the function of x bar and y bar this equals 1 minus x bar by a times 1 minus y bar by b psi 2 equals x bar by a times 1 minus y bar by b psi 3 equals x bar y bar by a b and psi 4 equals y bar by b times 1 minus x bar by a or I can write it in 1 single expression that psi i for the e-th element psi i could be written as minus i to the power of i plus 1 into 1 minus x bar plus x i by a times 1 minus y bar plus y i by b.

So, this is the generic formula if I use different values of i, I will get these four relations. So, this is these are the expressions for approximation functions associated with the 4 nodded rectangular element and here, these approximation functions are valid for the local coordinate system only if my local coordinate system is oriented in a parallel way to the is parallel to the global coordinate system and the rectangular sides are aligned with the access of the local coordinate system. So, if I have a tilted triangular then these relations will not necessarily work out.

So, this is what I wanted to discuss in context of approximation functions and now that we have discussed how to develop approximation functions for a triangular element and also for a phone nodded rectangle. We can use similar thinking to develop approximation functions for a 6 nodded rectangle which is the quadratic 6 nodded triangle which is a quadratic element or 8 nodded rectangle which is the second order rectangular element. So, with that what we will do is we will move on to a somewhat different topic and, but the overall theme is still remains the same.

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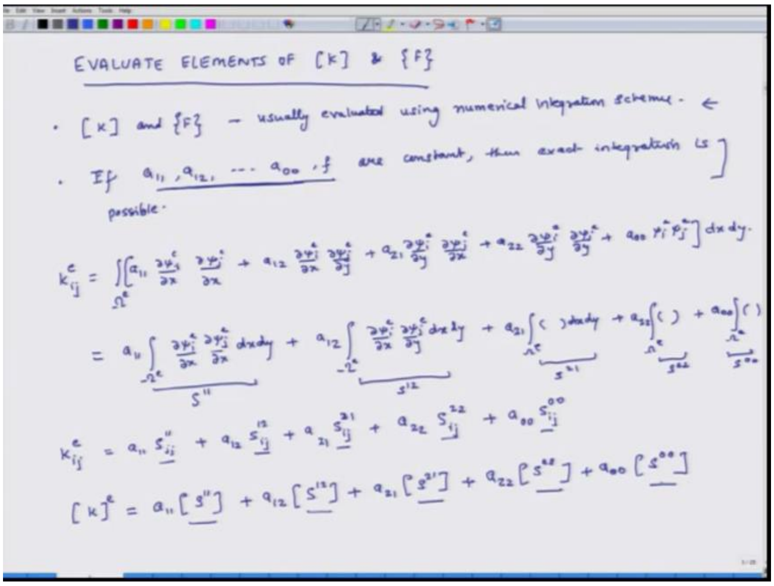

And what we will start discussing is that how do we Evaluate Elements of K metrics and F vector, how do we evaluate this. So, the first thing is that elements of K and this vector F there are usually evaluated using numerical integration scheme. So, in second and third week of this course we have discussed different ways to do numerical integration right a specifically Gaussian Quadrature and Newton cotes method of numerical integration. Now those integration methods we had discussed were in context of 1 dimensional system.

But when we are integrating we first integrate with respect to x then we integrated with respect to y. So, the same thing same Quadrature approach could be used to integrate, this numerical integration for 2 dimensional systems also because first when you are integrating with respect to x then you do that using the same approach then you move on to y and then again you do the same thing. So, that is there, now this numerical integration becomes really effective if closed form integration are not possible now when does that happen now we know that K elements of K metrics involve terms like a 1 1 a 1 2 a 2 1 a 0 0 and so on and so forth.

If these terms are not constants or if they are complicated functions, then it may be possible that when you are trying to integrate the entire expression for K you will not have a exact integration formula. So, in that case you resort to numerical integration schemes, but if a11 a12 and also a00 and F they are constant, than exact integration is possible within you can integrate the term in a close form way. So, that is what we will do in next couple of lectures let we will develop exact expressions for the specific case when these terms a11 a12 a22 a21 a00 F they are constant and the requirement here is not that these terms have to be constant over the entire domain. They have because what we are interested in is finding out the values of K and F matrices for each element. So, as long as these terms are constant over specific element we can get exact integration values. So, it can vary from one element to other element. But as long as these terms are constant for a specific element we should be able to get exact integration solution.

So, that is what we are going to do now we know that we had developed an expression K i j for e-th element and what was it a11 del psi i over del x del psi j over del y d x s y plus a12 del psi i over del x del psi j over del y plus a21 del psi over del y del psi j over del x plus a22 del psi i over del y del psi j over del y plus a00 psi i psi j d x d y. And this we are integrating over the domain which is the e-th element and each of these corresponds to e-th elements. So, I am putting a superscript e here, plus the second term a12 plus the third term plus the fourth term plus a00 term right.

So, I can break this up and are the reason I am taking a11 a12 all these outside the integration sign is because these have Constance over the over the element they can change from one element to other, but over that element they are we consider them as Constance. So, this integral I can call it as s11 this integral I can call it s12 this 1 as s21 this 1 as s22 and this one as s00. So, K i j for the e-th element is nothing, but K11 s11 i j plus a12 s12 i j plus a13 s i j 2 1 plus a22 s i j 2 2 plus a00 s i j 0 0. So, these are components. So, calculate the overall stiffness term k i j I have to calculate these individual integrals s11 s12 s21 s22 s00.

If I can calculate them then i have to just multiply then by these Constance and it will give me the value of k i j. So, or in matrixes form I can write it as k for the e-th element equals a11 times s11 matrix plus a12 times s12 matrix times a21 times s21 matrix plus a22 s22 plus a00 s00. So, what we will do is now we will develop expressions for integrals of s11 s12 s21 s22 and s00.

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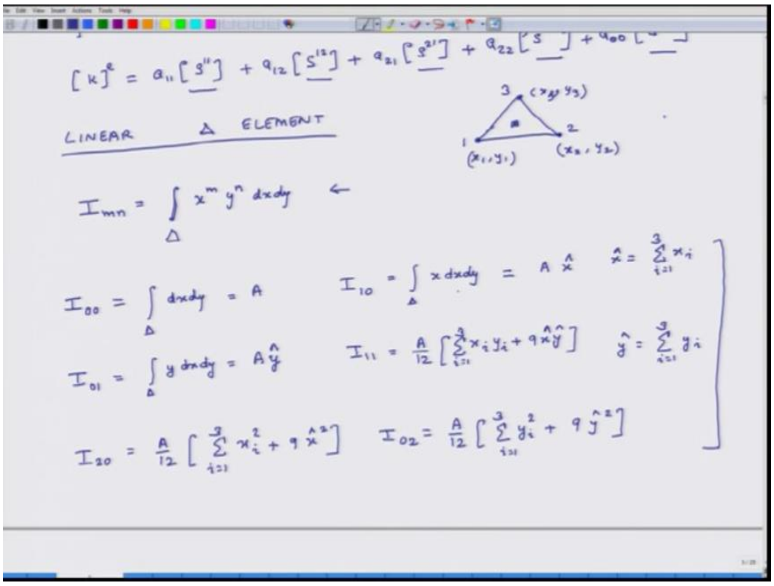

So, we will develop that for a linear triangular element. So, what does that mean? We will develop these expressions for an element which is 3 sides 3 nodes 1 2 3. So, it is a 3 nodded triangle. So, it is a linear element its coordinates a first node are x1 y1 second node x2 y2 and third node is x3 y3 and before we do that I wanted to use some identity standard mathematical formula, so if I have this integral x to the power of n and I am integrating it over the area. So, this is the standard mathematical formula we will use this formula to compute s11 s12 s21 22 so on and so forth.

So, going to just write down these formulas and later we will use these to compute these things. So, this integral i m n is basically an integral of x to the power of m times y to the power of n times d x d y and I am integrating it over the domain of the element and in this case the domain is triangle. So, that is why I am writing this triangle. So, what is the value of this integral I00 which means m is 0 and n is 0. So, when I00 means x to the power of m and n both are 0. So, x to the power of m times y to the power of n becomes 1. So, this is equal to dx dy and what is the value of the this integral.

So, what is dx dy? dx dy is small piece of element in the triangle, it is a small piece of element which is d x y and d y long right and if I add up all these small areas over the whole triangle what do I get area of the triangle. So, this is equal to area of the triangle. I10 equals what x here m is 1 n is 0. So, it is x dx dy and this you can do this maths it is equal to area times x hat were x hat equals 1 2 3 x hat is sum of x1 x2 x3 divided by 3.

Similarly, I01 equals when am integrating the over the area of the triangle is y dx dy and this equals area times y hat. And y hat equals yi i equals 1 2 3 and I11 equals a divided by 12. So, I11will be what integral of x y over the area of the triangle. So, this is equal to x i y i plus 9 x hat y hat i is equal to 1 2 3. So, what is the I11 it is a by 12 times x 1 y 1 plus x 2 y 2 x 3 y 3 plus 9 x hat y hat. So, these are standard mathematical formula we are going to use this and then I20 equals a divided by 12 times x i square plus 9 x hat square i is equal to 1 2 3.

And finally, I02 equals a divided by 12 times 9 y hat square. So, these are 1 2 3 4 5 6 standard formulas they do not have anything to do with finite element method, there mathematical results based on understanding if geometry and we will use these results to compute s11 s12 s21 s22 s00 and that is something which we will work on in the next class.

So, that completes the discussion for this lecture and will continue this discussion in the next class.

Thank you very much.