**Basics of Finite Element Analysis – Part II**
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**Lecture - 18**
**Gauss Quadrature**

Hello. Welcome to Basic of Finite Element Analysis. Today is the last day of this third week of this particular course. And in the last class we had just started discussing numerical integration.

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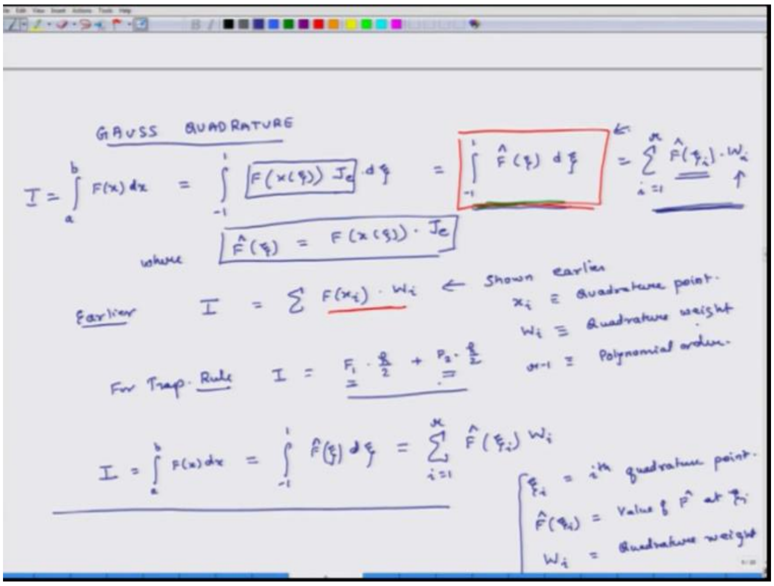


And what we are going to discuss today is the gauss quadrature method of numerical integration so again our aim is that we want to compute this integral, F of x d x. And this I can perform the same integral in the natural coordinate system, by doing 3 such things. Changing the limits to minus 1 to 1 expressing f in terms of zeta so we know that x is a function of zeta so this is what we do and then the third thing we do is that d x gets replaced by the Jacobean for the eth element and times d zeta.

Now, this entire thing could be written as another function f hat. And this depends only on zeta times d zeta. Where F hat zeta equals F of x which is the function of zeta times

Jacobean of eth element. Now earlier we had shown that, so this is an integral. So I can call it some i right. This is an integral earlier we had seen that for trapezoidal way of integrating and Simpson's rule of integrating, I could express i as nothing, but a sum of F of x i times wi's.

This we had shown earlier where x i was what, it was the quadrature point and w i was the quadrature weight. And r minus 1 was the polynomial order. It was the polynomial order. So this integral can also be expressed in same terms. This integral can also be expressed in similar terms in terms of weights and quadrature weights and the functional values of the function at quadrature points. What is F of x y it is the value of that function at quadrature point right.

For instance, for trapezoidal rule, what was the function I was F 1 times h by 2 plus F 2 times h by 2, where F 1 was the value for that function at first node, in the x coordinate system. See this is about natural coordinate system, but the trapezoidal rule works in the x coordinate system. So in the x coordinate system F 1 was the value of that function at first node and F 2 was the value of that function at the second node. And h 2 h is the element link h by 2 is the first quadrature weight, and h 2 is the second quadrature weight.

Similarly, here also, but here the domain is the national coordinate system, but the same taught process can work here also, just that the domain has changed from minus 1 to 1. So I can write it as t is equal to 1 to r, just putting the new index F of zeta i and it should not be F, but it should be F hat, where f hat is this thing, F of x expressed in terms of zeta times Jacobean, times oh I am sorry actually I will keep it I will write i itself times w i.

So, if we know these Gaussian quadrature weights, and if we know the location of these points, see in case of trapezoidal rule the location of the points was the first point and the last point, that is where we had to calculate in case of Simpson's rule the 3 points were first point midpoint and last point right in Gaussian quadrature also I mean using Gaussian quadrature also the same thing is there. We have to find out the value of at some number of points and also the associated weights for those points.

So, if we know this information, then I can very easily calculate the integral of this function, F x d x over the domain. So I will again write it here, i from a to b f x d x can also be written as minus 1 to 1 F zeta d zeta. And this equals F hat which is the function of zeta i then f hat evaluated at location zeta i times quadrature weight. And this summation is going to happen from 1 to r which is happening going to happen from 1 to r. So this r is a third parameter see, n was one parameter it was about the transformation, n was second parameter it was about the variable.

How it varies here r represents how many points we are going to integrate it. This right represents how many points we are going to integrate it. I can integrate it at say 1.2 0.3 0.4 and so on and so forth right. So that is why it is the third index r. So here zeta i is ith quadrature point, F hat zeta i is value of F hat at zeta i and w i is quadrature weight. Now in trapezoidal rule in trapezoidal rule, so there is a very fundamental difference between gauss quadrature and trapezoidal and Simpson’s rule and some other integration schemes.

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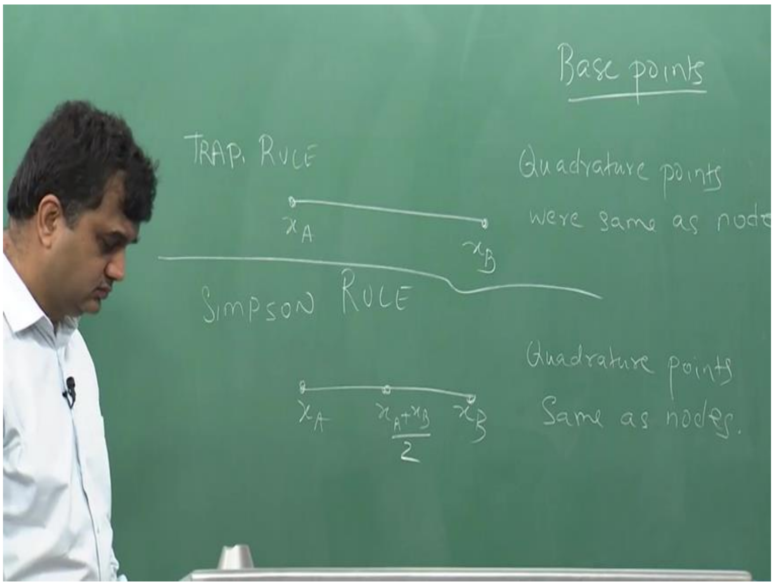


So, in trapezoidal rule, if this was the element, the quadrature points, so this is for trapezoidal rule, were same as nodes right. For a linear element the trapezoidal is rule is where we have 2 nodes. So nodes were x a and x b and that is where we calculated F 1 and F 2. In Simpson's rule x a, x b and this is the mid node. And it is location was x a plus x b by 2. Here also same as nodes in Simpson's rule also, the same thing is true. These quadrature points are known as base points they are also known as base points.

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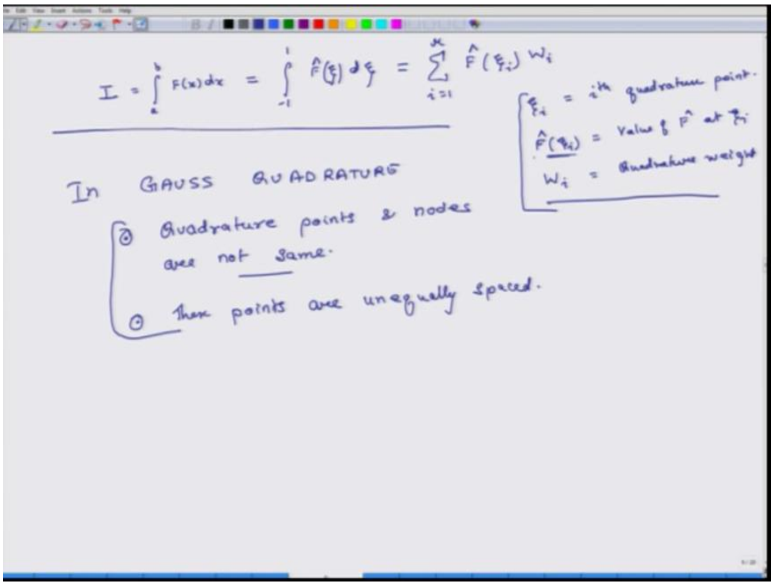


But in gauss quadrature, in gauss quadrature first one is quadrature points, and nodes are not same, we will show that. This is very important. Second is these points are unequally spaced unequally spaced.

We will show that I will give some example; we will show that they are not equally spaced. So quadrature points and nodes are not same it is a very important distinction. So this zeta i value will not be same as that of the node. It will not be same as of node. It is important to understand. So we will write this, so what are these points and what the weights are. So like in this case, we had 2 quadrature points then we had 3 quadrature points we can have also a rule for cubic, we can have another where we will have 4 quadrature points and so and so forth.

So now, what I will do is so basically when you are increasing the number of quadrature points what is changing, r is changing when I am increasing the number of quadrature.

And as r changes the integration becomes more and more accurate becomes more and more accurate. And if r exceeds a particular value then it becomes exact we will see that.

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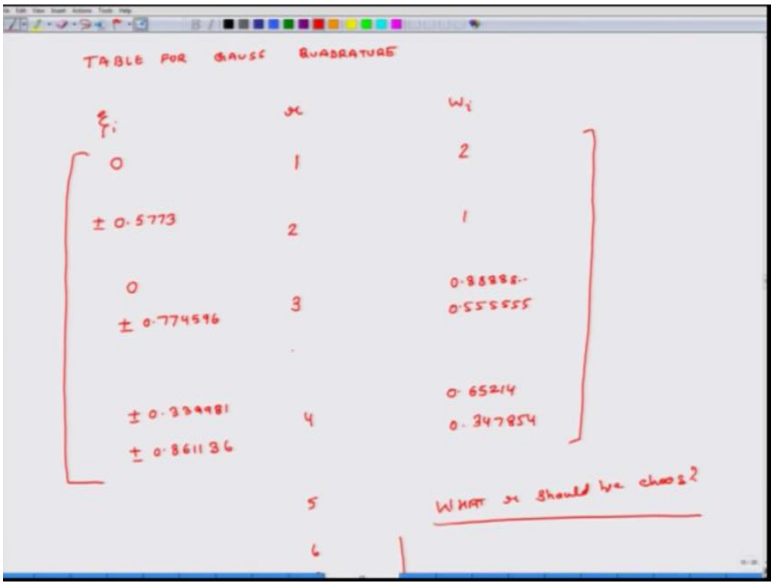


First we will look at this table. Table for gauss quadrature, so zeta values ah varies in the from between what limits, zeta minus 1 to 1, r and these are weights. So if r is equal to 1 if r is equal to so, so first case will be r is equal to 1. If r is equal to 1 it essentially what constant right order of polynomial will be n minus 1 right and r is equal to 1, the location at which you have to calculate this f zeta that location will be zero zeta is equal to 0. So it will be in the center of the element and the weight will be 2.

So, if you are using one quadrature point to do the integration, what will you do? If you are doing one integrate f hat evaluated at 0 times that weight function weight, weight number which is 2 understood. Then you say oh I do not like it, I want to be more accurate. So r could also be made 2. And this table can grow very far. So if r is equal to 2, then you have to evaluate the function at 2 locations, the first location is zeta equals 0.5773 and the second location is zeta equals minus 0.5773. And the weights are 1 for both of them.

So in this case suppose we are using 2 gauss quadrature points to do the integration, then what will you do, you will use this formula again? You will evaluate f hat at how many locations? 2 locations, the first location will be when zeta equals minus 0.5773 and you will multiply you will get some number you will get multiply that number with 1. And then you will add that to f hat evaluated at plus 0.5773 times 1.

So, that is there. Then you can have 3 gauss quadrature points. Here the first gauss quadrature point is 0. And the second gauss quadrature point is plus minus 0.774596. Now how do these common in we are not going to discuss in this thing, but just as we said that Simpson rules, we directly explained that must since we directly wrote down the Simpson rule, but if you find you can go and explore literature to find the mathematical basis for this, but so this is the third. So when r is equal to 3 then you have 3 gauss quadrature points one point is located at 0 and other 2 points are located at plus minus 0.774596.

So, they are not equally spaced. It is not that between the 2 nodes all the gauss quadrature points are equally spaced right. Because here the first node is at minus 1 and the first gauss quadrature point is at minus 0.77. So there is a difference of 0.23. And then the next gauss quadrature point is at zero. So the difference is 0.77. So things are not equally spaced. And the weights are 0.88888. And for these 2 guys it is 0.55555. And so on and so forth. If you if your function is more complicated, then you can use gauss quadrature for r is you know r is equal to 4.

Here we will have 4 gauss quadrature points. These 2 points are 4 are plus minus 0.339931 or 8 1. And plus minus 0.861136. Again they are not equally spaced. And the weights are 0.65214 and 0.34754. And there are you can have 5 6 7 and so on and so forth. But the point is that you can use this table, if you have this table, or you can give this table to your computer program. And you tell it that to the computer program, what is the value of r you want integration for then for? Those values of r you can have implement this formula very easily. Right all you have to do is find out this value of f hat at gauss quadrature points, right f hat at gauss quadrature points multiply that by w i wi is given here. And once you have multiplied it by wi those products you add them up and you will get your integral you will get your integral.

Now, the question is what r should we choose? So the answer for that is, so if I choose r is equal to 100 may be the answer will be accurate, but if I increase the number of gauss quadrature point what happens. I have to do more computation because we are talking about may be lakhs of elements right. So for each element if I go from r is equal to 1 to 2, number of computations double. Here they go 3 times, when I go to r is equal to 3. So I have to make sure that I take use the value of r which is mini mum and it still gives me accurate results.

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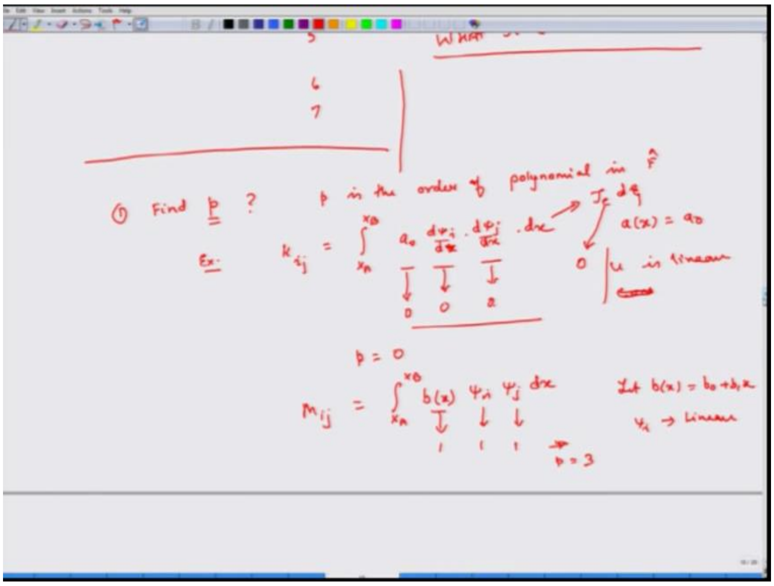

So, the choice of gauss quadrature points, depends so what l what r should we choose. So you choose, so first thing is find p. What is p; p is the order of polynomial in f hat. If F hat it is the order of polynomial. So that you can figure it out. For instance, example, suppose your k i j is equal to x A to x B, a x d psi, i over d psi d x times d psi j over d x, times d x.

Suppose this is your stiffness matrix definition, and let say that a x is constant. So I can replace this a x by a naught and if I am assuming see what are this psi's they represent the variation of u over the domain. You know if I assume that u is linear, then what is the variation of d psi over d, d x, constant you know d psi over d x is constant, so it is because of d psi over d x, my polynomial order is 0. Because of this d psi over d x i, I still have 0, because of a I still have 0.

So, overall polynomial order is constant. So now, then in that case 0 plus 0 plus 0 is and what is d x, d x is j e times d zeta and what is j e it is a for a straight element, it is constant we saw the we wrote that earlier so j e is also constant. So this this is order 0 here so what; that means, is that p in this case, is equal to 0. If p is equal to 0, so this is the first step. Another case it could be let say m i j is equal to x A to x B. Let say b x psi i psi j d x. So in this case let say let b x is equal to b naught plus b 1 x suppose. Then what is this first order psi i.

Let say psi i is linear we choose inter linear interpolation function.

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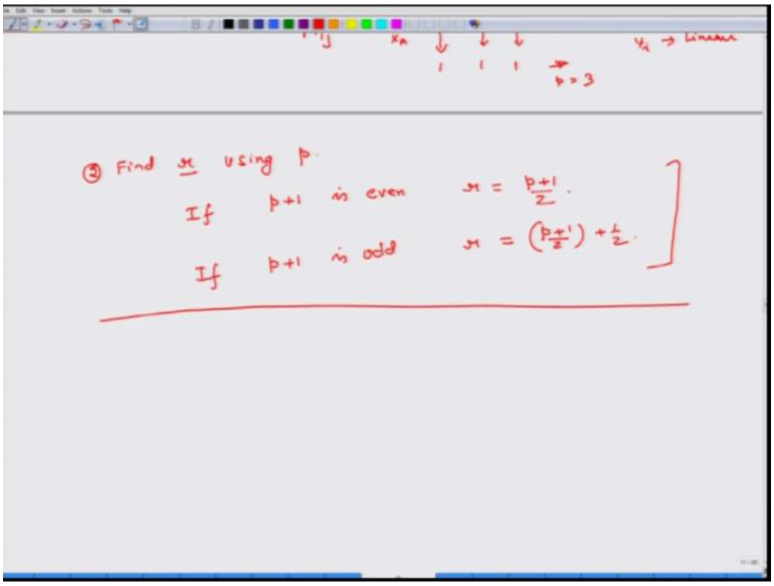


Then this order is 1, this order is 1. So total order keep now it becomes p is equal to p becomes cubic in this case right. So first step is you find out p. Then second find r, using p, if p plus 1 is even then r equals p plus 1 by 2. If p plus 1 is odd then r, is equal to p plus 1 by 2, plus half. Because r cannot be a fraction so you go to the next order. So this is how you figure out r. So I think this closes the discussion for today's week.

We will continue this discussion may be for one or at the most 2 more lectures in the next week, about gauss quadrature. And we will actually do some examples so that all of you become familiar and more comfortable of how to use gauss quadrature for integrating functions. And once we have done then we will move to some other topics in FEA. So thanks a lot for your time and I look forward to see you tomorrow or in the next week.

Thank you very much, bye.