Basics of Finite Element Analysis – Part II
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Lecture - 17
Approximations - Part III

Hello. Welcome to Basics of Finite Element Analysis MOOC course. This today is the fifth day of the current week and our theme for the discussion today. We will continue to be the same which we were doing in the last class that is the notion of Jacobian and how it influences and the role it has in computation of integrals of functions over an element domain.

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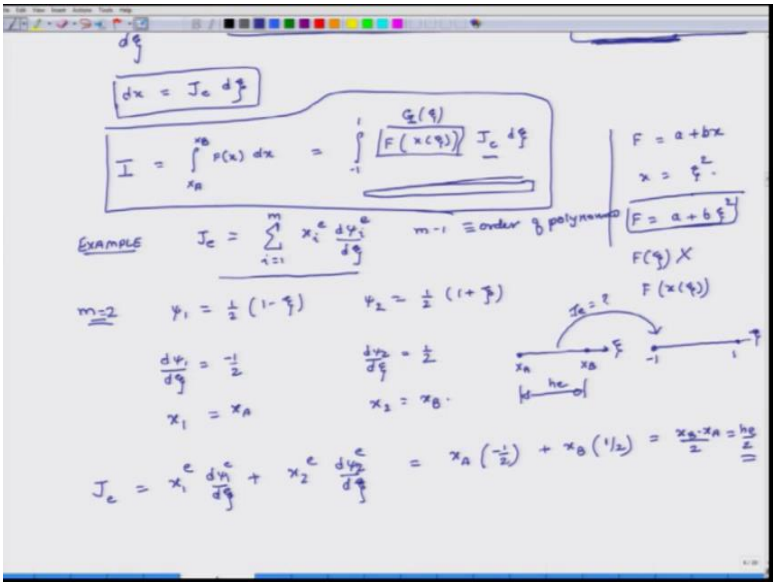

So, we had shown in the last class that the derivative of x at least in 1 dimensional system with respect to zeta is known as Jacobian of the x to z or the zeta transformation and it can be explicitly calculated using this relation. And it is symbolized in this course as J subscript e. So, d x can be written as J e times d zeta and then if I am interested in finding out the integral of F x over the domain x A to x B. Then in the national coordinate system it can be expressed in this form. So, this is the form which we will use when we will actually do the numerical integration using Gaussian quadratural scheme. Now, we still are not there at Gaussian Quadrature, but we are getting close to that end,

but before we discuss any new topic, I will like to explain you this Jacobian by an example.

So, we know that J e equals summation x i. So, these are the constants, x i is a constant times derivatives of psi i for the eth element with respect to zeta and here i ranges from 1 to m and m minus 1 is order of polynomial, that is the order of polynomial. Now we will do an example. Let us say m equals 2, which means that the approximation of geometry is of what type? It will be of linear type. If m is equal to 2, then approximation of geometry will be of linear type. So, the first approximation function psi 1 will be half of 1 minus zeta.

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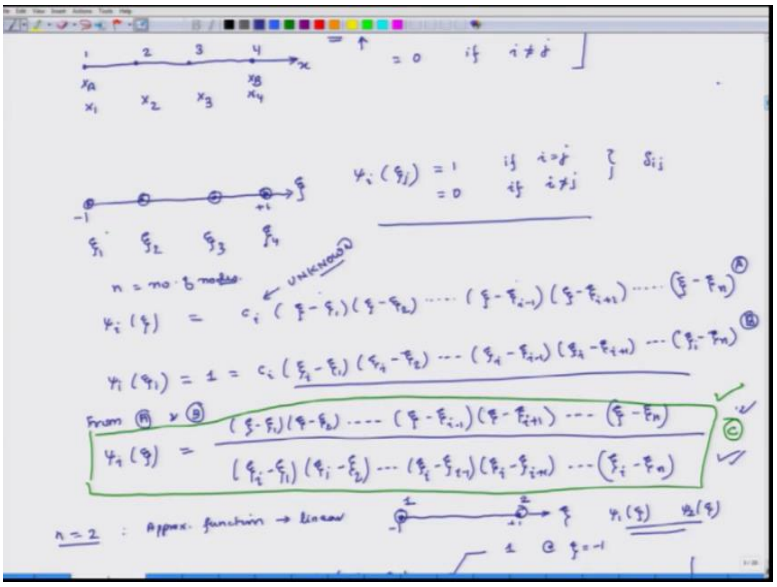

And how did we get this? We got this relation essentially by following or using this equation c, which we had developed, couple of lectures earlier. This is the first approximation function; psi 1 equals half of 1 minus zeta and then psi 2 equals half of 1 plus zeta. So, then d psi 1 by d zeta equals minus 1 by 2 and d psi 2 over d zeta equals one by two. So, for this transformation, a two nodded element, x A to x B is being transformed into a two nodded element in zeta domain minus 1 1 to 1. So, this my zeta coordinates, this my x coordinates and I am transforming it from here to here right.

So, I have to compute the Jacobian of transformation that is our aim. So, our aim is what is Jacobian? So, the relationship for Jacobian is summation of x i these d x i over d zeta. So, and what is x 1. x 1 is x A and x 2 is x B right. So, the Jacobian for eth element is

this expression and if I expand it will x 1 e d psi 1 over d zeta plus x 2 e d psi 2 over d zeta and of course, I have to put superscripts. So, this is equal to x A times minus one by two plus x B times one by two. So, this is equal to x B minus x A over 2 and that is equal to h e over 2. This is h e, element length. So, in this case the Jacobian is nothing but half the element length of the original element in x domain. You can try the same thing for m is equal to 3 also. So, in that case psi 1 e will be half of zeta minus zeta 2 zeta minus zeta 3 psi 2 is equal to minus zeta minus zeta 1 zeta minus zeta 3 and psi 3 is equal to half of zeta minus zeta 1 and zeta minus zeta 2.

Now, you can again compute d psi 1 over d zeta, d psi 2 over d zeta, d psi 3 over d zeta and then what is x 1 in this case? x 1 equals x A x 2. So, x 1 equals x A for m is equal to 3. What does m is equal to 3 look like? I have a three nodded element, node 1, node 2, node 3 and this is x A x B and this mid point is x A plus x B by 2 and this I am transforming to zeta coordinate system. So, it is minus 1 0 1and nodes are 1 2 and 3. So, here I am calculating the Jacobian for this transformation. So, Jacobian is specific to the order. So, you do all this calculations. So, what you will find is that the Jacobian in this case, is equal to x 1 e d psi 1 over d zeta plus x 2 e d. So, actually before I write this relation x 1 is x a x 2 is x A plus x B over 2 and x 3 is x B right. So, now, I using this I write down the relation for Jacobian. Jacobian is J e equals x 1 d psi 1 over d zeta plus x 2 d psi 2 over d zeta plus x 3 d psi 3 over d zeta and these d psi 1 over the we can calculate, we can do differentiation.

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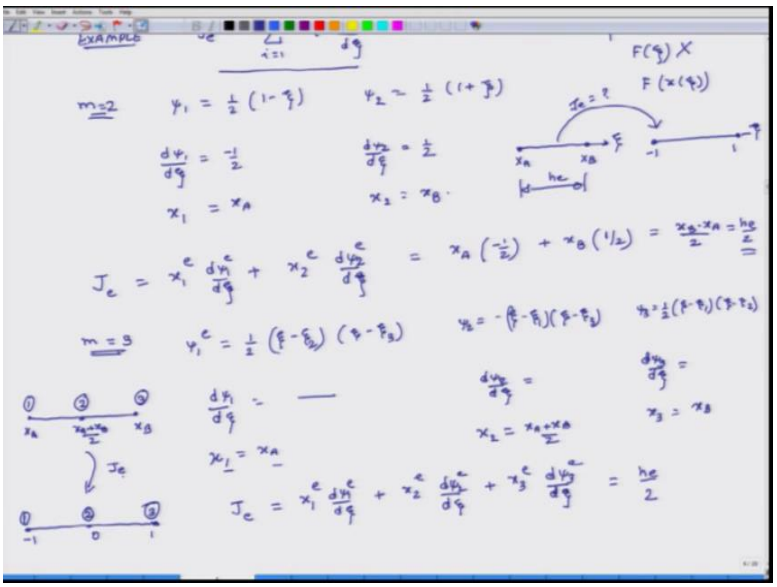

And we can do that. So, if we plug all this in here we will find that even for m is equal to 3, Jacobian is h e over 2.

I will make a very important observation, for all straight line elements J e is equal to h e over 2 regardless of m but if there are third elements involved, then J e is changing it changes from for. It depends on the value of m, but for all the straight elements J e is equals to h e over m h e over 2.

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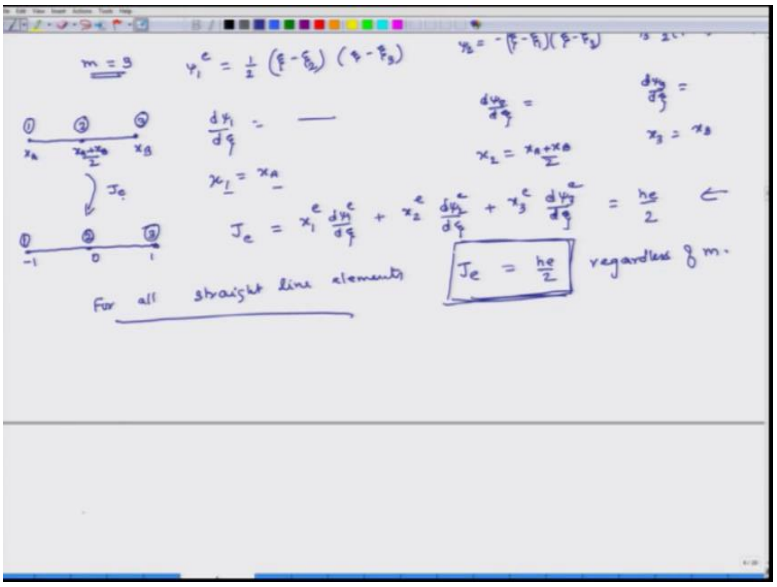

This is important to note and you can verify this statement by doing this calculation and we can do it for m is equal to 4 5 or whatever. You will always end up with Jacobian of this transformation as half of the length of the element. This is another concept and another observation which is work remembering.

The next concept I wanted to talk about is types of formulation.

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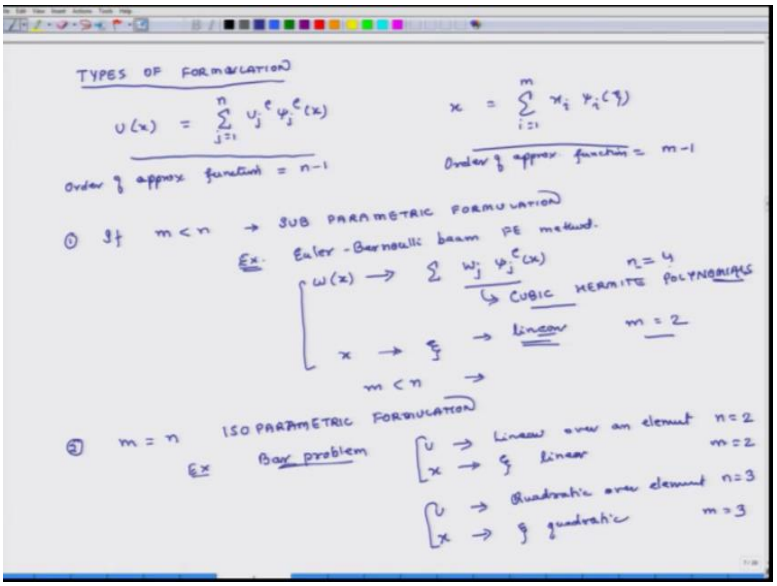

So, we had said that in FE, we do two types of approximation. One is the approximation for the variable. So, we had a for instance, if the variable is u as a function x, then we can express it as j is equal to 1 to n u j e psi j e x. This is the approximation of the unknown variable. There is a bar problem, then u is the unknown variable and u j's are the values of u they are constants and the values of u at the nodes the nodes of the element. And then the other approximation is of geometry. So, x we have said that it is x i psi i which is a function of zeta. i is equal to 1 to m.

So, let us make it a little more clear, here summation is happening till n and here summation is happening till m. So, here, order of approximation function is equal to n minus 1 and for geometric approximation, order of approximation function and these functions are also known as shape functions, for x x to zeta transformation. They are also known as shape functions. The order of approximation function is m minus 1. If here in, if here m is equal to 2, then it will be a linear transformation. If m is equal to 3, it will be a quadratic transformation. So, like this. So, there is one order of a approximation function for unknown variables and then another order for geometric approximation where the transformation from x to zeta is involved. So, these are two in different things and we should not get confused. We should not get confused that is why they are very clearly spelt out here. But there is no rule that they have to be always different. So, we can do FE in three scenarios. So, first one is if m is less than n, if m is less than n, then the FE Formulation is called Sub Parametric Formulation. Sub Parametric Formulation.

Example, earlier in part one, we had developed Euler Bernoulli beam we had developed a method for doing FE for Euler Bernoulli beam. There, the approximation what was the unknown variable here, the deflection of the bean, which is w. w is the unknown variable and it is a function of x. There, w was nothing but w j psi j e x and these were Cubic Hermite Polynomials.

So, w was a cubic function. In w, we had size js which were cubic functions. Please go and check that it was cubic functions. But x to zeta transformation, you can still have linear transformation or quadratic. So, if you do this then, in this case, n is equal to what? 4. n is equal to 4 and here m is equal to 2. So, m is less than n. So, this is Sub Parametric Formulation. Sub Parametric Formulation. So, the x to zeta transformation can still be linear, but the variation of w over x can be a cubic function. It was actually a requirement of that FE Formulation that it had to be a cubic function.

So, it could. So, it is. So, here n was 4 and m was 2 and in such a case m is less than n, so, it is a Sub Parametric Formulation. The second case is m is equal to n. Here this is called ISO Parametric. A lot of FE is of this type – ISO Parametric Formulation. For instance, if I am trying to solve this bar problem, I can say that u can be linear over an element. I can say that u can be linearly varying over an element and in this case, n is equal to 2 and at the same time, I can say that x to zeta transformation is linear.

So, m is equal to 2. So, this is one scenario. Other scenario could be that u is Quadratic over element. I can assume that than n is equal to 3 and I can consciously choose x to zeta transformation as a quadratic transformation. Then nothing will stops me. Then m is equal to 3. So, if this is the situation, then it is again ISO Parametric Formulation and so on and so forth.

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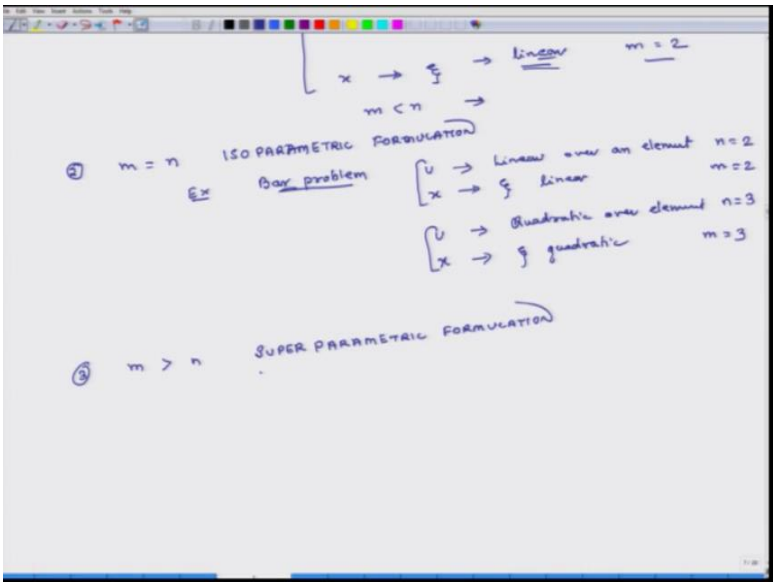

So, this is an ISO Parametric Formulation and then the last one is m is more than n. This is Super Parametric Formulation. This is not used much. This is not used much - Super Parametric Formulation. So, we can have the FE Formulation and the equations developed in different ways. I can have a sub parametric Formulation. Where m, which is the approximation order for, which tells us about the approximation order for the coordinate transformation; m is about coordinate transformation that is less compared to the degree for unknown variables. If both of these parameters are same, then it is ISO Parametric and if m is more than n, then it is Sub Parametric Formulation.

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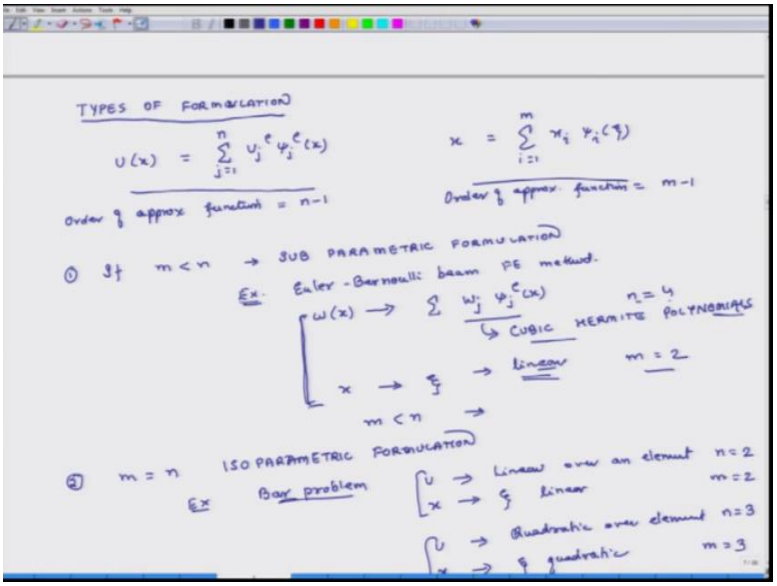

Now, one thing I wanted to mention in this context is that the Sub Parametric Formulation, at least in case of Euler Bernoulli beam, why did we do that? See I can have this, x to zeta transformation as linear because you have a straight element. You can map it to a straight element. No problem. But we know that w it varies in a very complicated way, in beam w, it varies in a complicated in a beam. So, that is why we choose Cubic Hermite Polynomials.

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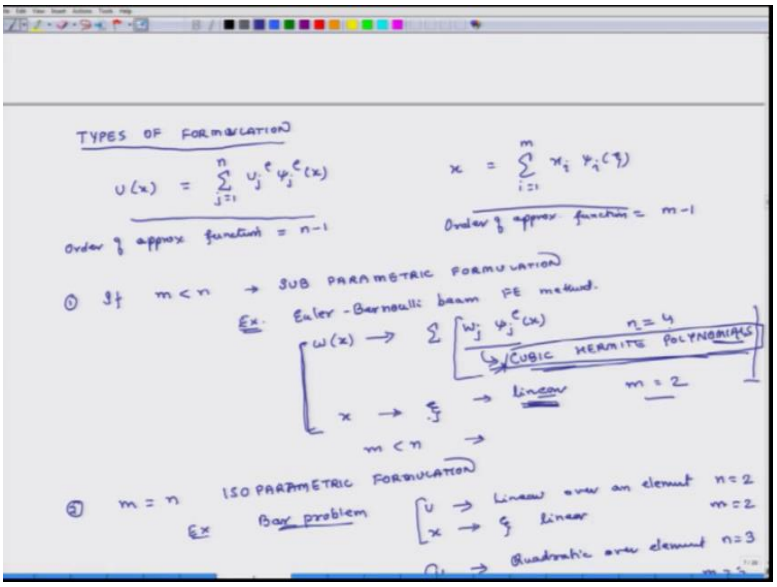

So, that requirement from mathematics side was, that we have to go for cubic functions, for w, because we have to make sure that the each for the being there are two boundary conditions at each hs. One is w and also about slope. And associated with these boundary conditions at each location or each end could be w, is associated with force and w, will be what? And slope will be associated with moment. So, anyway, the variation of, so that is about Sub Parametric Formulation, these are the three different types of Formulation and at this stage we can start discussing about Numerical Integration.

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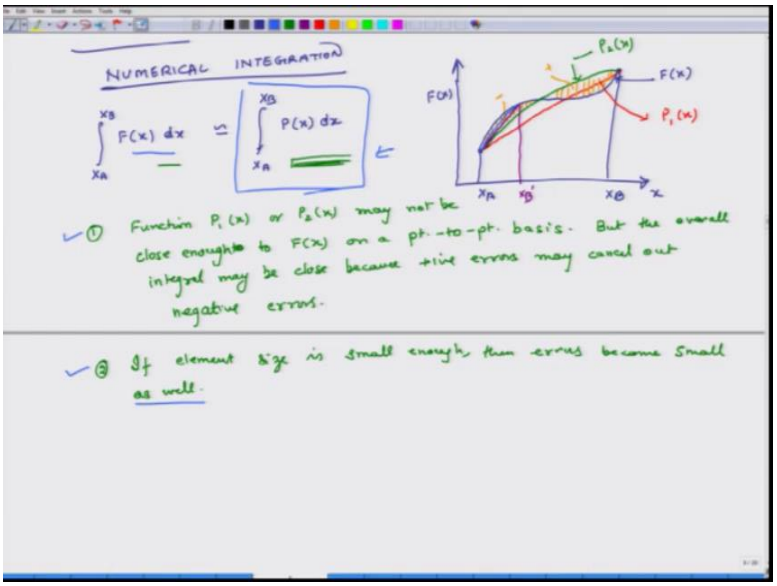

So, now with all this back ground, we will revisit our original problem. Our original problem is that, we are interested in integrating this function F x d x over the limits x A to x B. And we are not interested in integrating exactly because we cannot. We do not know exact nature of Fx. So, we cannot exactly integrate it. So, then we say if we get an approximate integral, we will be happy. So, so we integrate it between the limits x A to x B and because it is approximate.

So, if we choose slightly different function, P x what does this mean? So, this is F x, this is x, this is my F x. I can say that I do not how this x exactly varying, but if these are the limit for the element x A to x B, I can say that I will approximate the variation between a and b by a straight line. So, this is one option for P. P 1 x it is a state variation right linear variation. So, I will get an approximate integral. The exact integral will be under the purple line. Approximate integral will be under the red line. I can make another approximation, like this. This is another approximation, here I am assuming that its varying quadratically.

So, I am in finite element analysis, I am interested in this site of the integral, because I know that I cannot achieve exact integral. So, I am interested in finding the integral either under the red line or the green line and , while I am doing this I realize that, this is important to note that, first the Function P 1 x or P 2 x may not be close enough to F x on a point to point basis. See on a point to point basis, this red line and purple line they may

not be close they are having a same points at couple of places, but at other point there is a significant variation, but the overall integral may be close because positive errors may cancel out negative errors. You understand this. Positive errors may cancel out negative errors. Let us look at this. So, consider this green curve. Here, we have this is negative error and this is positive error. The green curve by itself may not be fairly close to the purple curve, is defiantly not close here at some several locations, but we are not interested in that curve.

We are interested in integral, because that is what we are trying to compute when I am trying to create k metrics or F matrix and things like that. So, I am interested in integral and what is see that even if it is approximate function, this error this is positive error, this is you can call it negative error and this is positive error. They somehow cancel out and the integral may be recently close. So, maybe, this p 2 will work for me or even p 1 will work for me this is 1.

Second is, if element size is small enough then errors become small as well. Errors become small as well right. So, if x B was not this far, but x B was let us say here, if this was the x B then I could have just taken a linear line and this would have been much more accurate or if I had taken a quadratic line it would have been even more accurate. So, two points are, that there may be positives errors and negatives errors and approximation function may still be good and if I make my element size small then this approximation become even better.

So, because of both these reasons, reason one and second reason two, when we are doing numerical integration, we are focused on finding these approximate integrals and we say that if we keep on refining the element size to smaller numbers. This value will still be close to F x d x integral and we will be happy with that approximation. So, this is the overall theme. So, in the next class we will now start discussing Gaussian Quadrature method of numerical integration.

Thank you very much and we will meet tomorrow.