Basics of Finite Element Analysis – Part II

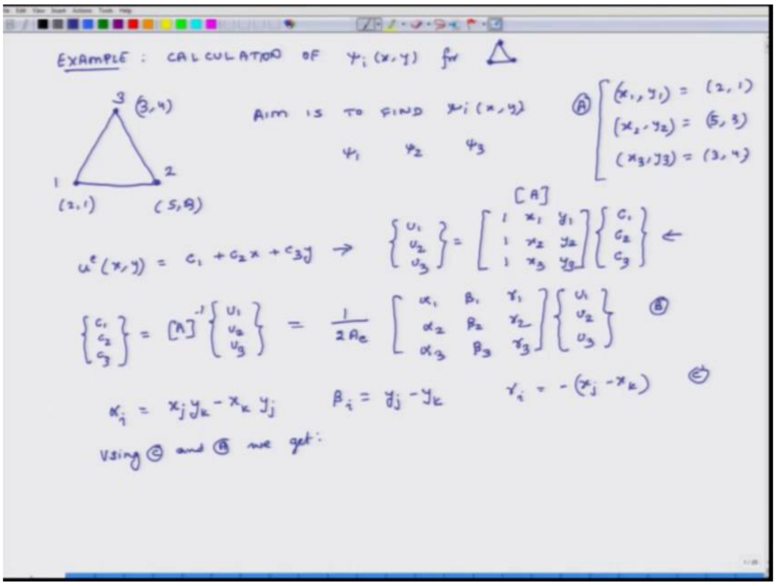
Prof. Nachiketa Tiwari
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Lecture - 31
Interpolation Functions for Triangular and Rectangular elements

Hello. Welcome to Basics of Finite Element Analysis Part II. This is the beginning of sixth week of this course and what we will do today and also in the remaining part of this week is work out lot of details on how do we go around computing element level stiffness numbers and also element level values for force vector and queue vector. So, that is what we plan to do today and it is a continuation of the discussion which we had in the last week. In last week when we closed the discussion on FEA, we were discussing the shape functions for triangular elements. How do we calculate the interpolation functions for triangular elements?

So, starting from today we will continue that discussion and specifically today what we will do is we will actually work out an example for a triangular element and calculate the functions psi 1, psi 2 and psi 3 and once we are done with that and we get a better understanding of that through actually worked out example, then we will move to working out the details of a rectangular element; a four nodded rectangular element. So, that is what the agenda is for this week.

(Refer Slide Time: 01:46)



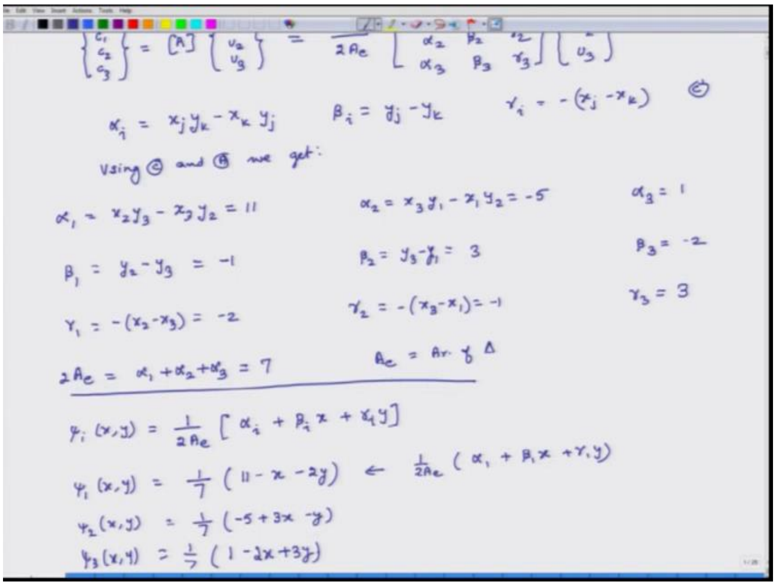
So, what we will do today is we will do an example. So, example it involves calculation of psi i which is a function of x and y for a three nodded triangular element. So, let us consider this triangle and so this is node 1 this is node 2 this is node 3 and the coordinates of the node 1 are 2 and 1, coordinates of node 2 are 5 and 3 and coordinates of node 3 and 3 and 4. So, for this specific triangle our aim is to find psi i. So, this specifically we have to find psi 1, psi 2 and psi 3 and all of these are functions of x and y.

Now, in earlier week we had said that the approximation that this field variable u it could be represented as C1 plus C2 x plus C3 y and from this we had developed three equations that the well. So, U1, U2, U3 equals a matrix which we called as A matrix 1, x1, y1, 1, x2, y2, 1, x3, y3 right and here x1 is the location of point 1 node 1, x2, y2 is the location of point 2 which is node 2 and x3, y3 are the coordinates of the third node. So, we had developed this expression. So, U1, U2, U3 is equal to so this sector equals this square matrix times another vector C1, C2, C3 and from this equation we can compute C1, C2 and C3 in terms of U1, U2 and U3 and the relation which we had developed in the last week was this C1, C2, C3.

So, let us say this is matrix A then this is equal to A inverse U1, U2, U3 and we had actually calculated the value of this inverse. So, this is equal to 1 over 2 A e times these numbers alpha 1, beta 1, gamma 1, alpha 2 beta 2, gamma 2 and alpha 3, beta 3, gamma 3 times U1, U2, U3 like this. So, alpha i and here we had said; So, these alphas can be written in terms of x1, y1, x2, y2 and x3,y3 and the expressions for these are alpha i is equal to x j y k minus x k y j, beta i equals y j minus y k and gamma i equals minus x j minus x k this is what we had developed.

Now, in this case we know that x1,y1 corresponds to 2,1, x2,y2 corresponds to 5,3 which is the second node and x3,y3 equals 3,4. So, let us call this equation A, let us call this equation B and let us call this equation C. So, using C and A we get we can calculate all the values of alphas betas and gammas. So, that is what we are going to do now.

(Refer Slide Time: 07:49)



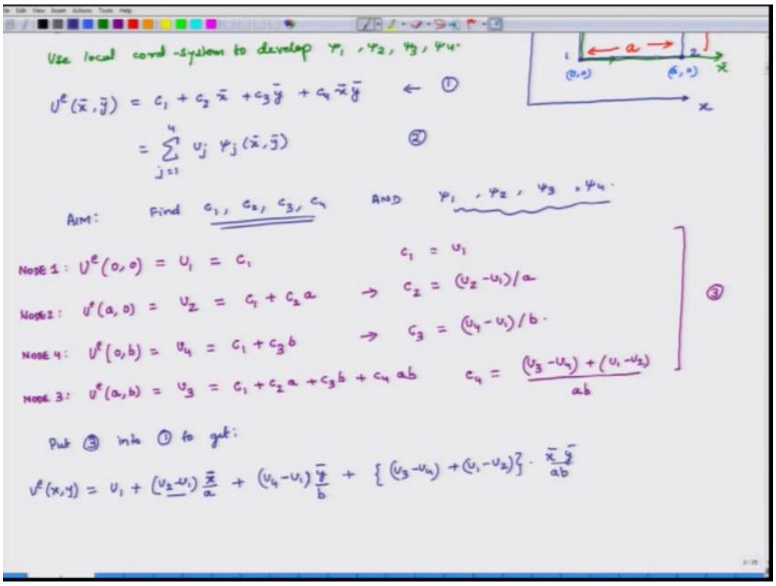
So, alpha 1; so i equal 1 then j will be 2 and k will be 3. So, this will be alpha 1 equals x2 y3 minus x3 y2 and that equals 11; similarly alpha 2 equals x3 y1minus x1 y2 and that if y2 the calculation is comes out to be minus 5 and then alpha 3 using the same type of the formula we get it is value to be 1 .

For beta 1; beta 1 is y2 minus y3 and that works out to be minus 1, beta 2 is y3 minus y2 and that comes out to be 3 and similarly I can calculate beta 3 as minus 2 and finally, gamma 1 is equal to minus x2 minus x3 and that is equal to minus 2, gamma 2 equals minus x3 minus; I am sorry this should be y1. So, gamma 2 equals x3 minus x1 negative of that and that is equal to minus 1 and gamma 3 equals 3 and finally so we have calculated alphas, betas and gammas and A e equals alpha 1 plus alpha 2 plus alpha 3 and that equals 7.

So, please note that A e is equal to area of this triangle; it is equal to the area of the triangle and 2 A e is the inverse of that determinant. So, these are our alphas and gammas. So, then we had said that psi i which is the function of x and y equals 1 over 2 A e alpha i plus beta i x plus gamma i y. So, psi 1 which is the function of x and y which is associated with node 1 equals 1 over 7, 11 minus x minus 2y because this is equal to 1 over 2 A e alpha 1 plus beta 1x plus gamma 1y; similarly psi 2 which is the approximation function associated with the second node this is equal to 1 over 7 minus 5 plus 3x minus y and psi 3 x, y equals 1 over 7, 1 minus 2x plus 3y. So, this is how we can calculate psi functions for a triangular element so that is what we have shown.

So, the next thing; so if we have a discritize system with the lot of triangles then we have to program the (Refer Time: 12:02) in a such a way that for each triangle we can compute psi functions and then we use these psi functions and their derivatives to compute elements of k matrix and also the force vector. Till so far we have been talking about triangular element a three nodded triangular element so it is a linear triangular element.

(Refer Slide Time: 12:29)



The next thing we will discuss is a linear four node element and specifically; so a four node element could be a quadrilateral but in specifically what we will talk about here will be a rectangle. So, how does it look like? So, these are my global coordinates x and y and I have a rectangle. So, I have a rectangle node 1, node 2, node 3, node 4 and it is oriented in such a way that its sides are parallel to at least one of the axis which is x and y. So, this is my x and y is my global coordinate system and here I will also put a local coordinate system which is x bar and y bar.

So, to develop the interpolation functions for this rectangular element in this case what we will do is we will use a local coordinate system to develop. So, we will use local coordinate system to develop psi 1, psi 2, psi 3 and psi 4. So, U e can vary with this is the function of x and y or if I am in local coordinate system then it is the function of x bar and y bar. So, that equals a constant C1 plus C2 x bar plus C3 y bar plus C4 x bar y bar and; so this is the linear expression and like incase of a triangular element I can also express U as a function of U j's which are the field variable at specific nodes times approximation functions which are; which depend on x bar and y bar and my index of summation will be 1 to 4 in this case.

So, our aim like in the case of triangular element is to find C1, C2, C3, C4 and also find out expressions for psi 1, psi 2, psi 3 and psi 4 as functions of x bar and y bar which is the local coordinate. So, to do this; so, before we start doing this we will also put some dimensions on this rectangle. So, let say the dimension of this rectangle is a in the; along the horizontal axis and this dimension is b. So, in the local coordinate system the coordinates are 0, 0; it will be a, 0 this will be a, b and these coordinates are 0, b. So, these are the four coordinates in the local coordinate system for the element.

So, we write down expressions U e at 0, 0. So, the value of this field variable; unknown this field variable U e it will be U1 right because the contribution from all other nodes at the first node will be zero and this equals; so let us number these relations; let us call this equation 1 and let us call this equation 2. So, if I use equation 1 then at the first node; so this is node 1; so for node 1; U is U1 and that equals C1; and C2 x bar y bar and x bar zero x bar y bar zero. So, all the terms involving C2 and C3 do not exist; they vanish.

Then for node 2; the value of u at a, 0 equals U2 and if I use equation 1 then this U2 equals C1 plus C2 times a. The terms involving C3 and C4 will not appear because y bar is zero at node 2. So, this gives me C2 is equal to U2 minus U1 divided by a. This gives me C1 equals U1.

Next we go to node 4. So, for node 4; U e for node 4 the coordinates are zero and b and the value of U at node 4 is U4 and this equals C1 and at node 4 x bar is zero. So, C2 term will not exist and C4term will not exist. So, what we get is C1 plus C3 times b. So, this gives me C3 equals U4 minus U1 divided by b and finally, I do for node 3; U e a, b equals U3 and this equals C1 plus C2 a plus C3 b plus C4 a b and I have already calculated values of C1, C2, C3. So, from this I can calculate value of C4. So, if I do the math what I find out is C4 equals U3 minus U4 plus U1 minus U2 divided by a b.

So, let us bracket these relations and call them 3. So, now what we do. So, we have calculated the values of C1, C2, and C3. So, this part of our aim is now completed; now our next aim is to find out the psi functions or the approximation functions. So, to do that what we do is we put 3 into 1 and then we get; so what do we get from that. So, U e x, y equals; C1 is equal to U1. So, U1 plus C2 is U2 minus U1 divided by a; plus C3 times y and C3 is U4 minus U1 divided by b. So, it is U4 minus U1 and this should be x bar y bar divided by b plus C4 times x y; so, it is U3 minus U4 plus U1 minus U2 times x bar y bar divided by a b.

Now we are pretty close to getting the approximation functions. So, what we do is now we reorganize these terms. So, that is something which we will do in the next class. So, that is what I wanted to say today.

Thanks a lot and we will look forward to see you tomorrow.