Basics of Finite Element Analysis – Part II

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Lecture - 24
Two dimensional one variable FEM problem

Hello. Welcome to Basics of Finite Element Analysis Part II. Today is the last week of last day of this fourth week and what we will do today is developed the details of this two dimensional one variable problem. And specifically we will do that in context of boundary value problems. So, first we will develop write down the model equation.

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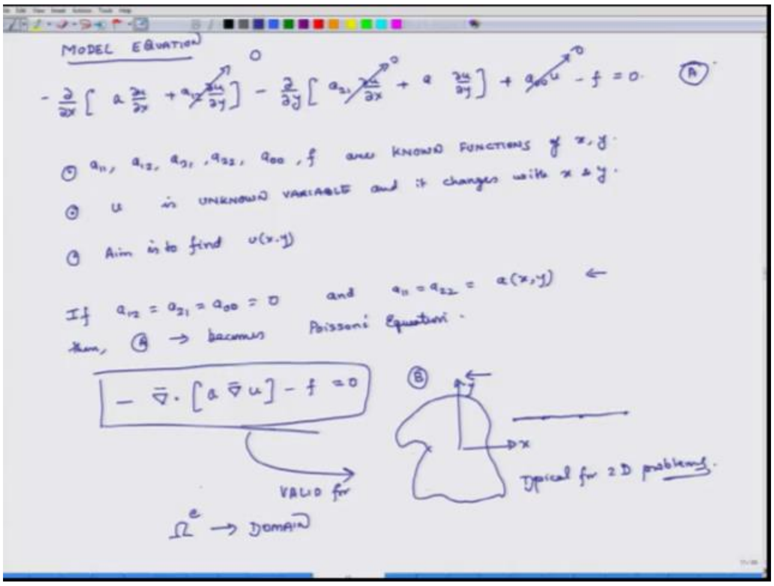

The governing equation and this is the equation we want to solve. So, the governing equation is del over del x a 1 1 del u over del x plus a 1 2Vdel u over del y and there is a negative sign here minus partial derivative with respect to y a 2 1 del u over del x plus a 2 2 del u over u del y plus a 0 a 0 0 times u minus f is equal to 0.

So, this is the most general two dimensional linear equation partial differential equation most general form in two dimensional. So, here a 1 1, a 1 2, a 2 1, a 2 2, a 0 0, f are known functions they are known functions of x and y. So, they can change with respect x and y, but how they change we know that. So, that is the first thing. Second u is unknown variable and it changes with x and y.

So, our aim is to find what is our aim it is to find this function u x and y by integrating this equation a, and when we integrate this. So, this is one equation with one variable right. So, we should able to solve for it, but the problem is when integrated this is the second order equation, it is second order equation right. So, when we integrate it we will get some additional entities. Now because, it is a parse when we integrate a ordinary differential equation then, when we integrate it we get integration constants this is a partial differential equation when, we integrate a partial differential equation, what do we get either functions of x or y or something right.

So, we will get additional functions. So, we will the unknown thing is not only u, but those also additional functions. So, those additional functions and use will be calculated based on this equation a, which is one equation and also additional boundary conditions which we have to apply. So, that how we get the entire solution. So, that is important to understand. So, now, here if a 1 2 is equal to a 21 is equal to a 0 0 and this is equal to 0 and a 00 is equal to a 2 2 is equal to some function of a x and y. So, if this is the case then if you put all these then, a becomes Poisson's equation. So, that is what we get. So, this term goes away this term goes away this term goes away and this becomes a then this equation then this equation b as same as equation a right.

If, we do the operations correctly we can clearly show that equation b is as same as equation a, another thing I wanted to mention in this context is that for one dimensional equations for one dimensional equations our domain was a line right. So, we broke this line into elements and it was a straight line because, we were talking about one dimension. So, the only thing which could vary was x. So, the line could not bend or curve it was a straight line, but in two dimensional for two dimensional equations the domain could be complicated right. So, this is my x, this is my y the domain could be whatever shape as long as it does not change in the z direction. So, domain is this is a typical domain for 2D problems. So, all of a sudden the complexity of the geometry becomes significantly large it not only is changing a not only is there in a two dimensions, but shape could be not a rectangle or square or circle, but it could be some complicated shape. So, that is there.

So, this equation this is valid for this domain for this domain and in two dimensions and three dimensions and we can also use it one dimension this symbol is used for domain, if the domain is an element then I can put subscript superscript omega e, but the symbol omega is lot of times it is used to symbolize domain.

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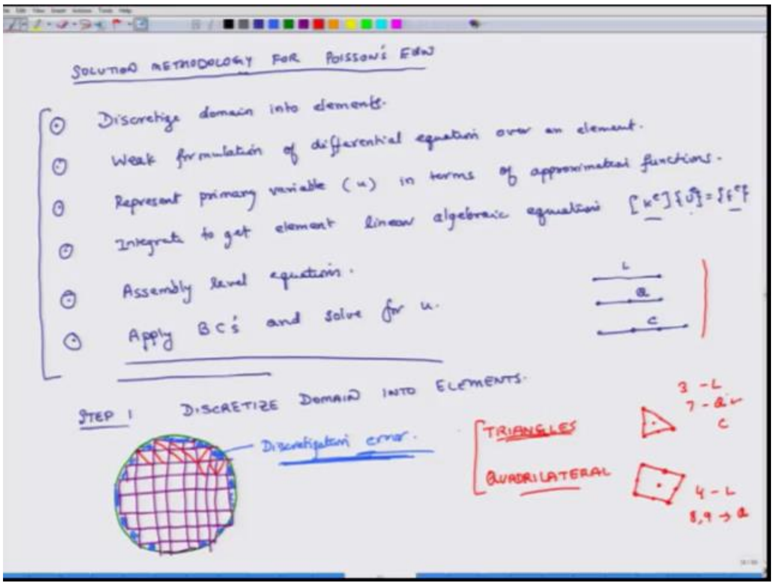

Now, we will develop a solution approach solution methodology for Poisson's equation, what was the first step for 1D problems what did we do discretize. So, the steps are the same. So, first step is we discretize domain into elements the sequence of steps does not change, but the implementation of each some step become complicated compared to the one dimensional problem. So, we discretize the domain into elements then we apply the equation for each element because the equation is valid for all points in the domain. So, we applied for each element. So, what was the second step we did in 1D problems we develop the weak formulation.

So, then the next step is weak formulation of differential equation over an element because then we go down to the level of element, apply the equation on that element multiply the error by a weight function weaken the requirement for the variables and that is how we get the weak formulation for a specific element, then what do we do? Now before we integrate it here, we still have u we represent primary variable is u in terms of what approximation of approximation function right only once, we assume this function then only we can integrate otherwise how can we integrate then, we integrate to get element level. What kind of equations is it a partial differential equation is it ordinary differential equation is it an algebraic equation what kind of equation is to when, we integrate we get linear algebraic equations of for what k e times u vector is equal to f e these are linear equations this is known k f is known u is not unknown.

Then what we do. So now, we have equations for all individual elements then we do assembly. So, we develop assembly level equations and then we apply b c’s and solve for u solve for u. So, this is what we do? So, the approach is same, but each step at least first yeah almost all the steps are much more complicated, then if we can figure out how to do these steps for two dimensional systems then we can easily use the same approach for three dimensional systems. But jumping from 1D to 2D involves significant increase in the level of complexity of the problem. So, that is why important that we understand how 2D works in very rigorous way. So, this is the methodology and as we discussed here the first step is that we discretize that domain into elements.

So, let us. So, step one discretize domain into elements. So, consider let us consider circle for purposes of discussion and I can discretize it like this as a lot of rectangles right I can discretize it an. But we see that when I am discretizing it in terms of rectangles or quadrilaterals there are some areas left which are not covered in my geometry. So, this is the discretization error this is the discretization error you can discretize this domain into elements, and what we see something very fundamentally different in context of 2D is that the domain geometry is not necessary accurately captured by all of our elements right.

So, all these blue areas they represent the discretization error of domain, discrete error related to domain approximation related domain approximation. So, this is one additional way error which comes into FEA for 2D and 3D cases this error is not present in 1Dcase because there was no domain approximation error. So, this is important to understand. So, as we discretize we have to make sure that our element size is reasonably small. So, that the amount of this error introduce because this is as small as possible, this is one thing. Now I can discretize it not only in terms of rectangle, but I can also use triangles to domain you know and so on and so forth. So, I can.

So, two very common topologies or elements which are used are triangles or a quadrilateral or a quadrilateral. So, we can use either only triangles or only quadrilaterals or a combination of those to discretize the domain these triangles and quadrilaterals. So, in 1D problem we could have a two nodded linear element, this is linear element or a three nodded quadratic element or a four nodded cubic element and so on and so forth.

Similarly we can have a three nodded linear element. So, if you have three nodes for a triangle then it is a linear element you can have additional elements, here not elements additional nodes here to get an may be one in the centre to get a 7 nodded quadratic element and if I increase the number of nodes then I can also cubic or quadratic or pentic and so on and so forth.

So, I can have triangular elements of different varieties. So, whenever you do finite element analysis we have to make sure that not only you decide what the shape of your elements is. But also what order these elements are they linear or quadratic and so on and so forth. So, this is one thing same thing is true for quadratic elements also a 4 nodded quadratic element quadrilateral element is linear in nature. But if I have eight nodes or 9 nodes 8 or 9 then, this is quadratic in nature we will develop the mathematics for this, but at least at this stage you should be aware that this concept of linear quadratic cubic is not only valid for 1D elements, but it is valid for triangles quadrilaterals and if we go to 3D it is for bricks wedges and. so on and so forth.

So, this is where we have to make choice that what is the amount of triangles and quadrilaterals which we want and then the other thing is that what is there order. So, if it is quadratic element it is a high order element linear is low order element cubic is even higher order element. So, this is all languages associated with it.

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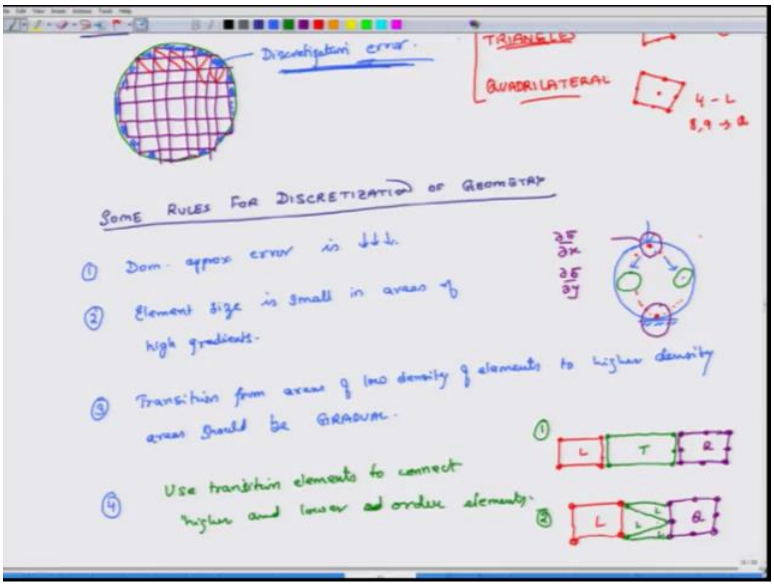

So, some rules some rules of when we are discretizing the geometry first thing is that, we have to make sure the domain approximation error is as low as possible which actually means that our element size should be small.

Second things second thing is that suppose you have let say a disk and this disc is fixed at one end and I am applying a point force here. Theoretically if it is a point force the stress here will be what if I am applying a point force on a surface. What is theoretically the stress at that point it will be infinite because at that point there will be finite force divided by 0 area infinite forces. That means, is that stress will be extremely large here reality that does not happen because material will yield and it will.

But theoretically stresses will be very large here. But as you move, so stresses are going to be very large here, but as you move away stress will become, less as you move stress is going to become less and when you come here what will be the stress here at the point where it is rigidly fixed again, if it is fixed at just one point all the reaction force will be witnessed by only one single point.

So, again you will have a singular point here and as you move away stress will become less which means that, in this region the gradient of stress which is suppose the gradient of stress or even the gradient of u displacement, displacement will also be very large here right. So, gradient of stress and gradient of stress in and y directions at this point it will be high or low it will be high. So, in this region gradient of the solution is high in this region in the green region gradient is not that high?

So, if I am doing discretization we have to make sure that element size is small and how small you have to do several iterations on FEA and whether you are getting the results in areas of high gradients because if you take large element then, you will not be able to capture the sharp variation of the gradients. So, you have to take small elements which means that when you are meshing this the purple area you have in that area you have to take element sizes, as very small in green area you can take it you can make extremely small, but it will not give you some extra benefit and the larger of number of the elements what does it mean computation time becomes large.

So, when meshing is going on especially in areas where you expect gradients to be high you have make sure that element sizes are small and as you move away you can gradually reduce it. So, that is the second thing third thing transition from areas of low density of elements see in this areas in purple area density of elements will be what high because element size is small. So, there will be lot of elements. So, number of elements per unit area will be high and in green area it will be low. So, the transition as you move from here you will have lot of elements, small elements as you have in green will have less number of elements, but large elements. So, this transition from areas of low density of elements to higher density areas should be gradual this is important.

Otherwise you may have numerical errors which may be large. So, these are some rules and then there is another very important thing suppose you have a triangular element and let say it is a linear order element. So, it will have three nodes it will have three nodes. Now suppose the adjacent element, let say this is the adjacent element and then there is a third element here actually I will this will be better. If I use rectangular elements, suppose there is a linear rectangular element.

So, if it is linear element it as four nodes and if in my mesh I have a quadratic element I have a quadratic element. So, I have a linear element here and I have a this is linear element that is a quadratic element and suppose there is some element which connects these two and I will not physically join them. But in reality they will be touching each other, but I have put some gap to make things clear.

So, this is the connecting element green is connecting element now the question is see if the linear element has 2 nodes at the common interface the between green and red the quadratic element has three nodes. If I use a linear element here then this will be the situation, but then this node, when I am doing my assembly operation I have to make sure that when there are common nodes, I have to that I cannot do here at the interface of the green and purple right and if I use a quadratic element in case of green the same problem will come between red and green.

So, what do I do either I use in a mesh only linear elements or only quadratic elements then they would not be problem. But in al lot of cases I want in some areas to get more information. So, I want linear quadratic elements of cubic elements in that area and in other area probably solution is simpler. So, I can live with linear elements, but then what do I do? How do I connect linear elements with quadratic elements is it important to understand.

What do we do in that case one option is you use an element called a transition element and these are special elements? Where it is having some quadratic variation at one edge and at the other edge it is linear variation. So, these are transition elements. So, use transition elements to connect higher and lower order excuse me elements this is one thing we can do there is another approach we can do. So, this is this option on the other option could be I still have linear here and here, I have quadratic and then I can still without using any fancy transition elements. I can still connect them with linear triangle this is another linear triangle and this is another linear triangle this is another way to connect them.

So, these are some of the important rules for our how do you connect elements and how do you generate meshes? I think this is pretty much it for today and also for this week and we will continue this discussion over next several days probably, closely two weeks next two weeks. So, that we develop a very good understanding of two dimensional problems with one variable because graduating from one variable to two variable is not that much complicated. So, that this brings to the closure for today and thanks lot and we will meet once again next week.

Thanks, and have great week end.