Basics of Finite Element Analysis – Part II
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Lecture – 20
Gaussian Quadrature – Part II

Hello. Welcome to Basics of Finite Element Analysis Part II. Today is the second day of this fourth week, yesterday we had done a quick recap of Gaussian Quadrature scheme and what we had discussed was that depending on the nature of integrand, we have to calculate the number of Quadrature points and to ensure that are numerical integration if use Gaussian Quadrature is exact.

So, that is what we discussed yesterday and today what we are going to do is continue that discussion on Quadrature and specifically we will actually calculate the value of K i j of a I will show this whole method works.

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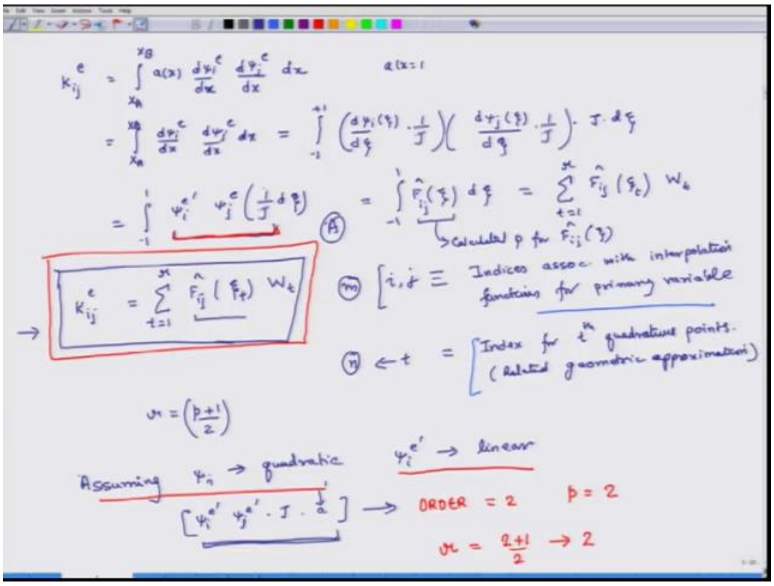

So, we had shown that K i j equals X A to X B a x d psi i e over d x, d psi j over d x excuse me times d x. Suppose this is the definition and lets assume that a x is equal to 1 then what I get is X A to X B, d psi i over d x for e-th element times d psi j for e-th element over d x d x. And this now I transform it to the zeta space or natural coordinate system. So, the first thing is I change the limits from minus 1 to plus 1, then I express d psi i as a function of zeta and d zeta times J right and then this is d psi j as a functional zeta over d zeta times j and d x is J d zeta. So, this gives us minus 1 to 1, psi i e prime psi j e prime times J times P zeta and this is equal to.

So, I call this entire thing as F hat, as a function of excuse me. So, this entire things psi j prime times j I call it F i j hat, if I call it F i j hat and if I have to use Quadrature method then what do I do. I rarely express it as a summation of F i j hat, but I will evaluate this F i j hat at which point? At the Quadrature points; so, the index for those Quadrature point is different which is t and this is and the weight for t-th Quadrature point is Wt. So, I will add this up over index t and t varies from one to r. So, this is very important to understand and make sure that you do not get confused with these 3 different indices.

So, if I am using Gaussian Quadrature, this is the formula we have to use, and this formula has 3 indices i j and t, the indices i and j where do they come from? Where did they come from? They are associated with the psi functions which are interpolation function for way what? Because we remember we had discussed that we have interpolation functions for geometry and interpolation functions for variable. So, i j are interpolation functions are indices associated with interpolation functions for what? Is it geometry or primary variable? Functions for primary variable important to understand.

And t is the indeed index for 40th Quadrature points. So, this is related to geometric approximation. In this i and j is related to the approximation for primary variables. So, if i and j varies from 1 to 4, then this gives us the value of m, we saw it in last week and this gives us the value of n and if m equals n then what kind of formulation is this? Isoperimetric formulation. If m is less than n then it gives us sub parametric formulation and if m is more than n then it gives us super parameter. So, this is important to understand.

So, while we are doing all this mathematics or are we are developing code we have to make sure that we carefully handle these indices otherwise will get confused and will get bad answers. So, this is very important to understand. So, this is the formula we have to use to calculate K i j. So, I will again make this more explicit. So, we have already written that these depend on you and these depend on Quadrature points and so on and so forth. And we had seen that r is equal to p plus 1 divided by 2. So, when we have to calculate p we have to calculate p for this function, F i j as a functional zeta we do not have to calculate p for d psi i over d x or d psi j over the x or a x, we have to calculate p for this function calculate p for F i j zeta.

Because it is in this function everything else is embedded, a is embedded, j is embedded, derivatives of psi are embedded, if it is a mask matrix then psi and psi j may be embedded. So, this is what we are interested in we have to calculate p for this function then we can correctly calculate the value of r. Now, we will actually calculate K i j for a quadratic function. So, we are assuming psi i to be quadratic, this means psi i e prime is linear. So, that means, psi i e prime times psi i psi j e prime times J, times a which is 1 this entire expression is this is the integrant right for this equation. So, this is equation 1 or I will call it equation a.

So, this for this equation, this entire integrant what is the order of this; if psi i is quadratic then psi i prime is linear this is the entire integrant we have to calculate this is my F i j zeta right. What is the order of this? Order is 2 that mean, p equals 2. Which means the number of Quadrature points should be 2 plus 1 divided by 2 and that gives me 2 so that means, that r should be 2 or higher I can use r as 2, but I cannot use r as 1, I can use r as 2.

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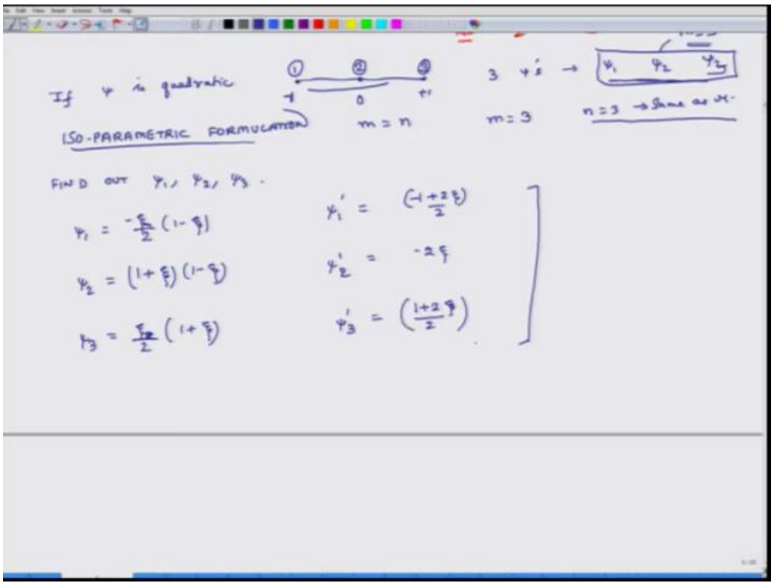

So, now if. So, second thing is if psi i is quadratic, what does that mean? How many points it has in a quadratic elements, it has 3 points right, in a quadratic element this is node 1, node 2, node 3 right. Which means for each element how many approximation functions we will have, will have three approximation function, for a linear element it has two nodes there are two approximation functions, psi one and psi 2, the value of psi 1 is one at node 1 and 0 at node 2 has a value of psi 2 is one at node 2 and 0 at node 1.

Similarly for a quadratic it is element there are 3 nodes, it will have 3 quadratic functions psi 1, psi 2, psi 3 right. So, we will have 3 psi or psi 1 psi 2 and psi 2 psi 3. And the value of psi 1 is going to be 1 at first node and it will 0 at nodes 2 and 3 value of psi 2 will be 0 at 1 in 3 and 1 at node 2 and value psi 3 will be 1 at node 3 and 0 at 1 and 2. So, remember yes. So, if have to do isometric for iso parametric formulation. So there are 3 sizes if I have to do now suppose we want to do iso parametric formulation.

So, in this case m is equal to 3, m is not the order right m is m plus 1 is the order no m minus 1 is the order. So, this means m is equal to 3 and m minus 1 is 2 which is the order of this polynomial expression which is quadratic it is important remember there is the. So, m is 3. So, the order is 2 because its m minus 1 and that is what it is right. So, if I have to do iso parametric formulation, m should be same as n. So, or m is equal to 3. So, n is equal to 3, n is same as r. So, how many Quadrature point we should be use for iso parametric formulation 3 n is equal to 3 which is same as r it is 3. For iso parametric formulation I have to use 3 Quadrature points we have to use 3 Quadrature points. If I use r equals 2, it not will be an iso parametric formulation, but I will get the correct answer.

What is the minimum number of Quadrature point required? So, this should be actually r not r, but r min. So, minimum number of Quadrature points is 2. If I use 3 Quadrature points I will be fine the formulation will be iso parametric and answers will be accurate. If I use r is equal to 2 will I get accurate answers I will get accurate answers, but the formulation will not be called iso parametric. So, that is what we will do. So, the first thing we will do is find out psi 1 psi 2 psi 3.

Now, in the last week we had discussed how we can develop formulas for these 3 psi. So, I am going to write down the expression directly not psi one these as I am sorry find expression for psi 1 psi 2 and psi 3 and these expressions are psi 1 equals minus zeta by 2 times 1 minus zeta psi 2 equals 1 plus zeta 1 minus zeta and psi 3 equals zeta by 2 in to 2 plus zeta. So, if we look at psi 1 if you put in psi 1 if you put zeta as 0 what do you get? No I am sorry if you put.

So, for the element zeta is minus 1 here and plus 1 here and 1 here. So, at the first node zeta is minus 1 if I put minus 1 then the value of psi 1 is 1. So, it is value is 1 at first node and its value is 0 at second node where zeta is 0 and its value is 0 at third node because at third node zeta is 0. So, you can check all these psi 1 psi 2 psi 3 need those requirements. So, now, I will calculate their derivatives with respect to zeta.

So, this is minus 1 plus 2 zeta by 2 this is minus 2 zeta, and this is 1 plus 2 zeta by 2. So, these are the 3 derivatives and now we have to use these derivatives to calculate f i j hat I have to calculate this f i j hat at which points at 3 Quadrature points which is zeta 1 zeta 2 and zeta 3 and those zeta 1 zeta 2 and zeta 3 are not minus 1 0 and 1 they are different values we find those values from this table.

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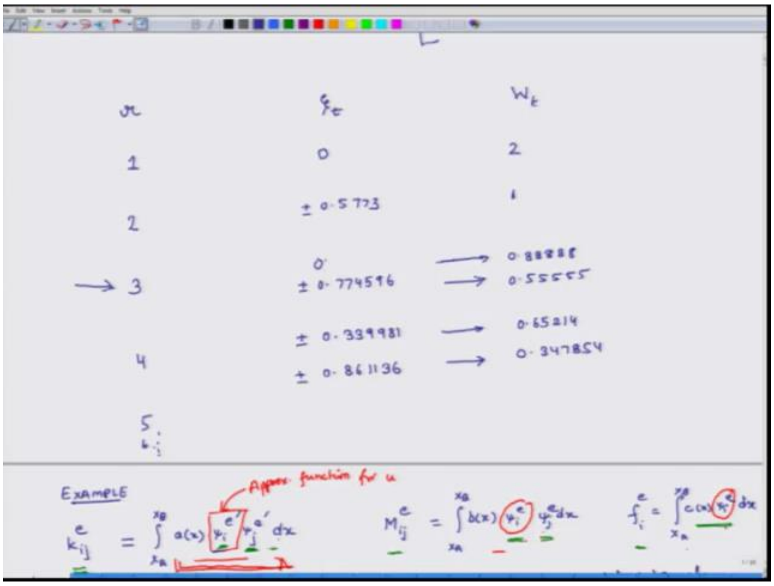

So, in this table if I have to use 3 Quadrature points then the coordinates of those zetas are what they correspond to this line. So, the first Quadrature point will be 0. So, I have to evaluate this function f hat at zeta equals 0 I have to evaluate it at zeta is equal to plus 0.774596 and I have to evaluate at minus 0.774596, these are 3 places where I have to evaluate zetas. The f hat function what we will do in the next classes we will actually calculate all these functions and find out the actual value of K i j. And we will also do it for some other values of r and then hopefully we will see in more precise terms how this numerical integration method is performed.

So, that concludes our discussion for today and we will look forward to seeing you tomorrow and discussing the same topic tomorrow again.

Thank you very much, bye.