Basics of Finite Element Analysis – Part II
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Lecture -15
Approximations-Part I

Hello. Welcome to Basic of Finite Element Analysis. This is the third week of this course and today is the third day. Yesterday we were developing or we have developed an expression for creating approximation functions of nth order in the national coordinate system.

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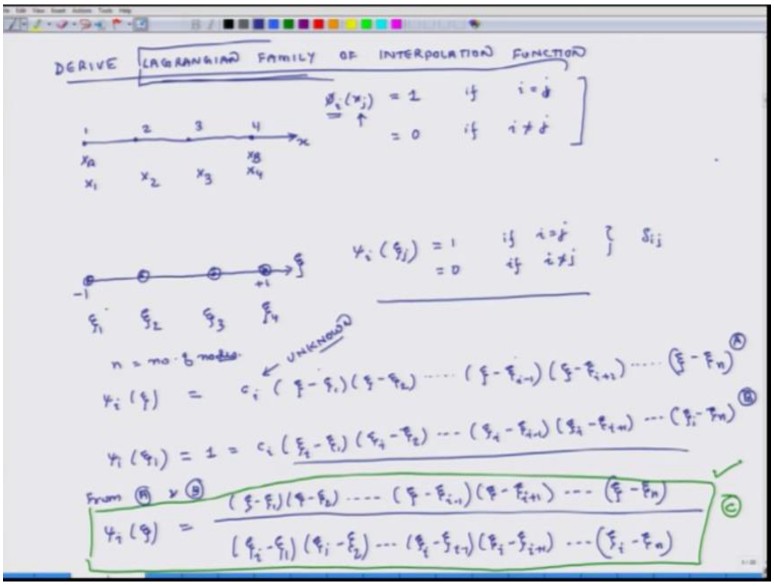

And this is the expression we had generated so the power of this expression is that we can generate very easily, any order approximation functions using this relation and it can be very easily computed. What we will do today in the first part is actually developed some of these approximation functions in natural coordinate system for 2 different values, so you can get little bit practice of how to use these functions in terms of how to use how to use this relation and generate approximation functions.

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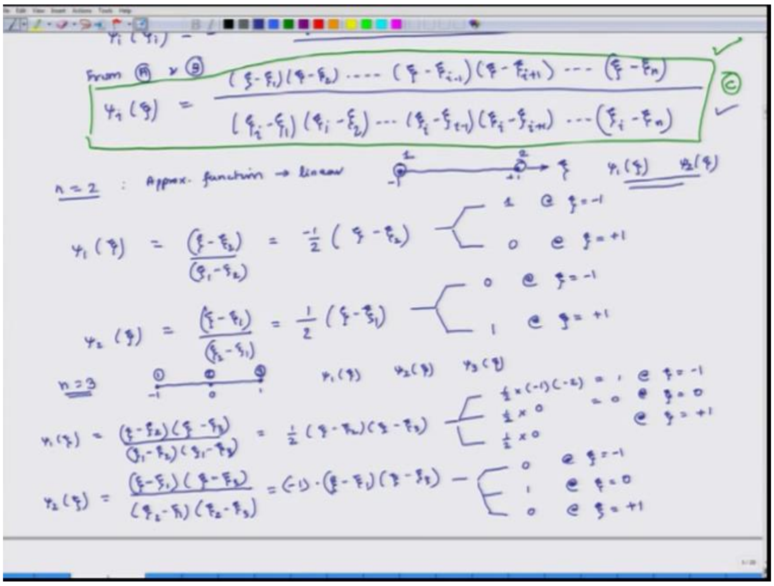

So, what we will do is, the first thing we will do is we will generate approximation functions when n is equal to 2. So this means that the approximate function will be linear. It will be linear in nature. So you have the zeta coordinate system and it is first node is located at minus 1 and the second node is located at plus 1 and it as only 2 nodes; node 1 and node 2. There is no intermediate node. So we have to generate psi 1 as a function of zeta and psi 2 as a function of zeta.

This is what we have to generate. So psi 1 zeta equals, right so what is the value of i in this case, 1. So if it is I is equal to 1 and how many if i is equal to 1, then I will not have psi 1 in to minus psi 1. Right because the value of psi 1 is at first node is going to be 1. So I will not have psi 1 minus psi 1 term. So that will be there. So I will have psi minus psi 2 psi minus psi 2. And will have psi 3, psi 4, and all that, I will not have, right because this index will only go up to 2. And in the denominator again I will not have psi 1 minus psi 1, but I will have psi 1 psi 1 minus psi 2 or zeta I am sorry zeta 1 minus zeta 2 in denominator. I will have zeta 1 minus zeta 2.

So, if I simplify, I get so denominator zeta 1 minus zeta 2 is minus 2. So that is my denominator into zeta minus zeta 2. And let us look at it is value at zeta equals minus 1, at zeta equals minus 1, what is the value of this. You please calculate it will be 1, minus 1, minus zeta 2 is plus 1 so minus 1 minus 1 is equal to minus 2 minus 2 into minus 1 is 2 2 divided by 2 is 1. So this is at it is value is at 1 zeta equals minus 1. And at zeta equals plus 1, it is value 0. Let us look at psi 2, psi 2 as a function of zeta is equal to so from equation c what is the first term will have it. Zeta minus zeta 1 and we will not have zeta minus zeta 2. Because when zeta becomes zeta 2 then this will become 0 so we do not have that term. And then divided by zeta 2 minus zeta 1, and in the denominator so the denominator is 2, and the numerator is zeta minus zeta 1.

Let us look at it is value again. It is value is 0 at zeta equals minus 1. And 1 at zeta equals plus 1. We will do another example then we will move on to this was for n equals 2. Now we will do n equals 3. So if n equals 3 then we have what kind of approximation function will be will we have, it will have we will have quadratic approximation functions. The nodes will be at 1 minus 1 and 0. This is node number 1, this is node number 2 this is node number 3. And we are interested in finding out psi 1 psi 2 and psi 3.

So, let us write an expression for psi 1, psi 1 as a function of zeta equals, so I have this this corresponds to the first node. So I will not have psi 1 minus psi 1 term right. And I will have first term will be psi 1 zeta 1 minus zeta 2. And the second term will be zeta 1 minus zeta 3. Right and in the denominator I will have sorry zeta minus zeta 2. So in the numerator the first term will be zeta minus zeta 2 the second term zeta minus zeta 3. And in the denominator it will be zeta 1 minus zeta 2 and zeta 1 minus zeta 3.

So, the denominator is this, zeta 1 minus zeta 2 is minus 1, and zeta 1 minus zeta 3 is minus 2. Right so this is equal to 1 by 2 into zeta minus zeta 2 zeta minus zeta 3. And let us look at it is values. So at it is value is when zeta equals zeta 1, what does the numerator become, 1 minus 0 into 1 minus excuse me minus 1, so what does it value become it becomes 1 by 2 into minus 1 into minus 2, in this equals 1 at zeta equals minus 1. At zeta equals 0 it becomes 1 by 2, into 0 at zeta equals 0. And this is equal to 1 by 2 into 0 at zeta equals plus 1.

The expression for psi 2 and that equals in the numerator the first term will be zeta minus zeta 1, and the second term will be zeta minus zeta 3, and the denominator it will be zeta 2 minus zeta 1, and zeta 2 minus zeta 3. And this equals minus 1 times zeta minus zeta 1 zeta minus zeta 3. And the value is 0, 1, 0 this is at zeta equals minus 1 at zeta equals 0 at zeta equals plus 1, any questions?

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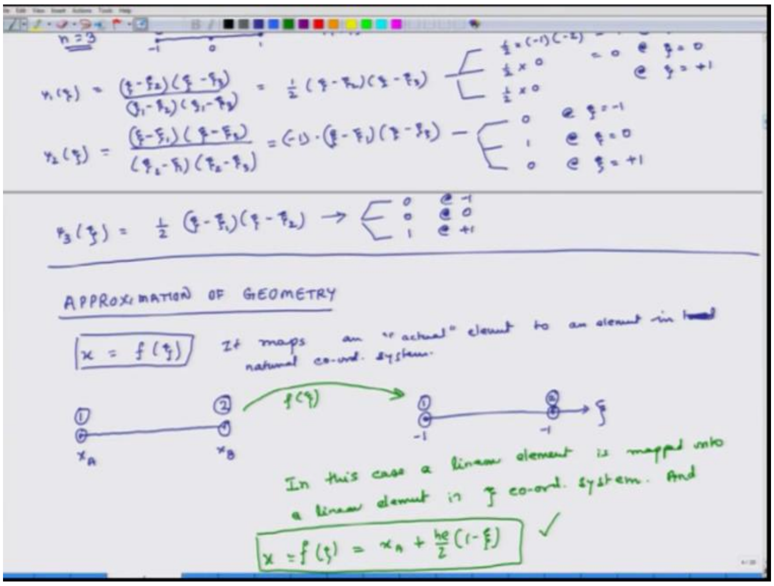

And likewise the third approximation function psi 3 will be I will directly write this it will be half zeta minus zeta 1, zeta minus zeta 2. And this value is 0. 0. 1 at minus 1, at 0 at plus 1.

So, these are the expression for quadratic functions approximation functions. These are the expression for linear approximation functions. So like in a very system way and very easily we can generate approximation functions of any order legend approximation functions. I am sorry Lagrange approximation functions of any order using this equation c. So now, what we will do is we will move on to the next topic and that is all about approximation of geometry.

So what is our final goal? Our final goal is that somehow given fx, we should be able to numerically integrated on the domain of actual element which varies from which as limits x a to x b. We want to integrated it, because when we have to construct stiffness matrix or k matrix k i g or f i the force vector. We have to perform this integration right. And we perform this integration over the length of an element.

Now, we have seen that if I can somehow map these elements into or on to an element in the national coordinate system. Then things become easier and more systematic. So what we have done till so far is we have developed mapping mechanism right this is what we did here.

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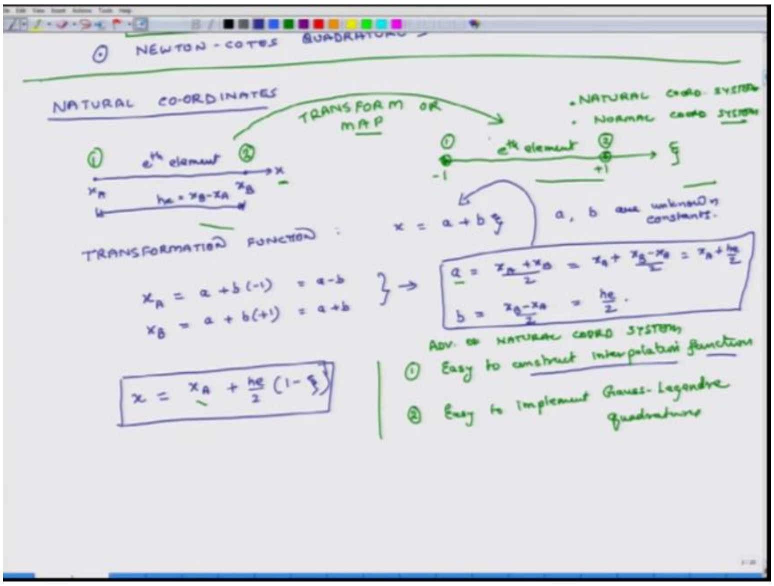

We have developed a transformation mechanism to map the e th element in actual coordinate system to e th element in national coordinate system. This is the first thing we did. The second thing is that if we are in the e in the national coordinate system we should be we should able to easily calculate the approximation functions there. We know because this this is the coordinate system we are going to use, and then because our integrals will involve lot of approximation functions, we should know how to develop those approximation functions in natural coordinate system.

So this is why we developed Lagrangian family of interpolation functions the relation for those functions in natural coordinate system, which is this relation c. So this is first part so the overall goal is to some integrated in national coordinate system. And for that the first requirement is that we should know how to generate Lagrangian polynomials in natural coordinate system.

The second thing is we should know how geometry transforms. How geometry gets approximated. So this is the second piece of the puzzle. So this is the second piece of puzzle. So let us look at a function x is equal to function of zeta. Now what is this function? So this is a lower case f it is not big f when we are integrating we were saying big f that is different than this lower f small f. So what does this relation do? It maps an actual element to an element in local not local national coordinate system. It helps us map and actual element into a local coordinate system.

For instance, if I have 2 nodes first one is located at x a second one is located at x b. This is node 1 this is node 2. And I have to map it to the zeta coordinate system the coordinates are minus 1 to 1. And this is zeta coordinate system right this is node 1 this is node 2. Then the transformation function is f of zeta. And in this case, in this case a linear element is mapped onto a linear element in zeta coordinate system. And f x equals what I am sorry f zeta, is x a plus h e by 2, zeta. So this is the expression we had developed earlier. This is not something new. In probably the first or second class we had developed this mapping right this is the mapping we have developed.

And what did we develop that how you transfer a 2 node element this important to understand 2 node elements to a nodded element in the zeta coordinate system. Is there was a 3 nodded element then this mapping relation would be different. This is important to understand right because then there would be 3 constants a plus beta b time's zeta plus c zeta square and then we will have to do 3 equations and we will have to calculate a b and c.

So, this mapping will be different it is important to understand. So that is why it is written in this case a linear element in x domain or x coordinate system is mapped on to a linear element in zeta coordinate system, and for this kind of transformation where we have 2 elements here 2 nodes here 2 nodes on the other side linear element on both sides. This is the transformation function it is important to understand. The other thing is here it is a straight line element and here also it is straight line element, so a straight line element is being mapped onto a straight line element.

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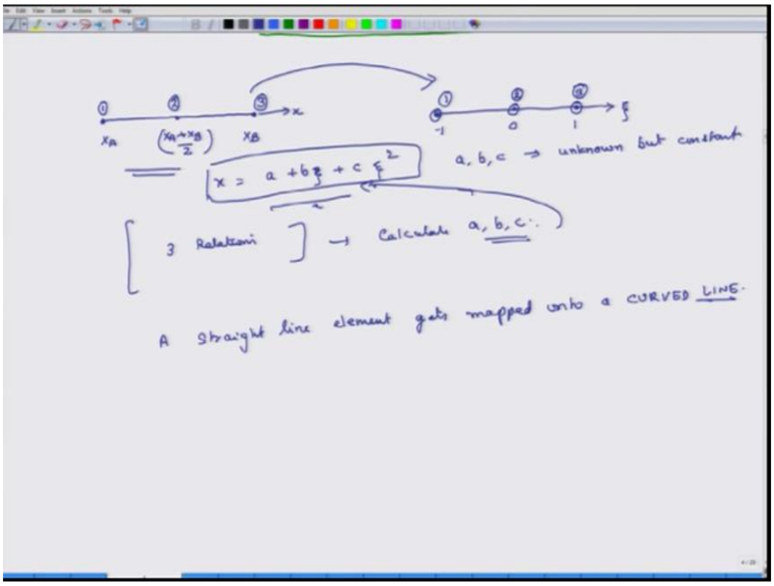

Now, consider this. Suppose I do not do this and I have a different type of element right. So this is node 1 this is node 2 this is node 3. And now I am trying to map it in the zeta coordinate system. So this is my x coordinate, and here the nodes are located at minus 1 0 and 1, this is node 1, node 2, node 3. So then what is happening is, so I can say that x is equal to a plus b zeta plus c zeta square.

So a b and c are unknown, but constant. And how do I calculate these a b and c. Again I see that node 1 has to map to 1 and so on and so forth. So I get 3 relations, between x a, x a plus, I essentially I get 3 relations. And from these 3 relations I calculate a b and c. And then I plug this a b and c back in this equation. I plug all these values of a b c back in this equation right. And that is how I develop the domain the mapping.

Now, when I do this kind of a transformation, then what is happening? See what is happening is that here you have a straight line, so a straight line element gets mapped onto a curved line gets mapped onto a curved line. Or you can do back and forth. Because x is a straight and zeta is 1 a plus b x plus c x zeta square right. So it is like a quadratic. So if you plot that x versus zeta, it would not be a straight line. Right the plot of this thing if I plot this equation, it is a linear transformation it is a straight line right. If on x axis if I plot x and y axis I will plot zeta I will get a straight line.

Here if I make the same plot this is a non-linear transformation, it is a curved line I will get curved line right. So using this kind of approaches I can also approximate curved geometries this important to understand. We will not do more about this, but just wanted to introduce this idea so this is there.

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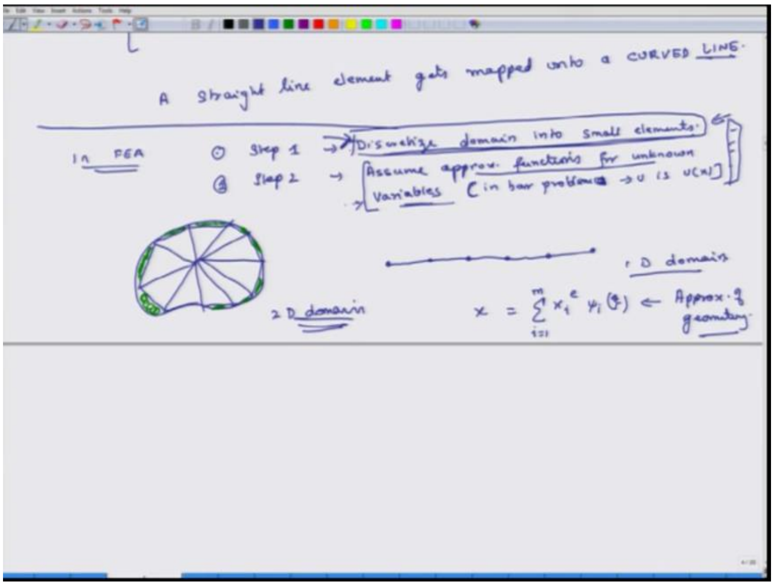

Now, the other thing is, so this is important to understand and remember now the other important thing to notice in FEA when we wanted do a finite element analysis what we will do. The first step we do is that we break the domain into small elements right. In FEA step 1 we had discussed this in our first week, the step 1 is discretize domain into small elements. And then the second step is we assume approximation functions or interpolation functions and this are assuming for what for unknown variables.

For instance, in a bar problem, we assume that u is unknown right in bar problem u is unknown, and we assume that this unknown variable how it changes how it varies over the length of each element right. This is what we assume and when we discretize this domain into small elements. So we do other 2 things to construct assembly element equations, assembly equations, impose the boundary condition and solve the equation all that we do, but these are the first 2 things.

These are the first 2 things, but when we are doing this we are first thing is we are discretizing the domain into small elements, and the second thing is we are for unknown we are so this is an approximation right this is an approximation. Why is it an approximation? If I have this kind of a domain, I can break it up into triangles, right each of these triangles could be an element. When I discretize a mesh right when I discretize a geometry each of these triangles could be an element. And this is the extra stuff which in this mesh I could not capture. This is a extra stuff, so when we discretize, now this problem does not happen, if I have a straight line if this is a domain ,1 d domain, this is 1 d domain this is 2 d domain, right so in 1 d domain if I break this up into smaller elements, this is the problem right.

The finite element discretization of the domain is exact. There is no error introduced because of discretization, but here when I am breaking up into small triangle or rectangles in 2 d domains I may have these green areas which are not captured in the actual thing right.

So, in finite element analysis, we not only developed approximation function for unknown variables, but we also develop approximation functions for geometry we also developed approximation functions for geometry. Right so you get introduction of a error here and also approximation functions when we develop for geometry. You get introduction of error here also right this is what we earlier also so approximation of geometry at least in 1 d domain, how does it look like that x is equal to, x i e times psi i zeta i is equal to 1 to m. This is approximation of geometry.

Now, in straight line it will not create a problem, but in triangles and things like that this approximation geometry creates some generate errors. So this is approximation of geometry. So what we will do is we will continue this discussion about approximation geometry in the next class also. And then we will start introducing things like Jacobean and stuff like that Jacobean. Jacobean transformation and related things, because all this concept will eventually help us perform the ultimate goal is which is to do numerical integration using Gauss, Legendre scheme or Gauss Legendre quadrature method that is what our goal is so.

Thanks, and we will meet tomorrow.