Basics of Finite Element Analysis – Part II
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Lecture – 19
Gaussian Quadrature review

Hello. Welcome to Basics of Finite Element Analysis Part II, this is the fourth week of this particular course and in the last week we had discussed the method of numerical integration. And we closed last week by discussing Gaussian Quadrature integration method or Gaussian Quadrature method.

So, what we will do today is that in the first part of this particular week, we will continue the discussion on Gaussian quadrature and we will actually do some examples which will make things little more hopefully clearer as to how this particular process conducted. And once we are done with discussing Gaussian quadrature and numerical integration, as a theme, then we will start discussing about 2 D problems having one variable.

So, that is what we intend to cover over this week and I hope this week like previous ones is a very beneficial exercise and endeavor for all of you.

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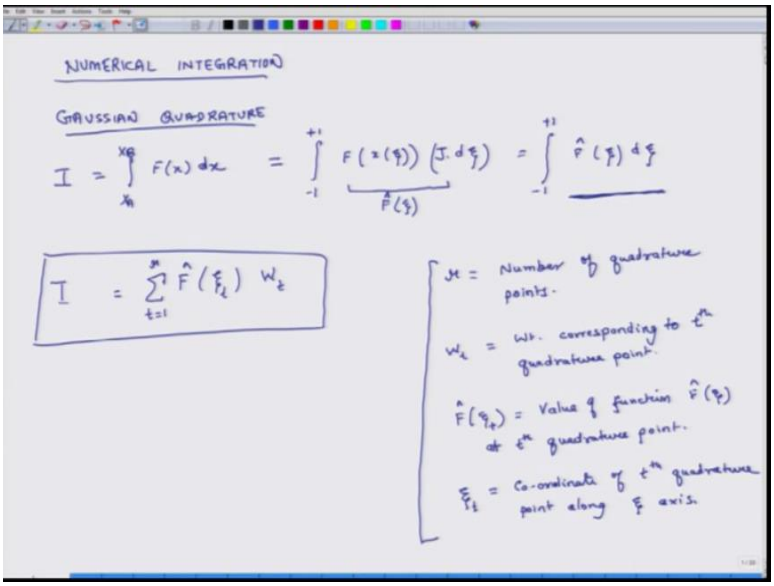

So, as said we will continue discussing numerical integration for starters. And if you remember we had said that for Gaussian quadrature, if I have to compute an integral I within the limits x B to x B and I have to integrate a function F x d x. Then the first thing is that I transform this integral into natural coordinate system. So, my limits become minus 1 to plus 1 then F is a still the same function, but x gets replaced by a function of zeta and then d x gets replaced by J times d zeta.

This is what we had seen and I can label this thing as F hat as a function of zeta. So, this becomes minus 1 to plus 1, F hat zeta d zeta. So, this is. So, now, that I have transform this integral in zeta space, then like other integrations schemes I can express this integral in terms of the values of this the integrand at specific points, times some weights and if I add up all those weights then I get the integral. So, this equals F hat zeta i times W i or actually instead of I now I will use a little bit different subscript. So it is excusing me, zeta t multiplied by a weight Wt and this index is moving from one to r.

Now it is important to understand that here, r is the number of quadrature points. For instance in trapezoidal rule we had seen that the number of quadrature points is two, we know the first point over the domain and the last point on the domain. If I go for Simpsons rule, the number of quadrature points is three. So, in Gaussian quadrature it depends we can choose whatever value of r we want based on how much accurate we want the integral to be. So, r is number of quadrature points. So, I can prescribe the value of r as 1 2 3 4 5 6 7 8 whatever, W t is the weight corresponding to t-th quadrature point, it corresponds to t-th quadrature point. And F hat zeta t equals value of function F hat zeta hat t-th quadrature point. And zeta t is equal to coordinate of t-th quadrature point along zeta axis.

So, these are the definitions. Now to recap we will see what these values r. So, as I said r could be anything it could be 1 2 3 4 whatever.

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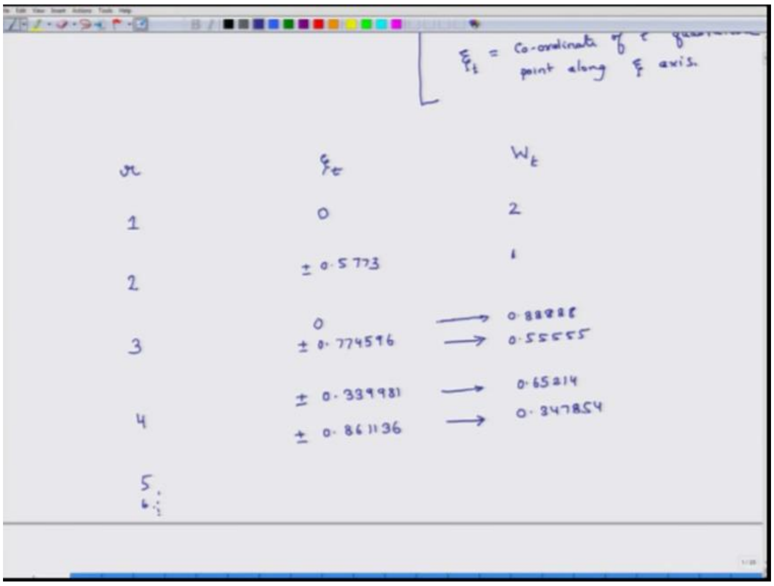

So, what we will do is we will reconstruct this table. So, r for specific values of r what are the values of zeta t and then what are the values of Wt. R as to be it will start from 1. So, when r is 1 then there will be only 1 quadrature point and it is value is going to be 0 and the weight will be 2 when r is 2, then zeta t could have either a value of plus 5 7 7 3 or minus 0.5773 and weight for both these locations is unity 1. If r is 3 then I will have 3 quadrature points, the first quadrature point will be located at zeta equal 0, the second quadrature and third quadrature point will be located at zeta equals 0.774596, and the weights associated with 0. Is 0.88888 and weight associated with the other 2 points is 0.55555, and i can also have r as 4 and the weights of the quadrature points are there are 4 points.

So, 2 points are plus minus 0.339981, and the other 2 quadrature points are plus minus 0.861136, and this is and the weights associated with these plus 2 points is 0.65214, and the weights associated with other 2 quadrature points is plus minus 0.347854 and similarly I can have r as 5 6 and so on and so forth.

So, now what we will do is we will actually calculate do some calculations. So, suppose for example.

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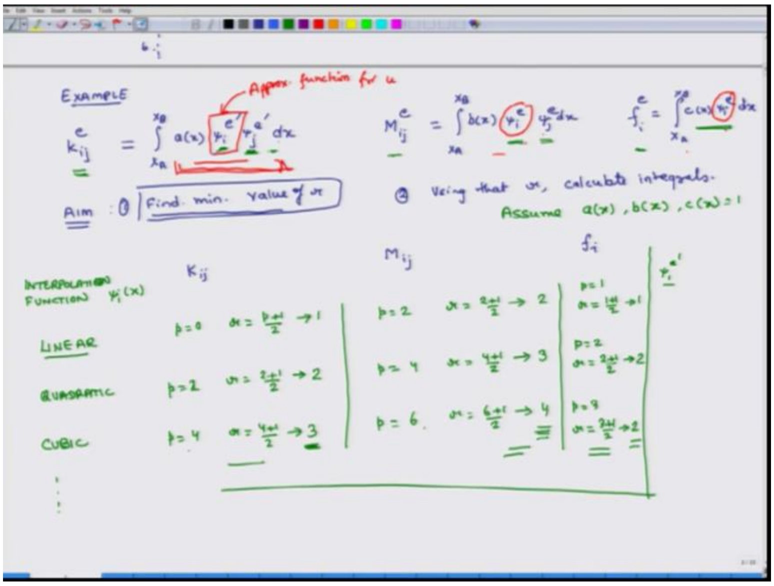

Suppose I am interested in finding the value of k i j right. So, it is a stiffness matrix associated with some differential equation and let say it is definition is integral from X A to X b. Some function a x psi i e psi j e and it is actually differential of this d x. So, what does psi i mean d psi i for d-th element over d x. So, this is what I am interested in finding. So, going to erase this another example cloud be that someone may be interested in finding mass matrix for a differential equation and if the definition of the mass matrix is such that it is equal to x A to x B, b x psi i e psi j d x. And third example could be a force vector f i e equals c x psi i e d x. So, again then i take this x A to x B.

Now, here our aim first is the first step is if i have to evaluate these integrals is what should be the minimum acceptable value of r. So, our aim is to find minimum value of r that is how many quadrature points should be used to calculate these integrals, so minimum value of r that is the goal; so minimum value of hopes of r. So, that is first and then second step will be using that r calculate those integrals. So, I have to compute K i j M i j and F i.

Now, psi i is what it is an approximation function. For whatever let say the variable is u. So, it is an approximation function for u, and this prime denotes that it is it is derivative right similarly here, also you have psi i j similarly you here, also you have psi i right. So, here you have psi i and psi j, but here you have only psi. So, for each of these cases you have to figure out what is the appropriate value of r. So, the appropriate value of r will depend on how complicated this integrand is. So, to figure out how complicated this integrand is we have to identify what is the order of polynomial the integral involves.

So, before we do that we also assume. So, all these are ones, suppose we just assume that a x b x c x everything is one, just to make things simple, but once we know how to calculate r will we can plug in a x and we can again recalculate the value of r. So, if a x b x c x is one, and suppose Interpolation Function. So, it could be linear, it could be quadratic, it could be cubic and so on and so forth. So, if it is linear then psi i e prime would be a constant. If the interpolation function is linear and when i differentiate it this will be constant.

So, will psi j prime also. So psi i times psi j will be a constant what about d x because when we are going to do this integration we are going to do it in natural coordinate system. So, d x gets transformed as j times d zeta and we had discussed in the last class that, if the line element is a straight, if the element is a straight line then the Jacobean or j is constant and it is h over two which is the length of the element. So, we have shown that. So, d x transforms into j times d zeta and the order of that is a still constant right order is a still 0 psi i times psi j is, the order of psi i prime and psi j prime is also zero. So, p is equal to zero in this case right which is how do you calculate p by adding up the order of psi i prime psi j prime and j times d x.

So, each of this is zero. So, p is zero. So, r is equal to p plus one by two. So, this is half, but we cannot have half. So, you go to the next 1. So, this gives you 1 now let us look at quadratic situation. If the psi function is quadratic then psi i prime will be linear. So, order will be 1 right order of psi i prime will be 1 order of psi j prime will be 1 order of j will be 0. So, p will be 1 plus 1 is 2 r is equal to 2 plus 1 by 2 and that equals 1.5, but you cannot use one and half quadrature points.

So, you go to two quadrature points. So what? That means, is that if you have to find this k i j and your interpolation functions are linear quadratic, then the minimum number of quadrature points which you need is 2. If it is cubic then what happens. So, if psi is cubic then psi i prime will be quadratic. So, that is 2 this is also 2 psi j prime and j will be still constant. So, constant means it is order is 0. So, p is equal to 2 plus 2 4 r equals 1 plus 1 by 2 and that gives us 2 point 5. So, i do not use 2.5 I use 3 which is the next integer.

So, this is. So, if i have to integrate to get k i j, then I can use r is equal to 1 for linear functions r is equal to 2 if psi is quadratic r is equal to 3 is psi s cubic and so on and so forth. Now let us look at m. So, of course, here we had assumed that a x is 1 or a constant now a x was a linear function. Then in the first case P will be 0 plus 1. So, then r will be 1 plus 1 divided by 2 it will still remain 1 right. So, accordingly we have to recalculate r it may or mean may not change.

Now, let us look at m. So, in case of m b the order of b is 0 because b is a constant, what is the order of psi i for a linear function 1, what is the order of psi j for a linear function 1. So, p is equal to 2 and r me is equal to 2 plus 1 by 2 and that gives us 2. Let us look at quadratic function for psi, in that case if psi is quadratic, then psi i order is 2 psi j order is 2 and order psi j is 2. So, this becomes 4 and r equals 4 plus 1 by 2, this gives us 3 and similarly if psi is cubic, then p is equal to how much 6, then r is equal to 6 plus 1 by 2 and that gives us 4 m is a 4.

Now, let us look at f. So, when psi I is linear then P equals. So, i am saying that c is I said that c is constant. So, it is order is 0 psi i order is 1 and this nothing else right. So, then here p is equal to 1 and r is equal to 1 plus 1 divided by 2 it gives us one next 1 .If psi is quadratic then p is equal to 2 and r is equal to 2 plus 1 divided by 2. So, it is 1.5. So, i go to the next number it gives me 2 and lastly if p is equal to 3 then r equals what? 3 plus 1 divided by 2 this gives me still 2. So, the point of all this discussion and this is something you are shown in the last class also is that for the same differential equation. If the differential equation gives us different terms involving k m and F, then while doing numerical integration we have to actually do some calculation and figure out how many quadrature points we need and accordingly prescribe the number of those quadrature points.

So, this is important. So, what it means is that if psi is cubic for k i have to use 3 quadrature points, for m i have to use 4 quadrature points, and for f i have to use 2 quadrature points. Now if you are want to be consistent we can go with four quadrature points for k m and f in all the cases because that will not or a lead to loss of accuracy, but if I use three quadrature points I will get correct answer for k I will get correct answers for f, but I will not get correct answer for m. So, this is very important to understand.

So, this is what I wanted to discuss in today's lecture, we will continue this discussion tomorrow and till then have a great day bye.

Thank you.