Basics of Finite Element Analysis – Part II
Prof. Nachiketa Tiwari
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Lecture – 25
2D Finite Element Problems with Single Variable
(Model Equation)

Hello. Welcome to Basics of Finite Element Analysis Part II. This is the 4th week of this online course and what we will do today is continue the discussion which we having last week that is how do we go around solving, 2 dimensional problem with one variable and specifically in the last week we had started discussing the Poisson's equation.

So, we will continue to do that discussion and hope fully by end of this week we would have formulated the problem using the finite element method and also developed interpolation functions for the same.

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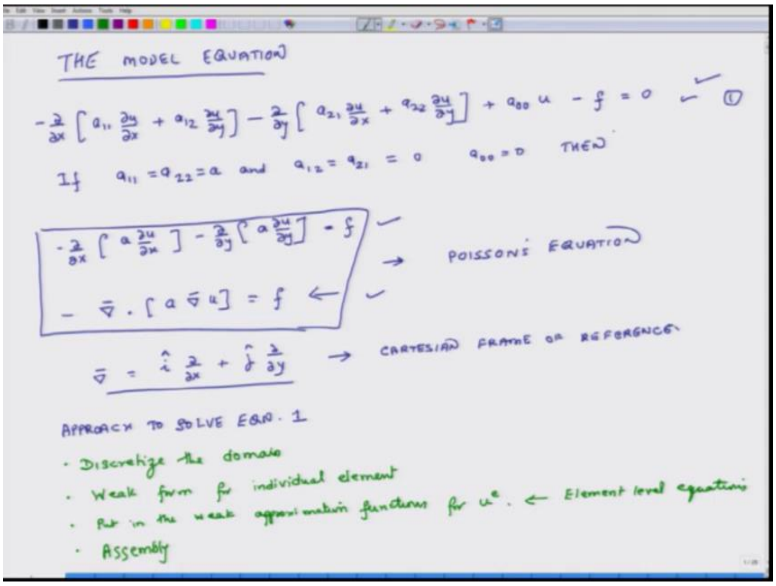

To recap the model equation was something like this. So, it is partial derivative with respect to x a11 del u del x plus a1 2 del u del y minus partial derivative with respect to y of a 2 1 del u del x plus a 2 2 del u del y plus a 0 0 times u, minus f equals 0 and we had said that if a11 is equal to a22 and a12 equals a 21 equals 0 and if a0 0 was also 0. Then we can express this equation in the following form.

So, a11 equals a 22 in this case if it is equal to some number a or function a, then I can express this as equals f or I can express the same equation in vector rotation. So, this is my operator diversion operator times a time gradient of u equals f. So, these forms, this form or this form is known as Poisson's equation and to recapitulate this operator is nothing but a unit vector times del over del x plus and the unit vector in the y direction times del over del y.

Now, this operator this definition is valid for Cartesian frame is valid for Cartesian frame of reference for cylindrical frames, spherical frames this definition changes, but we can still, but for other frames this equation is a still valid, but it just that the operator’s definitions changes. So, what we had said was that the way to solve this Poisson's equation or for that say given this equation, would be pretty much similar to the way we had solved one dimensional problems which we had discussed in Part 1 of this course.

So, the approach would be to solve. So, this is equation 1 if I have to solve this equation 1 then approach would be first we have to discretize the domain, then we have to develop weak form of this equation which is applicable to element of the system, not to the whole thing, but for element because that is where finite element comes in. So, weak form for individual element, and then we put in the weak form. So, in the weak form what we do? Approximation functions for ue. So, we develop the weak form and then in that weak form we put an approximation functions.

Then addresses stage we get element level equations and then in the next step is that we assemble the we perform assembly of equations, once we have done the assembly we apply boundary conditions and then we solve the problem, solve the equations for unknown variables and later if we want we can do post processing of that result which processing of the results.

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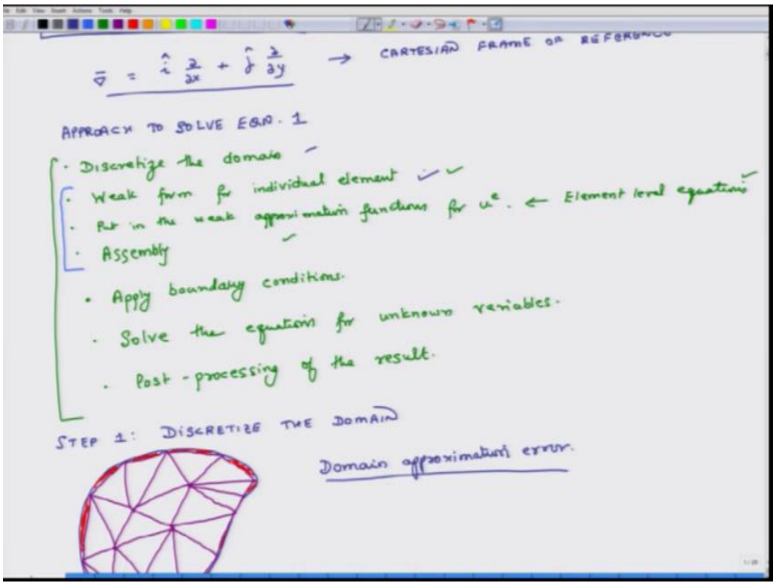

So, all these steps at a conceptual level they are same as what we had, encountered when we were doing FEA part 1, which was in the last semester. So, all these steps are same, but what differs when we move to 2 dimensional problems is significant difference in developing the weak form because here we when we do integration by parts we are talking about 2 dimensional domain in earlier system it was 1 dimensional domain. So, the results yield some different type of terms then the other important difference lies in developing the approximation functions and they are also significantly different then what we had seen in one dimensional thing and then also the assembly of the equations.

So, these 3 things are significantly different in 2 dimensional and if we learned how to do it two dimensional space, then it is easy to graduate to three dimensional system because there the concept is essentially is still remains the same. So, with that intent we will solve or we will develop finite element formulation to solve this equation one or it simplified version which is the Poisson's equation and the first step which we had said was we redevelop a we discretize the domain. So, this is again we are recapping and explain this earlier. So, if this is my domain, where I want to find the solution, let say I want to break it up into triangles. So, this is my discretized domain, but the important thing to notice that these areas of red when we discretize it, they do not get captured in the domain.

So, this is something very fundamentally different, and then what we had seen in 1 dimensional problem because there it was a straight line element. So, I could break it up into smaller line elements and the entire length of the line would be covered. Here the entire surface area is not covered especially if the outside parameter is a little complicated. So, this introduces what I had said was domain approximation error or discretization error. So, this is one additional error which comes into the picture, when we handle two dimensional or three dimensional problems.

So, this is what I wanted to recap in context of whatever we had done earlier. Now what we will do is we will start developing a weak formulation of the problem. So, we have talked about discretize the domain the next step is weak formulation of the problem.

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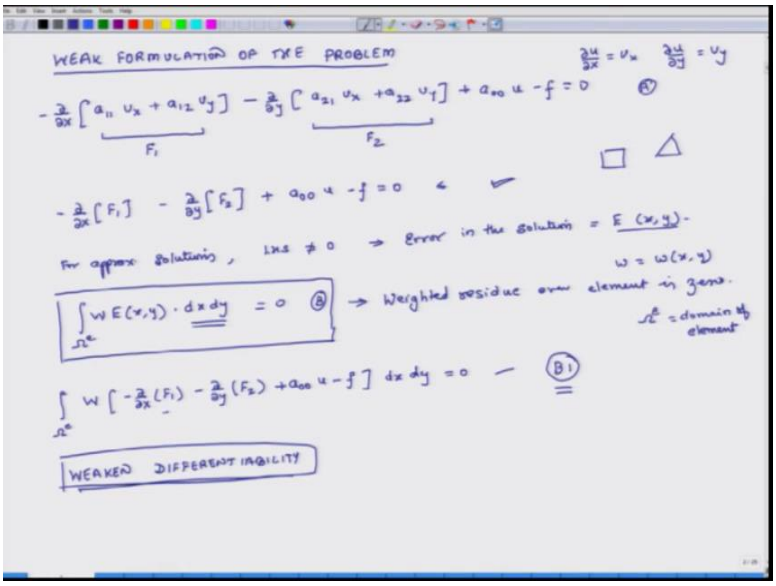

So, we will write down the equation once again. The equation is minus del over del x a1 1 and I will do is just to make things in brief sometimes I will also write del over del u over del x as u x and del u over del y as u y. So, so this is u x plus a1 2 u y minus del over del y a2 1 u x plus a2 2 u y plus a0 0 u minus f equals 0. Now I can call this entire expression as F1 and I can call this entire expression as F2. So, 1 minus del over del y F2 plus a00 u minus f equals 0. Now we are interested in developing it is weak formulation. So, what do we do? We find out. So, if it is my solution for you is accurate, then the left hand side will be equal exactly 0 right, at all points at every individual point in the domain at every location x y coordinate the value of left hand side of the equation will be exactly 0 which will equal the right hand side. So, in that case my solution will be exact and there will be no error in the solution.

Now, when we are talking about approximate solution such as finite element analysis this error is not going to be 0. So, for approximate solutions, left hand side is not necessarily equal to 0 and that introduces an error in the system in the equation and this error let us call this error as e this is the function of x and y. It changes from point to point it changes from one point to other point. So, when we are doing finite element analysis we say we realize that there is point to point error in the system, but what we want is that if I multiply this error with the weight function W and this weight function is also of function of x and y. So, if I multiply this error with a weight function and then I integrate it over each element and we want to develop a solution that it is error may not be zero of one a point to point basis, but the multiple of the error and the weight function when I integrated over the area of the element that should be equal to 0.

So, this is the weighted over element is 0. So, the error is not 0 at every specific point, but the weighted residue over the entire element it is 0. So, kind innocence and the error is 0 in an integral sense over the entire element. So, let us call this equation a and we will call these equation b. So now, what I do is I plug in this equation an equation b to get. So, before I do that, I wanted to make you understand or explain it clearly that in one dimensional set system we were integrating it over the length of a element, which was a straight line.

Here the element could be a triangle it could be a rectangle. So, we are integrating it over the area of the element. So, that is why we have 2 terms d x and d y and also this omega e represents domain of the element, where e stands for element. So, for eth element omega is the domain. So omega e, omega superscript e represents domain of the eth element. So, this equation is specific to an element this is important to understand.

For another element it will be a different equation. So, this is an element specific equation this is important to understand. So, the next thing we do is we plug in this equation in equation b. So, what we get is, integral over the domain of the element W times minus del over del x F1 minus del over del y F2 plus a0 0 u minus f d x d y equals 0. So, next what we do? Yes we weaken it is differentiability. Weaken the Differentiability of the system. So, before we start doing that i wanted to explain the term F1 has first order derivative of u in x and y; same thing is true for term F2. It is having u x and u y. So, it is having first order derivative of u x of u in x and y and F1 is being differentiated with respect to x and F2 is being differentiated with respect to y.

So, in this weighted residue statement which is this equation, I can call this B. In equation B1 we suppose we want we replace u with some approximate function of x and y then it should have at least what are done of derivative. We should have second order derivatives right it should have at least second order derivatives because if the order of derivatives is if it is only differentiable once then all the term these related to F1 and F2 will become 0.

So, it should have second order derivatives. So, in that context we will start developing in approach we will go, we will develop a weak form of equation B1 and then that case the differentiability requirement on u which right now as per equation B1 is to it will go down to 1. So, that is what we will do in the next class.

We will continue this discussion, in the next class and we will look forward to seeing you tomorrow.

Thank you.