Basics of Finite Element Analysis – Part II
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Lecture - 26
2D Finite Element Problems with Single Variable
(Weak Formulation)

Hello. Welcome to Basics of Finite Element Analysis Part II, today is the second day of this fourth week of the course, yesterday we had introduced the need to have a weak form of this 2 dimensional equation partial differential equation having 1 variable and that is what we will continue to today.

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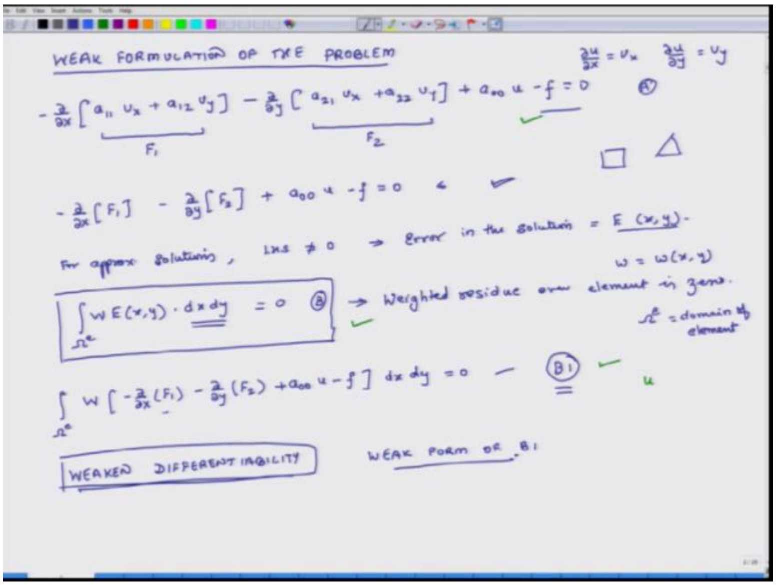

So, to recap what we have developed was this is r equation A is the basic equation which we will try to solve, for this equation we had developed a weighted residual statement which is equation B. And another form of weighted residual equation is equation B 1, and we have said that as per equation B 1 the differentiability requirement on u is of second order.

So, now we will try to weaken the differentiability by developing a weak form of equation B 1. So, were we will develop weak form of B 1? So, before we do that what we will do is we will write down 2 or three important relations.

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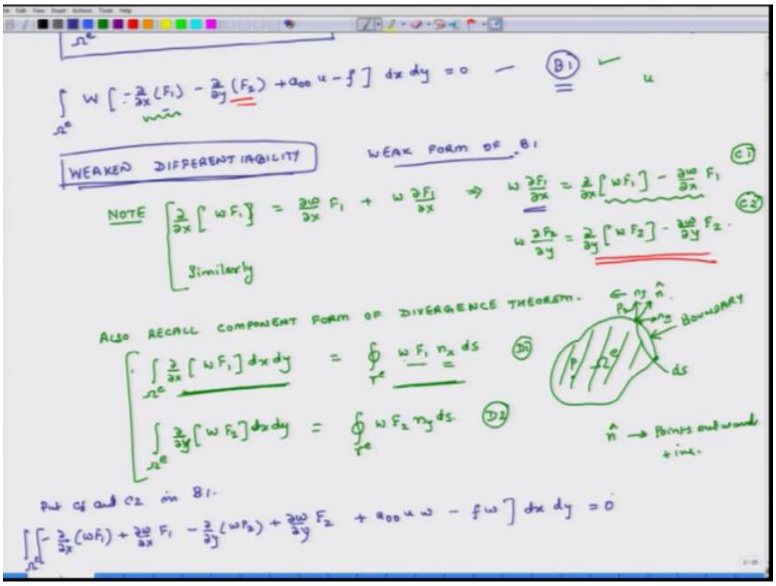

So, we will note, what we note that del over del x of w times f 1 is partial derivative of w with respective x times f 1 plus w times partial of f 1 with respect 2 x. So, this gives me w times partial of f 1 with respect 2 x equals this relation. And likewise similarly w times del f 1 over del y equals del over del y w times f 2 excuse me this should be. So, w time's partial of f 2 with respect to y equals partial with respect to y of the multiple of w and f 2 minus del w del y times f 2.

So, let us call this equation c 1 and let us call this equation c 2. So, this is equation c 1 and c 2 we get by rule of differentiation of 2 functions multiply to each other right differentiation by parts also recall the component form of divergence theorem. So, some people are also call it gradient theorem, and it has the same thermo can be expressed in a vector form or in a scalar form. So, scalar form is known as component form because a there we are taking individual components and write them a separate equations in vector form we assemble them in write it in to 1 single equation. So, this component form of divergence we had actually developed in FEA part one.

So, if you do not remember kindly go back and check some of our old lectures and you will see all the details. So, here what I will do is, I will directly write that results. So, the component form of that divergence theorem, says that if I have an integral of partial derivative of let us say any function w times f 1 then it is nothing. So, this left hand side is an integral over the domain of the element. So, this I can express it as a boundary integral of w time's f 1 times nx ds, and I will explain this phenomena it.

And similarly integral of del over del x del over del y w times f 2 dx dy equals boundary integral of w f 2 ny ds. So, what does this mean, what this mean says let us suppose I have a domain, now in realty this domain will be either a triangular element or a square element, but this equation. So, let us call this equations has d 1 and d 2 this equation is valid for any boundary, you know it can any complicated boundary, does not matter does not have to be a triangle or square of rectangle is valid for any boundary. So, what this says is let us try to understand what is equation d 1 the left hand side means that if I any point p.

I have value of f 1 because f 1 is a function of x and y right. So, if I take that functions that value f 1 I multiplying it by weight function that is also function of f n x and y. So, at any point p, I calculate this product w and f 1 and then if I differentiated with respect to x partial differentiate to this respect to x. So, I will gets some number right and then I integrate that function over the domain. So, this entire thing is the domain then that integral can also be evaluated by finding the value of w and f 1 on the boundary. So, this is my boundary. So, I do not have to find the all the values of f 1 and w in the domain, but I can also calculate the left hand side by using the right side right hand side equation By finding the values of f 1 on the boundary.

So, let us say this is another point p 2, this will be should clear let us say this is point p 2. So, what do I do I find out first thing is that I find out, what is the direction cosine? Of a vector which is normal to the surface. So, this is my n vector. So, for any boundary shape at every point, I can it is known this n is known, because I know the shape of the surface right I know the shape this edge of the boundary.

So, at every point I know what is the direction of this n vector which is a unit normal in the normal direction, then it will have nx component nx, and it will have in ny component. So, I can calculate nx and ny because the outside surfaces known. If outside surface or edge of the surface is known then at every point I can calculate nx ny because, I know the shape actual physical shape of the system.

So, I calculate at every point nx I multiply that nx with f 1 at point p 2 and also I multiply that by w at point p 2, and I multiply it by a small length this is ds. So, I do this our computations all along the boundary. So, I get a boundary integral which represents the left hand side. So, a surface integral which is of this form can also be calculated using a boundary integral; this is what the divergence theorem means. It can also be calculated using a boundary integral this something similar we also did when we were doing 1 dimensional equations, were we computed and integral along the length of a line in terms of boundary values at that termination points of the line right.

So, here also we are calculating the integral in terms of some parameters which exist on the boundary. So, this is what the divergence theorem says, that if I have a surface integral of this form not of any general form of this form which is shown here which is on the left side, then I can calculate the value of that by using a boundary integral by evaluating all these parameters w f 1 nx and integrating the multiple of this all along the boundary. So, this is what means it is very important theorem. And 1 thing I want to say that if this n point outwards then it is positive this is the convention, I have this point inverse then it is negative.

So, what we will do is we will first put c 1 and c 2 in equation B 1. And because of this term will get replaced by this term. And also this term will get replaced by this term this is the first step. So, that is what we will do. So, we put c 1 and c 2 in b 1. So, what we get w times del over del x of f 1 right this is this term. So, this get replace by this thing and this negative sign here. So, so I get minus del over del x wf 1 and then plus del w over del x f 1. So, I have put c 1 in equation B 1. And then I put equation c 2 in equation B 2. So, I get minus del over del y wf 2 plus del w over del x del y f 2. And then these terms do not change. So, I get I still have a 0 u w minus f w dx dy equals 0.

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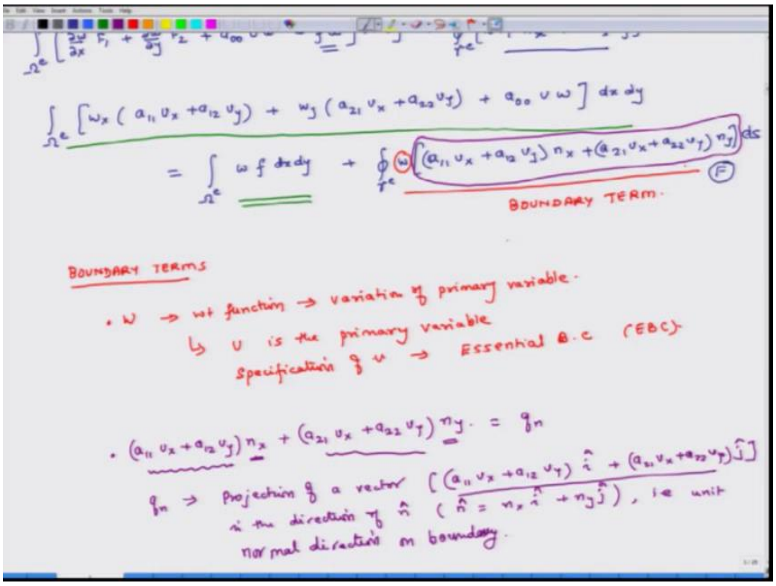

Next, what we do is put d 1 and d 2 in equation e. So, this is equation e. So, what does that mean that I am going to replace, this term is same as this term. So, it is a surface integral, I can replace it by a boundary integral. Similarly this term and this term are same. So, I can also replace these two surface integrals by two boundary integrals. So, first I will write the surface integral, terms and then I will write boundary integral terms.

So, I have del w over del x times f 1 plus del w over del y times f 2 plus a 0 uw minus fwd dx dy. These are the surface integral terms, and then I put boundary integral terms. So, this gamma e is represents the boundary of eth element, gamma e represents gamma is represents boundary e represents eth element. So, this equals minus wf 1 nx plus w f 2 ny ds equals 0. So, I can again rewrite this as integral of. So, f 1 now I am going to replace by the original definitions. So, this is wx times a 1 1 ux plus a 1 2 uy plus wy wy, means del w over del y times a 2 1 ux plus a 2 two uy plus, a 0 u times w dx dy equals this term and these terms, I will take to the right side. Boundary a surface integral of w times f dx dy plus. Boundary integral of w a 1 1 ux plus a 1 2 uy nx, excuse me plus a 2 1 ux plus a 2 2 uy ny ds. So, let us call this equation f. So, on the left side you have these boundary terms, excuse me surface integral terms and on the right side you have again surface integral term, but you also have a boundary term.

So, again to draw comparison in our 1 d formulation when we did a weak formulation we weaken the differentiability. Of the primary variable a variables right and when we did that in the process weakening of differentiability, we used to get some boundary terms we are seen that the same thing happens in the regular strong form of the weighted residual statement, this is the strong form because, the differentiability requirement is higher here there are no boundary terms, but when we weaken the differentiability we get some boundary terms, in process and these boundary terms help us understand what kind of boundary conditions we need to know to solve this problem.

So, let us look at the boundary terms. In this boundary terms the first thing we have is w. w is the weight function. And in context of 1 d problems which we have discussed, it could also represent as the. So, this is weight function and it could also represent has variation of primary variable. It could also be interpreted as a variation of primary variable. If the primary variable is known on the boundary then the variation is 0. If the variable, if at any point if the vary value is known then it cannot vary. So, then the variation will be 0.

What this tells us that this is primary variable. So, what this helps us understand is that u is the primary variable. And if I specify this variable then is specification of u implies that I am specifying essential boundary condition or EBC. In the boundary term either we specify w or we specify the other part. So, this is 1 thing, the other part in the boundary condition is this thing. This is a 1 1 ux plus a 1 2 uy nx plus a 2 1 ux plus a 2 2 uy ny. This I can abbreviate it as qn now this qn what does it mean it means. This is the direction cosine in x direction this is the direction cosine in y direction, these are the components of the direction unit normal.

So, this term represents the projection of an entity, see this term represents the projection of an entity in the x direction and this represents in the y direction right. So, what this means is that qn. It represents projection of a vector. It represents projection of vector, and what is that vector, vector will be a 1 one ux plus a 1 2 uy i plus a 2 1 ux plus a 2 2 uy j. If I project this vector in the direction of n right, if I project this in the direction of n y, because n is what nx i plus ny j. Then when I project it what will I get I will get the same thing qn, if I take a dot product of this vector. And this unit normal then I will get qn. So, it represents projection of this vector in the direction of n that is unit normal direction on boundary.

So, what that is. So, when we are specifying the boundary term either we specify u. If u is known then w will be 0. Because it represents the variation or if u is not known then I have to specify this thing. So, either we have to specify the boundary term has 2 terms w and this entire thing in purple. So, either we specify u or we specify this entire thing in purple.

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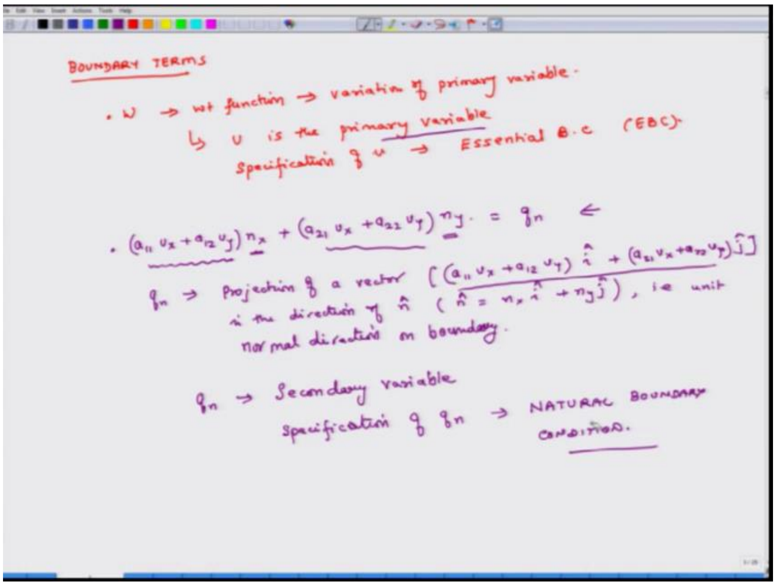

So, this is qn is secondary variable. And if I specify it I say that specification of qn, implies that I am specifying that natural boundary condition. So, this is what it means. So, I have 2 boundary conditions. 1 involves the primary variable which is u, the other involves and if I specify it, then I say that I am specifying EBC or essential boundary condition or is specify qn which involves natural boundary conditions.

So, this concludes our discussion what today, and we will continue this discussion on weak formulation tomorrow as well.

Thank you.