Basics of Finite Element Analysis – Part II
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Lecture - 27
Elemental Level 2D Finite Element Equations

Hello. Welcome to Basic of Finite Element Analysis Part II. This is the fifth week of this course and today is the third lecture and in the last lecture we were developing the weak formulation for two dimensional single variable problems. And we will continue that discussion today and once we are done with developing the weak formulation then we will introduce approximation functions in the weak formulation to develop finite element equations at an element level.

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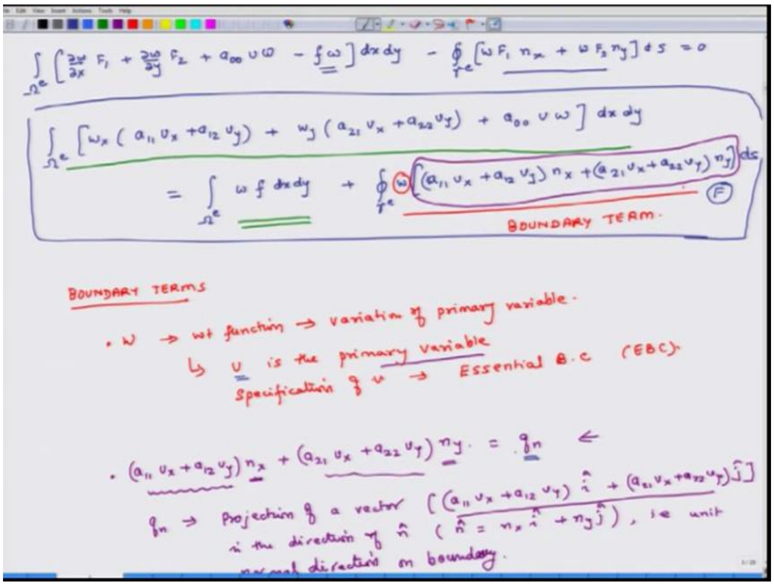

What we had discussed in the last class was that we have developed this particular equation F and in context of equation F we said that U is the primary variable. So, in the boundary terms, we get W which corresponds to knowledge of U, if it is known then the variation of U is 0. So, U is the primary variable and if we specify it then it is an essential boundary condition and then there is also a longish boundary term, which we can abbreviate as q n. That is the essential boundary condition. So, with that what we will do is we will rewrite this equation.

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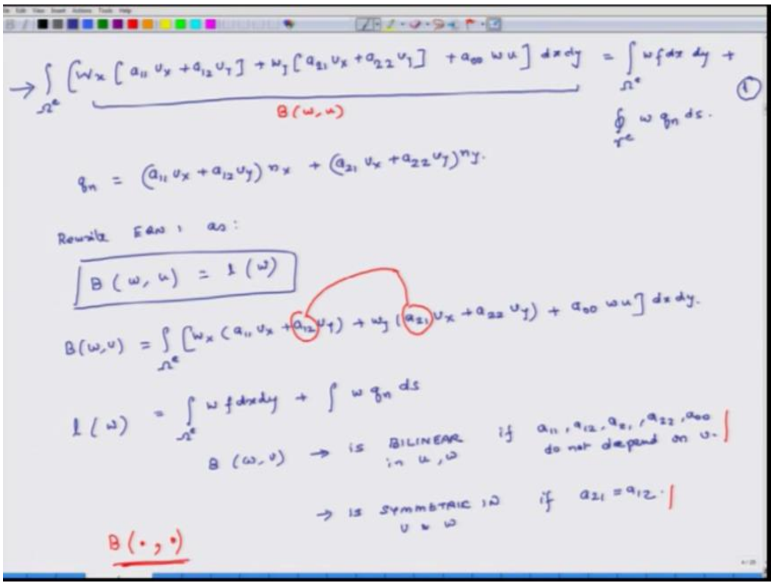

So, this is W x, a 1 1, U x plus a 1 2 U y plus W y, a 2 1, U x plus a 2 2 y y plus a 0 0 W u. So, this is a surface integral and this equals W f d x d y plus boundary integral of W q n d s. Here q n we had defined in the last class here right, this entire thing is q n. So, q n equals a 1 1 U x plus a 1 2 U y n x plus a 2 1 U x plus a 2 2 U y n y. So, let us call this equation 1. So, I can rewrite equation 1 as B some function B or functional B of w and u equals another functional of l w and I will explain this in a movement. So, B of w and u is this entire thing.

So, this equals. So, that is why B of w and u and then this l which depends on w is W f d x d y plus W q n d s. Now couple of things first thing B of w and u is bilinear, if a 1 1, a 1 2, a 2 1, a 2 2, a 0 0 do not depend on u. It is bilinear in u and also in w. It is bilinear in u and w. If a 1 1, a 2 2 all these things one of them depend on u then it will not be bilinear because then, when we plug in the value of a 1 1, they will be u square and terms like that. So, then it when all it will no longer be bilinear.

Also it is symmetric in u and w, if a 2 1 equals a 1 2. So, if these terms are equal then if you replace w with u, u and w then, you will still get the same result. So, these terms are then, it will be symmetric, but by default it is not symmetric, by default it is not symmetric, but it is definitely bilinear. It is bilinear if these conditions hold if it is symmetric if these conditions hold and a lot of times we write it as B dot comma dot. If it is symmetric then we write it as in this form because. So, with this understanding now what we will do is. So, we have developed the weak formulation.

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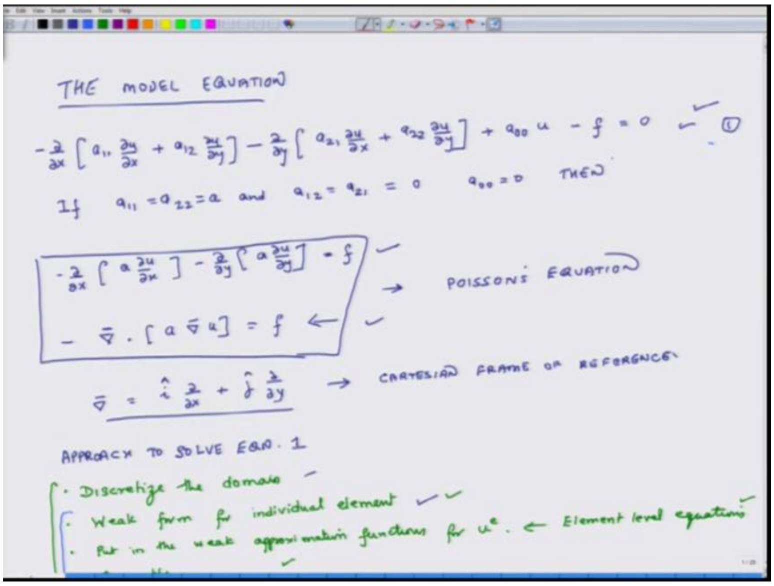

And then what is the next step? So, what we found was that first thing is we discretized, we do the weak form and then we plug in the approximation functions for this system. So, we are seen that the weak formulation when, we do the weak formulation surface integral, it also yields some boundary terms. So, it yields some surface integral terms and also some boundary integral terms.

So, at this stage so that was and the nature of this integration was different than what we did was in case of one dimensional problem. So, next thing we will do is develop finite element equations.

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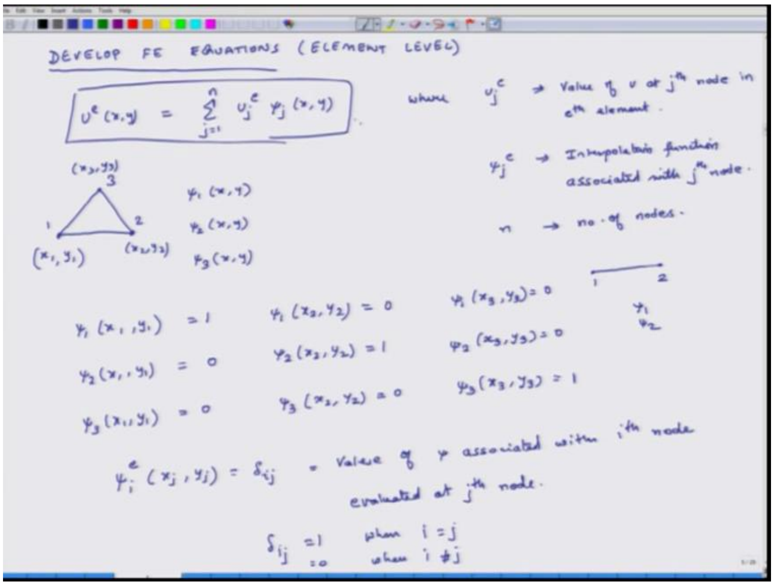

So, we will develop finite element equations at element level. So, this is my overall equation and I have to convert this into element level equation. So, what do we do? We first approximate U as we assume U as, some approximation function and what we do is that we say U for eth element which can vary with respect to x and y equals U j e psi j x y j is equal to 1 to n. Where, U j is value of U at jth node in eth element and then psi j is interpolation function. It is interpolation function associated with jth node associated with jth node.

So, if there are n nodes and n number of nodes. So, if I have a triangle I have 3 nodes 1, 2, 3 how many interpolation functions will be there in this case? 3. So, psi 1 x y will be an interpolation function associated with first node psi 2 x y will be an interpolation function associated with second node psi 3 x y will be interpolation function associated with third node understood? Let us say that these values are the coordinates are x 1 and y 1 coordinates of the nodes. These are global coordinates. So, this is x 2 y 2 and this is x 3 y 3. So, these are three different nodes. First node has coordinates x 1 y 1, these are global coordinates. Second node has coordinates x 2 y 2. Third node has coordinates x 3 y 3. So, in that case psi 1 x 1 y 1 and we will see why this comes out, but at these points of time we will just take it at its face value.

So, what will be the value of psi 1 at x 1 y 1? In one dimensional equation, in one dimensional element this is node 1, this is node two and we had psi 1 and psi 2. The value of psi 1 was what at first node 1 and the value of psi 1 at second node was 0, value of psi 2 at first node was 0 and psi 2 at first second node was 1. Similarly the value of psi 1 at first node will be 1, psi 1 at x 2 y 2 will be 0 and psi 1 at x 3 y 3 will be 0, psi 2 x 1 y 1 will be 0, psi 2 at second node will be 1 and psi 2 at third node will be 0 psi 3, at x 1 y 1 first node will be 0, psi 3 at second node will be 0 and psi 3 at third node will be 0 oh sorry 1.

So, I can generalize this by saying that, the value of psi i for eth element at node at jth node is chronicle delta i j. So, this is equal to value of psi associated with ith node evaluated at jth node, understood? What is chronicle delta is? Chronicle delta you must have read earlier that chronicle delta is equal to 1 when, i equals j and it is equal to 0 when, i is not equal to j. So, likewise we can also develop similar approximation functions for a rectangle. It will have four nodes there we will have psi 1, psi two, psi 3, psi 4 and same thinking continuous. I could have a 6 nodded element, triangle element and same thinking will get extended. So, this is my approximation function and this approximation function I have to plug it in this equation 1. So, I will call this equation 2, this is equation 1.

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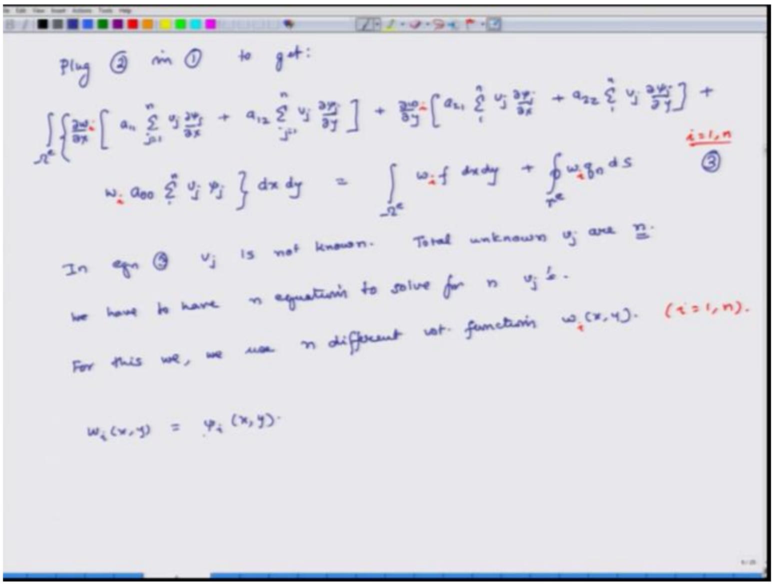

So, we plug 2 in 1 to get so, one is to plug equation 2 into 1. This is let us call this equation 3, this is the long equation which we get and this represents the weak form of the weighted residual statement.

Now, in equation 3 a 1 is known a 1 2 is known all as are known. These are some something which we know right, this relate to material property also psi js which are approximation functions we will develop them. So, we know these functions right. The only thing we do not know also W is a weight function. So, we will assign a weight to something, we have to know what weight we are assigning. So, W is known, the only things which we do not know. So, this first thing is that this is added over j is equal to 1 to n yeah. So, the only thing which we do not know is in this equation is what? U. U j is not known and how many U js are there? This is being added over n.

So, total and total unknown U js are n. So, this is one equation and we have u n unknowns. So, we cannot solve this equation right. So, what do we do? We have to generate n equations and this is at element level. So, we have to generate n equations at the element level and how do we generate n equations? So, we have to have n equations to solve for n U js. So, what do we do? For this we use n different weight functions. We use n different weight functions. So, if I have n different weight functions, so in first case I will use W 1 and write down equation 3. In another place I will have W 2 and write down equation 3 and so on and so forth. So, what I do is? Instead of W, I replace it by W i. So, the moment I replace W by W i and i is equal to 1 to n then, equation 3 represents how many equations? n equations.

So, i equal 1 to n. So, now, we have n equations and n unknowns. Now in variational formulation we had seen in part 1 also that, essentially what we do is that W i we say it is same as psi i x y. This is what we exactly did in one dimensional thing also. So, this is again same thing. So, then that case my equation. So, equation 3 becomes.

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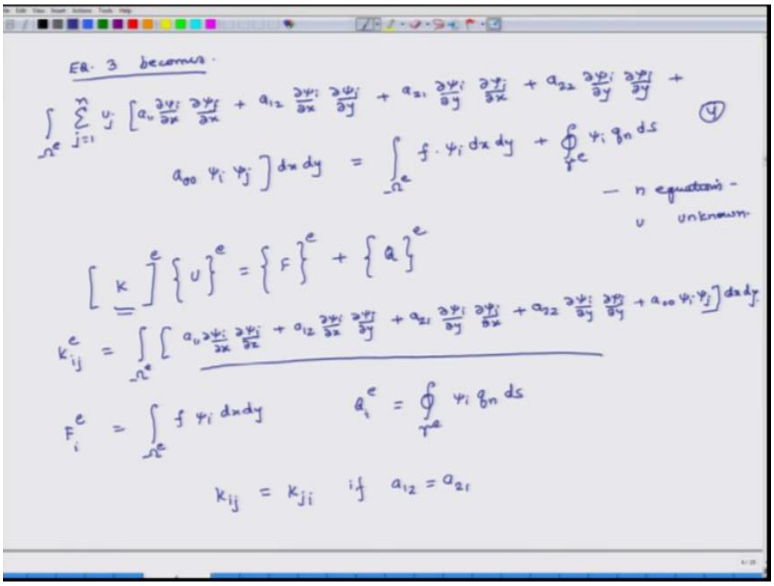

So now, what I will do is? I will take U j out of the differentiation term because U j is a constant. So, when I differentiate it, it will give 0. So, I can take it out. So, this is equal to j equals 1 to n plus oops sorry, now this now psi i psi j. So, that is equation 4. So, these are n equations and U j is unknown, I can also express this equation. So, these are n equations. So, I can also express this equation in a matrix form K for the eth element equals U for the eth elements equals a vector for the eth element and here K i j for the eth element equals integral over the domain of the element of this entire thing a 1 1 psi i partial with respect to x plus a 1 2 Del psi i over Del x Del psi j over Del y plus a 2 1 Del psi i over Del y Del psi j over Del x plus a 2 2 Del psi i over Del y oops sorry Del psi j over Del y plus a 0 0 psi i psi j d x d y and then F i for the eth equation equals integral of f times psi i d x d y plus q n psi i d s, this is a boundary integral.

Now, earlier we had said that this expression B is bilinear and it will be symmetric in U and W if a 2 1 equals a 1 2. That is what we had said? Now you please look here. This matrix, this K matrix will be symmetric if, a 1 2 and a 2 1 are same. Then K i j will be same as K j i. So, K i j equals K j I, if a 1 2equals a 2 1. So, in a lot of physical problems a 1 2 and a 2 1 is indeed the same entity. So, in those problems K i j equals K j i which means, K is a symmetric matrix and that makes things little bit easy when we are trying to solve the problems and the F vector I can also write it as f i plus Q i I should have actually, I will make a small correction I will call this Q and I will erase this. So, the F vector depends only on f and then the Q i is the boundary term which is psi i q n d s integrated over the boundary of the element.

So, these are the finite element equations and what we will do is we will close the discussion for today. In the next class what we will do is, we will develop these interpolation functions. We were right now we do not know, what is the nature of these interpolation functions? So, in next couple of classes we will develop these functions. So, thanks a lot and we will meet tomorrow.

Thank you.