Basics of Finite Element Analysis – Part II
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Lecture- 10
Convergence and Accuracy of Solution (Part – II)

Hello again, welcome to Basics of Finite Element Analysis. Today is the 4th Lecture of this week. In the last lecture we have discussed convergence the notion of convergence and we had also mentioned that FA in general, not in variably but especially for conservative problems in general it tends to under predict displacement over predict systemic energy and over predict stiffness. So, today and probably in next lecture we have going to expand on this little bit in detail and that is going to be the theme for today's lecture.

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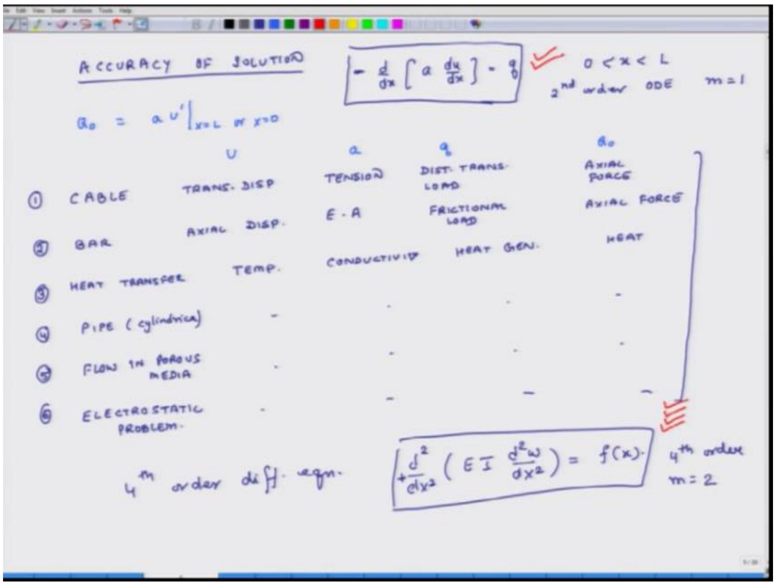

So, this is essentially continuation of last week lectures, which is accuracy of solution. So, before we start discussing this in detail let us look at this equation minus d over d x, a d u over d x is equal to q and this equation is valid for the domain x ranging between 0 and L, I want to make sure that you understand we have discuss this early also that the governing differential equation always valid in the domain. It is not valid on the boundaries, the conditions at the boundary are defined by the boundary conditions.

So, it is important on the standard, now this is the general second order differential equations. So, this is second order ODE and earlier we had said that two I am represents the order of the equation. So, here m equals one to I am represents its order is 2. So, m equals 1. Now the second order equation it represents a very large range of phenomena. So, we like to give you a flavor of different phenomena. So, fair instance we look at cable. So, we have a cable and some waiters hanging on a ten it is try to bend.

So, for a cable you represents transfers displacement, give represents transfer displacement and a represents tension in the cable and q represents distributed transfers load and finally q knot and what is q knot I will defined that here q not equals a times u time at let us say x is equal to L or x is equal to 0. So, or x is equal to 0. So, q not is axial force. So, the second order equation can described the behavior of a cable under tension.

If I consider u as transfer displacement, a as tension q is distributed transfers load and q knot as axial force.; this also represents displacement in a bar under axial tension and here u represents axial displacement and a represents Young's modules, times cross sectional area at position x, q represents frictional load and q knot represents axial force. So, this case 1, this is case 2 case three, will talk about heat transfer, heat transfer and heat transfer in a cylindrical pipe. So, it is one dimensional system cylindrical pipe since like that.

So, here u represents temperature and then, this a represents conductivity, q represents heat generator and q knot represents heat severally the same equation also represents flow of water in a pipe. Flow water in a cylindrical pipe and I can again map out different aspects of the problem here another way is another thing is flow in porous media and then it also the same equation also governs the behavior of a in for some electro static problem.

So, the reason I am talking about all these situations is and I can fill up all these things, but I do not want to spend any more time on this chart, but the reason I wanted to show you this chart is that the, second order differential equation there, m corresponds to value one, it is not just about one single physical problem. It can capture the reality of a very large number of physical problems and in this case the order of differential equation is 2 it is not 1 or 3 it is 2. Similarly, you have a fourth order differential equation. You have a fourth order differential equation an example of this could be d 2 over d x square, EI d2 w over d x square is equal to f x, this is the governing equation for a Euler Bernoulli bean sure on Euler Bernoulli bean.

Here this is the fourth order and m is equal to 2 and make sure that here I have a negative sign and here I have a positive sign. So, I have a second order differential equation an I have a fourth order differential equation so, but I do not have a third order differential equation because, I am not a wear of a physical process or system which can be represent by a third order differential equation.

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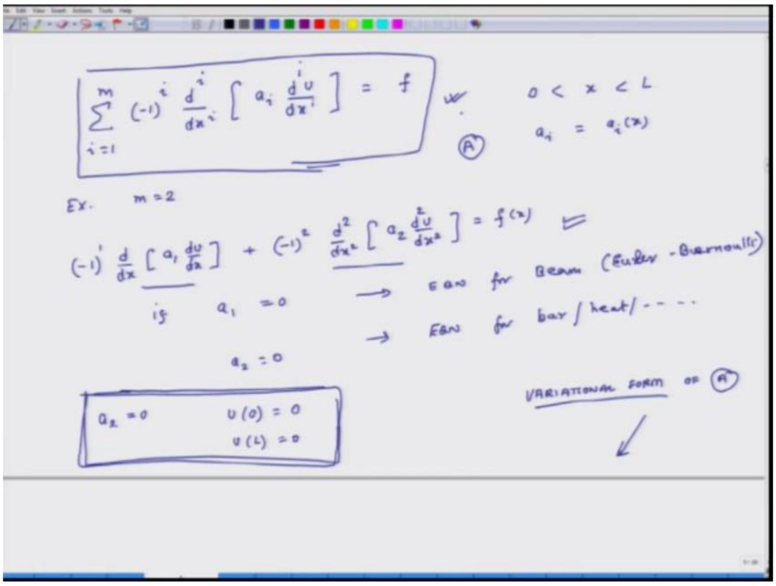

So, in general such types of equations can be represented in this form, i equal 1 to m minus 1 to the power of i d i over d x I, a I, d i u over d x i equals f . So, this a generic form of differential equation which captures both fourth-order and second-order right captures both fourth order and second order. So, here x varies between 0 and L right and a i is actually a function of x, a i is a function of x and u is the exact solution u is the exact solution when an f e solution then I will call it u h; now will do an example. Example let us say that m is equal to 2.

So, if m is equal to 2 then, this whole equation will have left hand side will have how many parts it will have 2 parts right. So, when m is equal to 2 then, this m becomes ya m is equal to 2 then, my first term is minus 1 to the power of 1 d over d x, a 1, d u over d x this is the first term right plus minus 1 to the power of 2, d 2 over d x square, a 2, d 2 u over d x square is equal to f, f can also be a function of x. Now if a1 equals 0 then I get a equation for bean Euler Bernoulli bean. If a 2 is equal to 0 then I get equation for bar or heat conduction, so many other problems which we discussed in this table. If a 1 is 0 then this term goes away if a 2 is 0 then, these terms goes away and also note that I have minus 1 to the power of 1, so sign is also taken care of in this format.

So, to discuss this accuracy and this convergence process, what we will do here is we will discuss this problem this feature of finite element detail, that why does it converse from bottom or top and so on and so forth. So, in our subsequent discussion we will assume that a 2 equals 0. So, will try to understand this concept of convergence in context of the bar problem and we will also assume that u is 0 at x is equal to 0 and u at x is equal to L is also 0.

So if that is the case, for these boundary conditions and for this condition the overall governing equation. So, first what we will do is initially we will be more general and later on we will implement these conditions. So, we will start from this equation or this can be abbreviated as this equation, will develop this equation further a later we will set m as one n and so on and so forth. So, we will start from this equation. So, this is the overall governing equation for a large number of physical processes and for this equation we will develop a form called a Variational Form.

So, let us call this equation a. So, we will develop variational form of equation a, that will discuss now. So, how do we develop it is variational form in f a one we have discuss this, but to recap what we do is we will multiply this equation a with a variation right with a variation small part of variation and let us call that variation v we will multiply this by v and we will integrated toward the whole domain and we will not only do that we will also see here the differentiability is on u is either or second order or fourth order.

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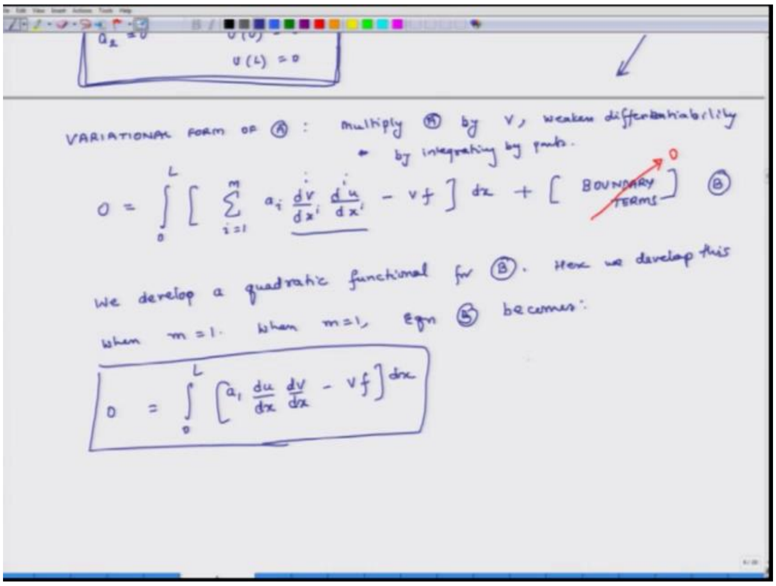

So, we will we can the differentiability and shift some differentiability to v. So, that is what we will do. So, the variational form of A, how do we get it? Multiply A by variation V then weaken differentiability and then we will also get some boundary terms right. So, what we get finally is 0 equals we can differentiability by integrating by parts. So, this is what we get. So, once we multiplied by v and integrated by parts this is a v can form and once we are integrating it phi parts. We will also get some boundary, will gets some boundary terms also and we assume that the nature of the problem is such that all these boundary terms are 0 you can assume that.

So, for instance if I have bar, I can assume that displacement at one end displacement other end is 0 or at one displacement is 0 and or the other end force is 0. So, it is a free end. So, in the boundary parts we have displacement times force term. So, if one of them 0 then 0, boundary term is 0. So, this is the variational form. So, this numbers this equation as equation B. Now we develop a quadratic functional, we develop a quadratic functional for b, now to understand what is quadratic functional I will suggest you go back and look at f a one lectures. But what we will do is in these lectures also we will explain a little bit of how we develop this quadratic functional.

So, for we develop quadratic functional, so here we develop this when m is equal to 1, if we solve the same procedure we can also develop it for 2, but here we are going to develop it when m is equal to one when m is equal to 1 this is what we are going to develop it. So, when m is equal to one equation b becomes 0 a 1, d u over d x, d v over d x minus v f, d x right.

So, let us call this c. Now I will first write and expression and then we will see that that expression is indeed related to this equation c.

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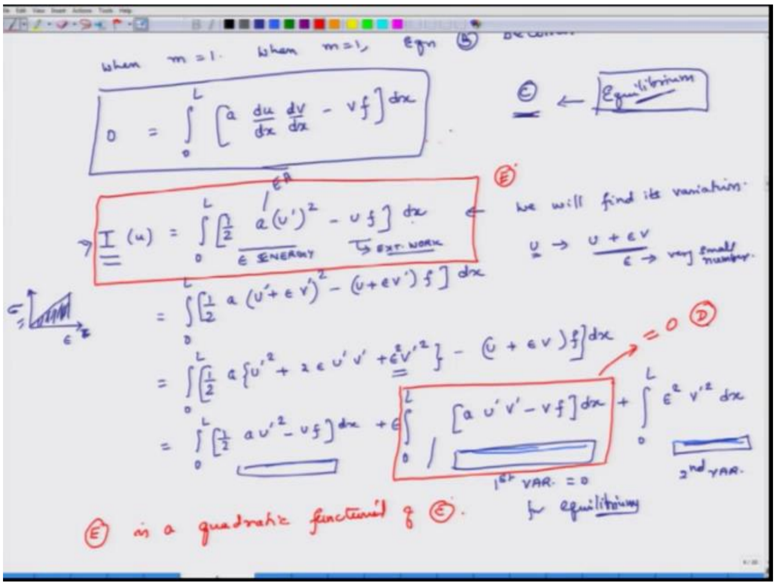

So, I will write this expression and this is a quadratic functional I it is the functional of u is nothing but 0 to L half, oh sake of simplicity I will remove this substitute 1. So, a u prime whole square minus u f d x, we will see that this I this is a quadratic functional, this I is related to equation we will see the relationship relational we will develop that relationship.

So, what we will do is for this relation. We will find it is variation and how do you find the variation? You replace u by u plus epsilon v, where epsilon is a very small number it could be 10 to the power of minus shift e extremely small number as small as you what you can think. So, very small number, once why replace u by epsilon v what does this mean what I am doing is that this is expression and I am replacing u by u plus some very small disturbance an you very small disturbance an you. So, then I will see how this I gets disturbed. I am finding out the disturbance, when you gets disturbed by an extremely small amount which is of v. So, this is equal to 0 to L half a u plus epsilon v and this is epsilon u prime epsilon is a constraint. So, it is derivative does not is 0. So, I have u plus epsilon v time minus u plus epsilon v prime.

So, now I expanded 0 to L half, oh there is a square time here right. So, have u prime square plus 2 epsilon u prime v prime, plus v prime square minus u plus epsilon v prime f d x. So, this is actually not u plus epsilon v prime, but u plus epsilon v. So, now, I reorganized all this stuff and I take first, I take terms half a u prime square minus u f d x this is one portion, plus 0 to L a epsilon u prime v prime excuse me I have to write again.

So, epsilon times a u prime v prime minus v f d x plus a third term 0 to L epsilon square, oh there is one thing this is epsilon square is also going to be here right epsilon square. So, epsilon square v prime square d x. So, what I have organized this entire expression broken this entire expression into part 1, part 2 part 3, in the first part there is no epsilon, in the second part, there is epsilon with the pair of one in the third part there is epsilon with the power of 2. Now we will see what all these means.

First thing we remember is that equation c, how did we get this equation c this equation c came from the differential equation and then we multiplied and then found it is variation. So, this represents the right. So, this represents equilibrium because the ordinary differential equation was the differential equation for equilibrium and equation c has directly come from that equilibrium situation right. So, this represents equilibrium if it is bar then it is equilibrium of the bar right now. Now we look at this term, this term is what if there is bar under tension then a is equal to e times a; a is equal to e times a, u prime is strain energy. So, strain times u prime is stress so, it is stress times strains and when you have stress strain graph right. So, it is stress times area. So, it is force. So, it is x no excuse me epsilon and stress.

So, the area under the curve is strain energy. So, e see a is equal to E times A, a times strain is stress is e times strain is stress this thing. So, what I am multiplying is stress times strain and half that is the strain energy density right and when I multiplied by area times d x it is the strain energy because, I am multiplying by it volume. So, this represents strain energy in the system, this represents strain energy this represents strain energy and what does u f represent u is displacement and f is force.

So, this is external work right. So, this is the total work done in the system right strain energy internal work and this is the external work and are principle of minimum potential energy says that, whenever the system will be an equilibrium when the potential energy of the system is that a minima right. So, this I represents the overall energy potential energy of the system and what I am doing is, I am finding it is variation when I replace u by u plus small variation then this is the basic potential energy this is the first variation because it is multiplied by epsilon and this is the second variation in the energy.

So, this is first variation, this is the second variation and wherever function or a functional has to be at minima what is the condition there that it is variation or it is first derivative for a function to be at a minima the condition is that, it is derivative has to be 0. Derivative has to be 0, first derivative has to be 0. Similarly for a functional, this is the functional I is the functional, for functional to be at a minima it is first variation has to be 0. So, this first variation has to be 0 for equilibrium right. For equilibrium this first variation has to be minimum when you look at this what are the terms here. So, since epsilon is constant I can since epsilon is constant I can take this epsilon out, I can take this epsilon out because, epsilon is a constant it is a small number and when, first variation is 0 which means that this entire thing should be equal to 0 this equation is same as this equation. So, let us call this equation d.

So, this equation is same as this equation. So, you see that this I is a functional it is a quadratic functional because, this I am having u prime square and it is first variation when it is equated to 0 I get equation c. so, let us call this equation E. So, I can say that E is a quadratic functional of c e is a quadratic functional of c and in reality at least in constant of the bar problem it represents the energy on the system. So, I can write that E is a quadratic functional of c.

So, I will close this discussion here and we will continue this discussion in the next class and then, we will show have all this understanding and lead to a conclusion that energy convergence in FA solution happens from above.

Thank you.