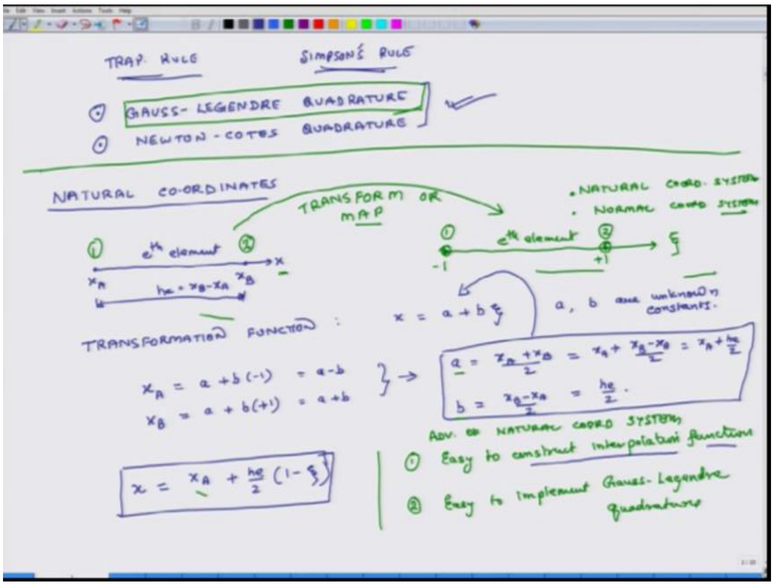
Basics of Finite Element Analysis – Part II
Prof. Nachiketa Tiwari
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Lecture – 14
Numerical Integration Schemes – Part II

Hello. Welcome to Basic of Finite Element Analysis Part II. Yesterday we had introduced the notion of numerical quadrature or numerical integration and we had seen 2 integration methods. One is the trapezoidal rule and the other one was the Simpson rule. Using these methods we could integrate a one dimensional function or a function which depends only on x over a domain which was from x A to x B. We had also introduced some other terms quadrature weights, quadrature points and values of the function f at specific quadrature weights. Using those 3 terms we had developed an expression a general expression for approximate integral, which is f of x i times w I, when you sum it over a series from 1 to r you get the approximate integral.

So, today we are going to extend this discussion further. So, the first thing I wanted to mention is that we had seen that there is one way to integrate.

(Refer Slide Time: 01:34)



That is known as trapezoidal rule. There is another way we can use Simpsons rule, but for when we are doing finite element analysis, we do not in lot of cases use these trapezoidal rule and Simpsons rule for numerical integration, rather we use 2 other schemes the first one is known as Gauss Legendre quadrature.

So, this is another scheme for performing numerical integration and it provides us with significant advantages compare to trapezoidal rule and Simpsons rule. What are those advantages we will see that later. So, this is very frequently used and somewhat less frequently used is another method known as Newton Cotes quadrature. So, these are more popular and among those one which is really popular is this Gauss Legendre quadrature it offers. So, in the remaining part of this week we will be actually developing the thinking which will help us understand how Gauss Legendre quadrature works and all the details associated with this thinking.

So, before we start doing that, I wanted to introduce this notion of natural coordinate system which is extremely important, natural coordinates. So, in the last class we had seen that when we are integrating we are integrating from point x A to x B that is the domain of the element. Now in reality this x A and x B these values can change, not can change they will change from one element to other element. So, it is a little more tedious to generalize this type of a numerical scheme because right, but consider a method where every element has a domain which is ranging from minus 1 to 1. So, that domain is fixed.

How that works out we will see that later, but every element as a domain the first point is minus one and last point is one all the elements. So, in that kind of a system if I have a numerical integration scheme it will be easy because I do not have to change my x A x B and all these things every time. So, that is where this concept of natural coordinate system comes in. So, what we are doing is that if I have an element and it is x coordinates so, this is in the x coordinate system right? It is in the x system the first point is x A and the second point is last point is x B and the length. So, this is my eth element and the length is h e and this is equal to x B minus x A.

Now I instead of using this element for doing numerical integration, I want to use a different element here. So, this is node 1 and this is let say node 2. So, let us right now will just talk about a 2 noted element, but we will extend this discussion for multi nodded elements also. So, here in this node 1 element in this different element this also as 2 nodes, but it is first coordinate is minus 1 and the second node is located at plus 1. So, this was x coordinate. I will call this what you call this zeta. So, we will call it zeta coordinate system zeta.

So, I have an x coordinate system, but I want to use an element which is this long it has 2 nodes again minus 1 to 1. In a different coordinate system called zeta coordinate system. My aim is to somehow transform or map this e th element in the x coordinate system to eth element in the zeta coordinate system such that, the first node which is located at coordinate x A moves to location minus 1 and the x B gets transform to location plus 1 this is my aim. To do this transformation I will say there could be a transformation function and what does this transformation function do? It connects zeta n x together through an equation. Let say that equation is equal to x is equal to a plus b zeta where, a and b are unknown constants they are unknown constants.

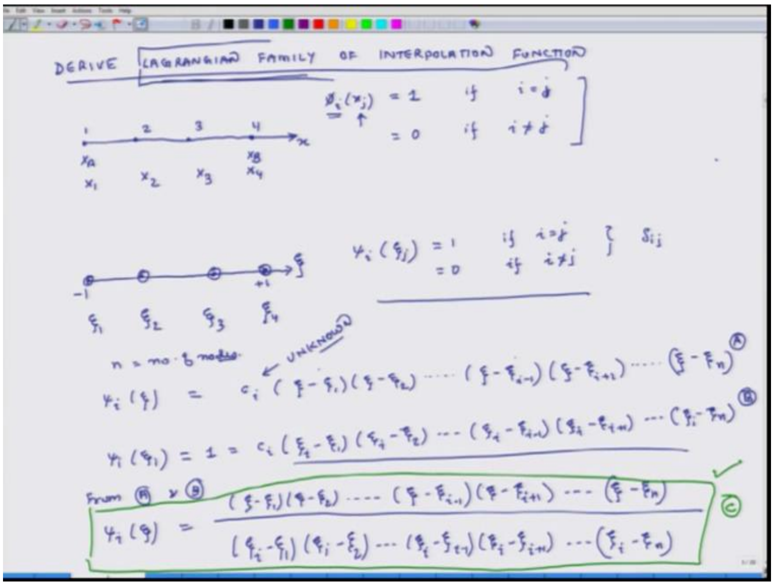
So, our aim is to figure out what is the value of a and b. Now we know 2 conditions about these 2 elements. Corresponding to x A what is the value of zeta corresponding to x A what is the value of zeta minus 1. So, when x is equal to x A value of zeta is minus 1. So, it is a plus b into minus 1 and when x is equal to x B the value of zeta is a plus what is the value of zeta plus 1. So, this is equal to a minus b and this is equal to a plus b. So, from these we can calculate the value of a and b which were unknown till so, far. So, a is equal to x A plus x B by 2 and this is equal to x A plus x B minus x A by 2. I can write it like this and this is equal to x A plus h e by 2.

Similarly, b is equal to x B minus x A by 2 and this is equal to h e by 2. So, what we do is, we plug these values of a and b back into this equation. So, my transformation function becomes x is equal to x A plus h e by 2 into 1 minus zeta. This coordinate system where I have zeta this is called natural coordinate system. It is also called normal coordinate system. So, the important point is that if I go from one element to other the zeta element does not change right, but this mapping function the transformation function changes because x A changes and h e can change. So, any 2 nodded element with whatever values of x A and x B can be transformed or mapped into an element in the zeta coordinate system, having first node at minus 1 second node at plus 1 and the length being 2 because the transformation constants a and b they vary from element to element.

So, how does this help? If I can integrate in the zeta coordinate system, then I do not have to the integration mechanism does not have to change from element to element basis right. So, most or lot of times in an extremely large number of times all the commercial codes they do not integrate in x coordinate system. They actually integrate in zeta coordinate system. How do we perform this integration? We perform it using these Gaussian Legendre quadrature we will understand the details later, but all those additions or numerical integrations they happened in the zeta coordinate system. So, this is very important to understand.

So, other advantages that in the zeta coordinate system it is easy to construct interpolation functions. This we have already discussed and the other thing is that it is easy to implement Guass Legendre quadrature. These are the advantages of natural coordinate system. Now we will continue this discussion further. The next related topic is

(Refer Slide Time: 14:31)



So, we had said that it is easy to construct interpolation functions in natural coordinate system. We will see why it is so. So we will now derive Lagrangian family so, we will develop a method, very easy to follow method. How do develop Lagrangian family of interpolation functions.

Consider an element. It is having x A first node second node is x B. This is in the regular coordinate system and let say that it has several nodes. Let say it as node 1, node 2, node 3, and node 4. So, it has 4 nodes. So, how many approximation functions it will have? If it is a 2 nodded element, it has how many approximation functions two. If it as having 3 nodded function for Lagrangian family, it will have 3 approximation functions. So, this will have 4 approximation functions. What will be the order of polynomial for these elements? It will be third order.

So, these approximation functions let us call them phi j which is functions of x j. What is the property of these functions? The value of this approximation function is 1. So, phi i x j and so, let me make sure that we get this right. This is x 1 this is x 2 this is x 3 this is x 4. So, x A is equal to x 1 then x 2, x 3, x 4. So, this x j can have how many values? It is a discrete the four values x 1, x 2, x 3, x 4. So, and phi will be having 4 values also. There will be one phi associated with node 1, node 2, node 3, and node 4.

So, the value of phi i it varies with x when x is equal to x 1 it will have some value. So, the value of phi i will be 1 if i is equal to j and this will be equal to 0. If i is not equal to j so, what does this mean? Suppose I am developing an approximation function phi 1 then the value of phi 1 at first node will be 1 and it will be 0 at node 2, node 3 and node 4. The value of phi 2 will be 1 at node 2. It will be 0 at node 1 node .3 and node 4 the value of phi 3, it will be 1 at node 3 and it will be 0 at node 1, node 2, and node 4 and similarly value of phi 4 is going to be 1 at node 4 and 0 at all other nodes. So, this is a property of Lagrangian family of interpolation functions. This is an important property actually it say property of all interpolation functions.

So, this is an x coordinate system. Similarly if I have to do work in natural coordinate system, then the length of the element is going to be 2. It will range from minus 1 to plus 1 and it will also have how many nodes? If I have 4 nodes in the x domain then it will have 4 nodes in zeta domain also. So, this is zeta 1, zeta 2, zeta 3, and zeta 4. Here I call these functions as psi just to differentiate that is all. These values of psi depends on psi j know zeta j and this is equal to 1 or zero. It is 1 if I equals j and it is 0 if i is not equal to j, actually, such a function can also be written as (Refer Time: 20:15) delta i j.

So, this is an important property so, when we are developing Lagrangian functions we have to make sure that the values of those functions is 1 at a node associated with that particular function and 0 at all other nodes. This is an important thing. So, in the zeta coordinate system what does that mean? That the ith function psi i, psi i means that it is associated with node. Its value will be 1 at which node ith node. So, this is the actually a function of zeta and it is value is going to be 1 at ith node and at all other nodes it is value is going to be zero.

So this, it will be of what order? Suppose the total number of nodes n is equal to number of nodes. Then the order of polynomial will be n minus 1. So, its value is going to be 0 at all other nodes except ith node. So, if at as to be 0 at all other nodes then it will have to be some constant times zeta minus zeta 1 because, it is going to be 0 at first node, zeta minus zeta 2 it is not going to be 0 only at ith node. So, first node it is 0 second node it is 0 dot dot dot and I keep on going till zeta minus zeta i minus 1 right? I keep on going I do not go till zeta i because if it is zeta minus zeta i then it will be 0 at ith node also. Then I continue further zeta minus zeta i plus 1 dot dot dot zeta minus zeta n. How many terms are these? These are n minus 1 term, because I did not have zeta minus zeta item.

So, this function will be 0 at all other nodes except the ith node because I do not have zeta minus zeta i here and this c i I do not know. C i is unknown right? It is unknown c i is unknown. What is the value of this function at ith node? At ith node it is value is what 1 and its value is 1 at ith node. So, we find the value of c i using this condition, so psi i at zeta i. This is first equation let us call this equation A. What is the value of psi i at zeta i zeta i is what ith node. So, it is at ith nodes psi i is 1 and that equals if I use equation is c i times what do i do in this equation? I replace zeta by zeta i right? Everyone agrees this.

So, from this equation in this equation B there is only 1 unknown which is c i everything else is known. I know the allocation of zeta i, I know the allocation of zeta 1, zeta 2, zeta 3 all these things. So, I can calculate c i from here. So, from A and B the relation for psi i is what? It is looks complicated, but it is pretty simple in the numerator I have zeta minus zeta 1. So, this is in the numerator and in the denominator it will be and I have to multiply this by c i which is basically 1 divided by this entire term.

So, in the denominator I get this. So, that is why function the definition of function zeta psi i in zeta coordinates. This is the definition of the zeta coordinates and I can using this formula I can have any approximation functions of nth order and could be 1 2 3 20 whatever. This is the pretty standard formula and I can easily automate it in the computer. So, lot of FEA theory I mean theory is there, but what we are learning in these lectures is how do we actually automated using computers.

So, that is why this type of definition of Lagrangian functions is easy to develop in computers because it can be very easily computed because, it does not change from element to element right? It does not matter what the value x A is and what the value of x B is because, we are in zeta coordinate system and there the limits are minus 1 to 1 that is all.

So, we will continue this discussion in the next class and till then have a great day bye.