Basics of Finite Element Analysis – Part II
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Lecture – 13
Numerical Integration Schemes - Part I

Hello. Welcome to Basic of Finite Element Analysis Part II. This is third week of this particular MOOC course. And in this week, we will be focusing in detail on something called numerical integration so what we know is that when we are doing finite element analysis for any problem, that is when we are trying to solve any differential equation using finite element analysis, then we get several matrix terms.

And the computation of this matrix terms involves integration of some functions. Now since we use computes to do all this integration, we should have some numerical based approaches to perform those integrations, so those types of integrations which use numerical based approaches are known as numerical integration schemes.

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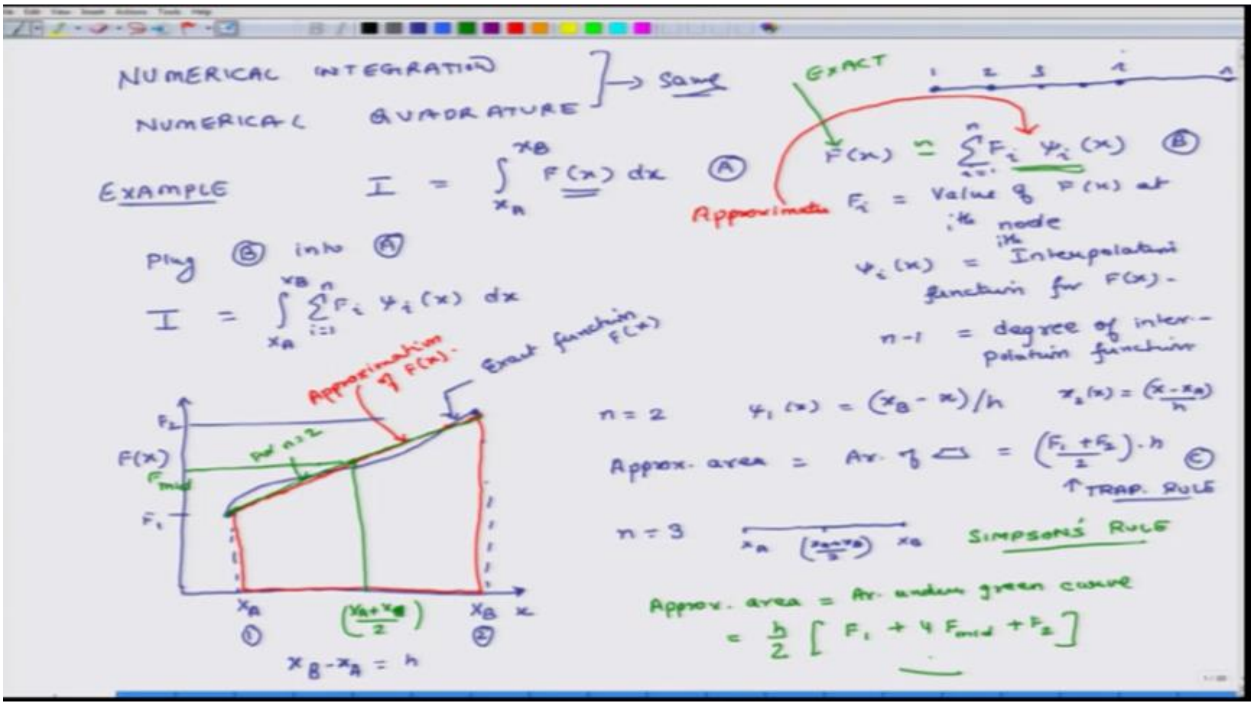

They are also known as, so there known as numerical integration schemes, or they are also known as numerical quadrature, so both of these are same. They mean the same thing numerical quadrature or numerical integration. They essentially imply the same idea, where function so let us say this x, and this is some function F x, and what I am trying to integrate is this function over the domain x A to x B. So analytically if i know f u this function x let us say F x is equal to x square plus 3 x. Then analytically I can integrate it using standard integration methods or if i have to do the numerical integration then there are several ways to perform this numerical integration.

So, let us look at an example. So suppose I say that this is an integral i and that is an integral of some function F of x d x, and I am integrating it in the limits x A to x B. Now when we do finite element analysis, a lot of times this F x lot of times this F x, is represented as F i psi i of x, right and here F i is the value of F x, at ith node. So suppose I have an element and let us say it has a lot of nodes. So this F x is sum of different F i times psi i. So I am doing i equals 1 to let us say n. Then i so this is node 1 node 2 node 3.

And if this is a very long element then maybe this is node n, n not node n, this is node i and this is node n. So value of F x at ith node is F i and psi i of x, is what the interpolation function. For F of x, and this is the ith interpolation function it is ith interpolation function for F x and it tells how this function so all these interpolation functions, when you add them up together, it tells how F of x is varying across the length of the element, that is what it means, and n or lower case n is the degree of interpolation function. So if it is a linear function, then I will have 2 sizes. It will be psi 1 and psi 2. IF i have a quadratic function then there will be 3 psis, psi 1 psi 2 psi 3 and so on and so forth.

Now, please note that this thing is approximation. Excuse me this is, so actually the arrow should not be pointed here this is approximation. And this one is exact, Fx is exact and F i psi i’s we are approximating it. That is why a we call them approximation functions so mathematically speaking if i am strict about it this is not an equal to sin, but rather it is approximately equal to F i times psi i. So this is important to understand, that the right side represents approximate function which may be very close to Fx, but the aim is to be as close as to Fx, that is how that is the basic idea.

So I have to do numerically integration what do I do? I plug, so let us call this equal a, let us call this equation b. I plug b into a this is the first step. So then i is equal to x a to x b, and instead of F x, I am putting it is approximate function and that is F i psi i of x d x. And this is some do where i is equal to one 2 n. So let us take a specific case in this context. So suppose this is x and on the y axis I am plotting F x. And let us say the function is varying like this. So my aim is to so this, is this is the exact function. This is the exact function, which is f of x. And my aim is to integrate this function over the domain right. Now in reality over the element I do not know how the variable is changing. Now this may be some theoretical case, but in reality when I am solving for finite element problem, I do not know how this function actually varies over the domain. So I really do not how this is varying.

So I have to guess I have to make some guesses, and I can and I can guess worst guess I can make is that the function varies linearly over the domain. So this is what this is the approximate, this is the approximation of F x. This is the approximation of F x. So if the exact value of this integral is all the area below the blue curve, and the approximate value of the integral is all the area below the red curve which is a straight line. So the approximate area is all the area below the red line exact areas all the area below the blue line or purple line.

Now, here I have assumed that the variation is linear. So in this case n is equal to 2. And so this means that I have 2 approximation functions. The first approximation function is psi one of x, which is associated with first node this is node 1. And this is node 2 then this is equal to so actually, this is x a corresponds to node 1 and x b corresponds to node 2. So corresponding to node 1 thus approximation function is and we have seen this in this course as well as in part 1 this x b minus x divided by h, and x 2 is equal to x minus x a, divided by h, very unique characteristic of these approximation functions are is that the maximum value they it has they have 3 important characteristics.

The maximum value is 1 right the second important characteristic is that associated with ith node, so if a there is an approximation function associated with ith node, and then it is value will be 1 at i, the node and it will be 0, at all other nodes. So psi 1 is associated with node 1, psi 1 is associated with node 1, and that is why when I put x is equal to x a it is value comes as 1, because x b minus x a, I am defining it as h which is the length of the element.

So, it is value is 1, at node 1, and at node 2. It is value is 0 similarly for x 2. It is value is 1 and the node 2, but 0 at node 1. Now the approximate area is equal to area of this trapezoid right. So this is the area of this trapezoid. This is the trapezoid and that is the approximate area. And that area equals F 1 plus F 2, divided by 2, times h. Right it is f 1 plus f 2 divided by h, where F 1 this is F 1 and this is F2. And earlier we had also defined that F i is the value of F x at ith node. So f 1 means the values of function at node 1 F 2 is value of function at node 2. So this is the trapezoidal rule. This is one example of a numerical integration scheme.

This is one example. So let us call this equation c. I can also have a case where n is equal to 3, in that case the element length will be still same. It is first point will be x a, second point will be x b, and there will be a node at midpoint. And here the coordinate of this will be x a plus x b divided by 2. So let us say so if it is n is equal 2 then it is the what kind of interpolation function it is a quadratic interpolation function. So I am just going to plot some quadratic function. So it is some quadratic function. So this is green curve is for n is equal to 2. It is for n is equal 2. And it is looks very close to a semi straight line, but leys say that it is have a has some curvature.

So, the midpoint is this. And it is x axis coordinates are x a plus x b over 2. And on the y axis the value of function is f, f mid, I will call it f mid. Then if i have to find this area under the green curve, there is another rule known as Simpson’s rule. So if they have we had 2 point points then we use the trapezoidal rule find to find the area under the curve. If the function is quadratic in nature, then there is another formula known as Simpson’s rule and we use that to find the area under the green curve.

And for that the area is, so it is again approximate area. And that approximate area is equal to area under green curve and that is equal to h by 2, into F 1 plus 4 F mid plus F 2. That is the so this comes from Simpson’s rule. This comes from Simpson’s rule. So to find the area under the green curve we have to know the 3 values of the function. The first value of the function which we have to know is F 1 the second fun value of the function we have to know is F mid and the third one is F 2, F mid. So this is important to understand we have to know 3 values of these things.

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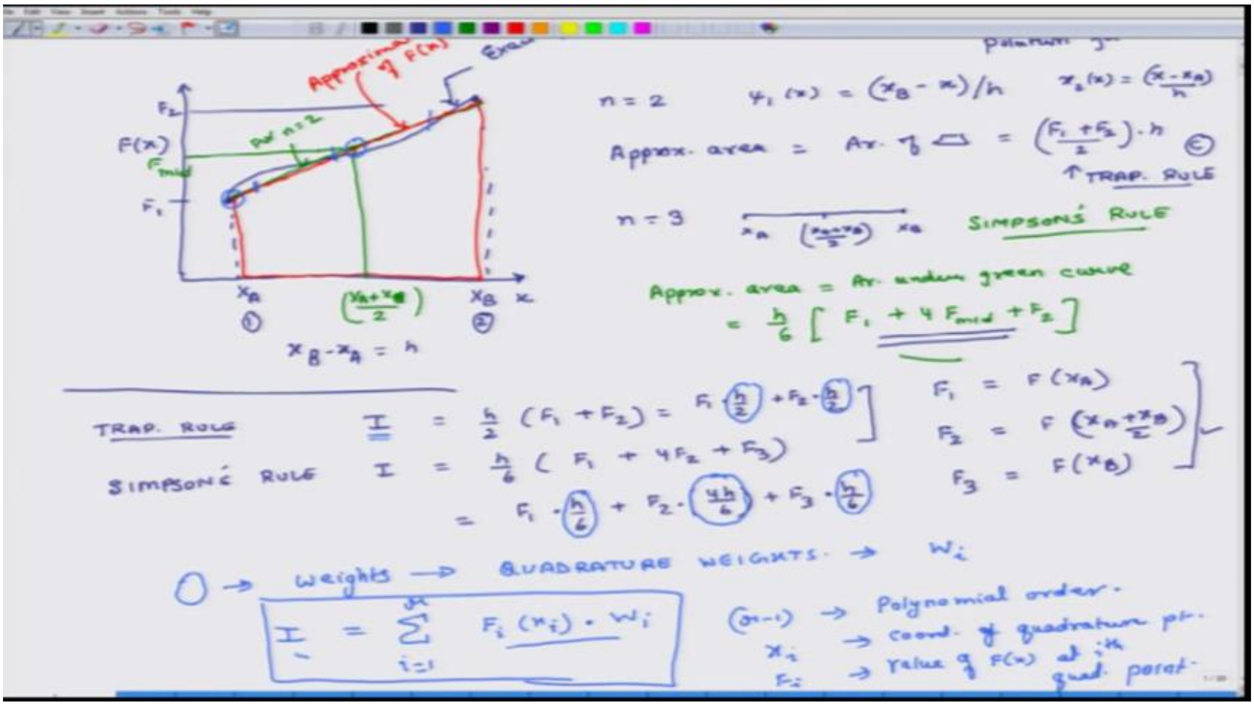

So, what will do is we will recap. So from trapezoidal rule, my approximate integral was h by 2, into f 1 plus f 2. From Simpson's rule the integral was h by 2, into f 1 plus 4 f 2 plus f 3. I have just re numbered some of these in this is so I will make it explicit.

In these 2 equations F 1 is for F of x a, F 2 is equal to F of x a plus x b by 2. So this is the midpoint F 3 is F at x b. These definitions are little bit different then this definition because here F 2 means the last point, but in these things F 2 means the midpoint, that is only difference. I can rewrite this trapezoidal rule. I can rewrite this trapezoidal rule in another form. I can write it has f 1 time h by 2 plus f 2 times h by 2, is the same thing.

Similarly, the Simpson's rules can be written as F 1 time h by 2 plus F 2 times 4 h by actually I am sorry this should not be h by 2, it should be h by 6. So 4 h by 6 and there is 6 here also. Plus, F 3 times h by 6. So it should be 6 here also. So what we are seeing here is that the integral is nothing, but some of the values of the function at specific nodes times constant this constant is h by 2 and h by 2 in trapezoidal rule. And as Simpson's rule this constant is h by 6 associated with F 1 it is 4 h by 6 associated with F 2 and it is again h by 6 associated with F 3. So these and circle constants they are known as weight functions they are known as weight functions or actually they are not functions.

They are weights they are not functions because they are constants so they are known as weights. And because they are used in quadrature schemes, because they are used in quadrature schemes, or numerical integration schemes they are known as quadrature weights. And let us designate them in general as w i actually I will it is w i. So, in general my approximate integral can be written as summation of F i w i, actually I will make little better F i at node i. So F i x i means it is the value of f at the ith node at the ith node. And the coordinate of that ith node is x i times the quadrature weight associated with that node. And this is being done from one to r. So I am in this using another index r. So r is the poly polynomial order. Actually it is not r it is r minus 1, r minus 1 is the polynomial order, x i is the coordinate of quadrature point.

So these nodes in this case in this case these nodes happen to be quadrature points. There is no rule that they have to be same as nodes. We will see later that we will use different points. Maybe we can use these as quadrature points, but in this case for trapezoidal rule and Simpson’s rules the nodes are same as the quadrature points nodes as the quadrature point. So x i is coordinate of quad quadrature point and F i is value of F x at ith quadrature point.

So what you have learnt today is this terminology known as quadrature, point quadrature weights. And what you have seen is that any approximate integral, can if i have a function over a domain then, I can have the integral of that function expressed as or the quadrature of that function expressed as summation of series which is F i times w i. Where F i represents the value of F at ith quadrature point and w i represents the weight associated with that point. And if i add these up then I get my approximate integral not exact integral. So this completes the discussion for today.

Tomorrow we will continue this discussion and extend this thinking further.

Thank you.