Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)
Course Title
Basics of Finite Element Analysis

Lecture – 09
Integration by Parts – Review

by
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Hello, we will continue the discussion which we started this week. And in todays and possibly tomorrows lecture we will continue to review some of the mathematical concepts, which are highly relevant to finite element method. So what we are going to do is, today review integration by parts and then try to explain, how this particular procedure is very helpful in context to finite element method.

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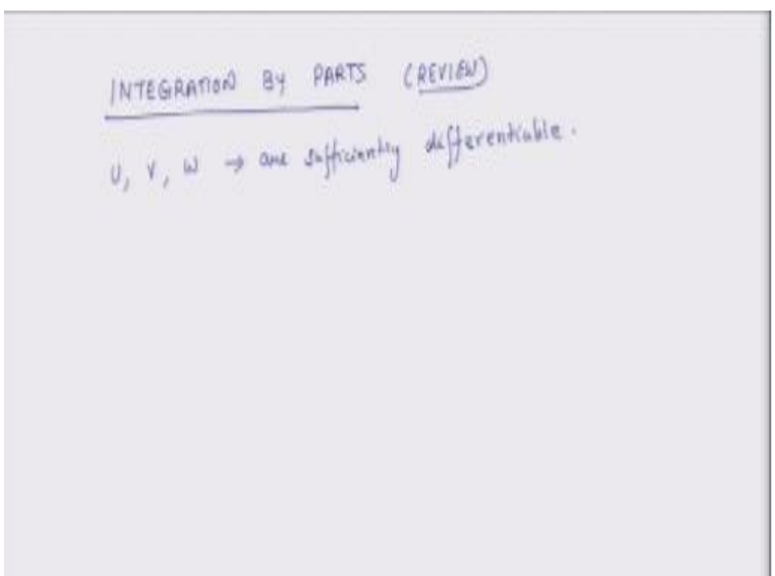

So we will discuss integration by parts. So this is the part concert review. So let us assume that u, v and w. So right now we are just talking about a one dimensional domain and they are, sufficiently differentiable. What is sufficiently differentiable? That if I need, a second order derivative it should exist. That is sufficient and it may not third order derivative. If my need is only up to second order then, an x^2 type of function is sufficiently differentiable. But if I need second order and if my function is just x then it is not sufficiently differentiable.

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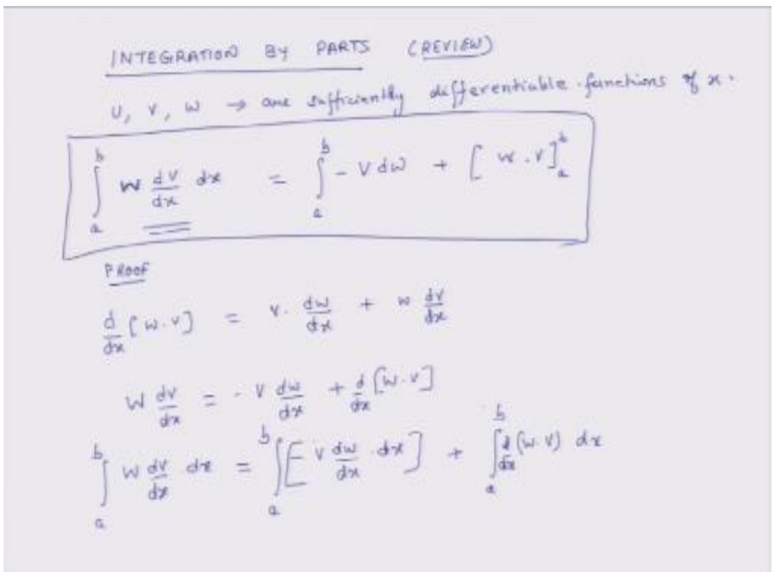

So these are all sufficiently differentiable functions, of x which is the independent variable. And suppose I want to integrate this guy, w which is a function of x times dv/dx and I have to integrate it over the domain a to b. So I want to integrate this thing, now this can be expressed as and will show the proof later today itself, is equal to minus v times dw. So dx, dx goes away, okay.

So this is v times dw plus function w, so w's and v's are functions they are not constants, is strictly speaking I should have written it like this. But for purposes of privacy I am not writing it. So w times v, in the limits a to b. This is the final answer and will get the final answer how we get it, we will review it, proof. And this is a very important thing at least in context of FEA. So we know that, derivative of two functions w and v equals v times derivative of first function plus w times derivative of second function which is v.

So I get w, because I have to get this on the left side dv over dx is equal to –v dw/dx + w. v d/dx, okay. So I will just reorganize this, now what I do is, I put the -- I integrate that okay. So I will write it on the next line. So now I integrate this equation, over limits a to b and this is equal to –v dw/dx. dx plus integral of w, v.

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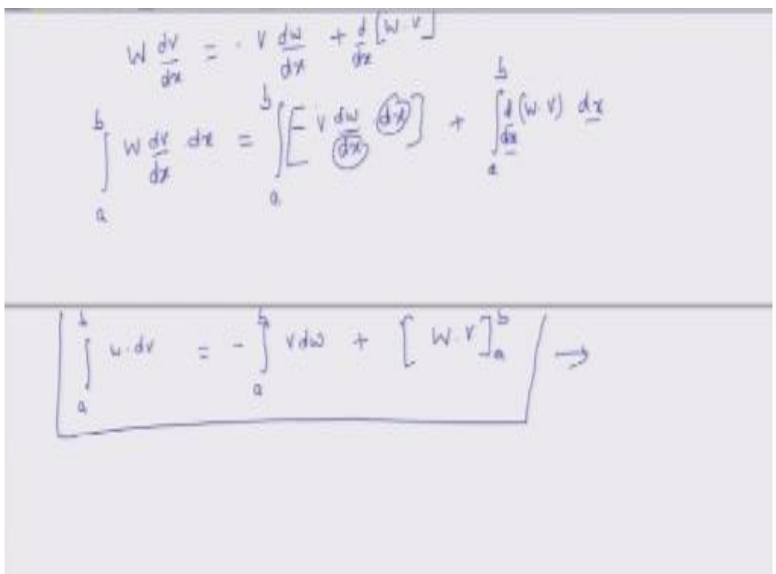

So now I continue this thing, so this is w times dv integral from a to b and this equals. So this dx and dx they can cancel each other. So I get basically a to b, v dw + again this dx and dx they cancel and essentially when I integrate this guy I get w times v, a to b, okay.

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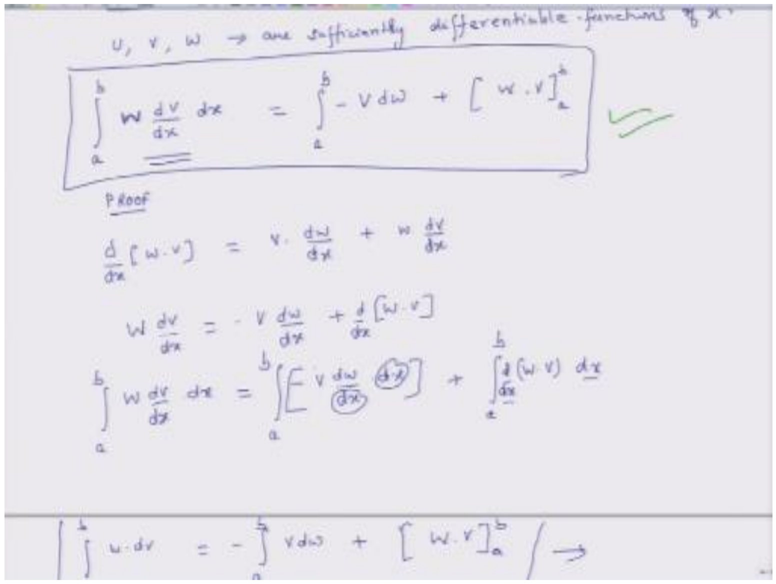

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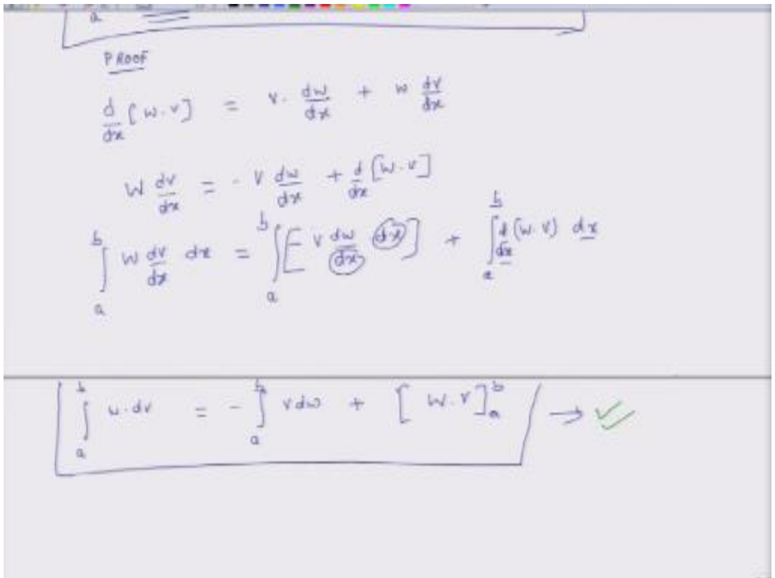

So this equation is same as what we had written earlier, this is the same relation as this relation. So we have shown that this equation is there, okay. Now this is a very important equation from standpoint of finite element method.

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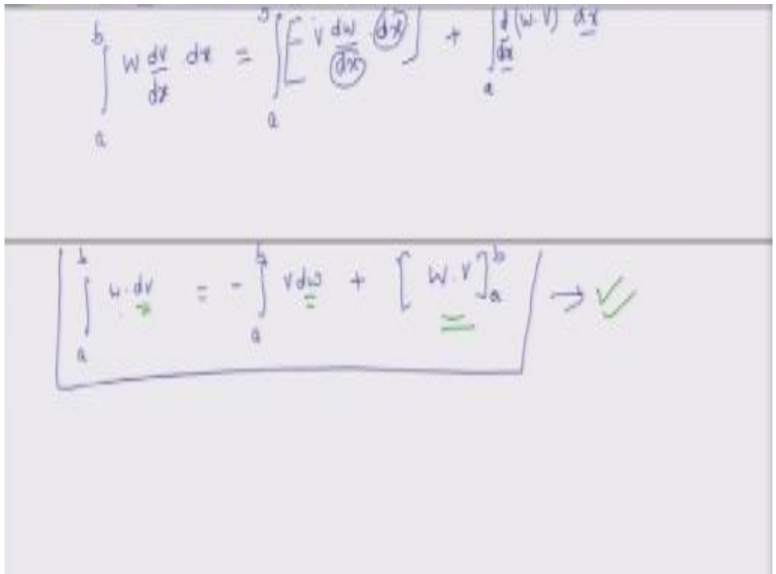

And what it does is and we will see its impact today itself, what it does is? That here it transfers the differentiability, here this has been differentiated v, that differentiability has been transferred to w, okay. And then of course we have the boundary terms. So we are transferring differentiability from v, in our case it could be that v is the residue, v could be the residue and w could be the rate function.

And through this method we are able to transfer differentiability from their residue to the weight function why that is important? We see it a little later. But this is something important to note.

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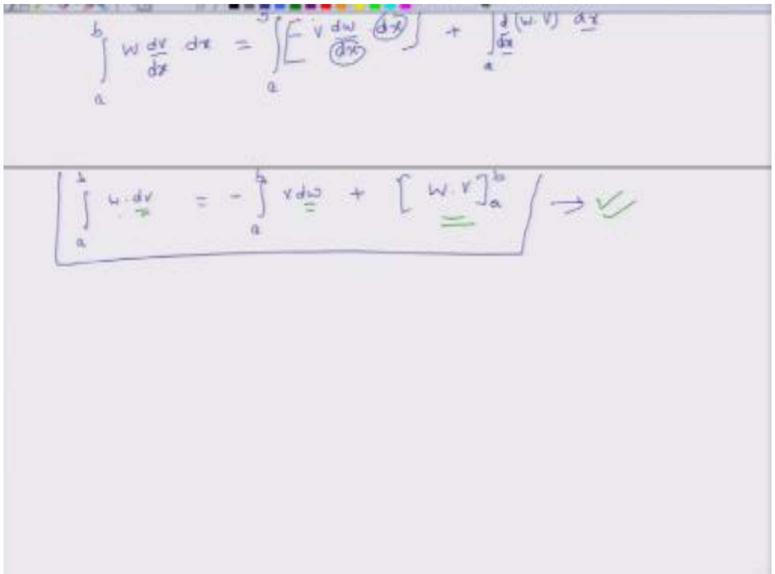

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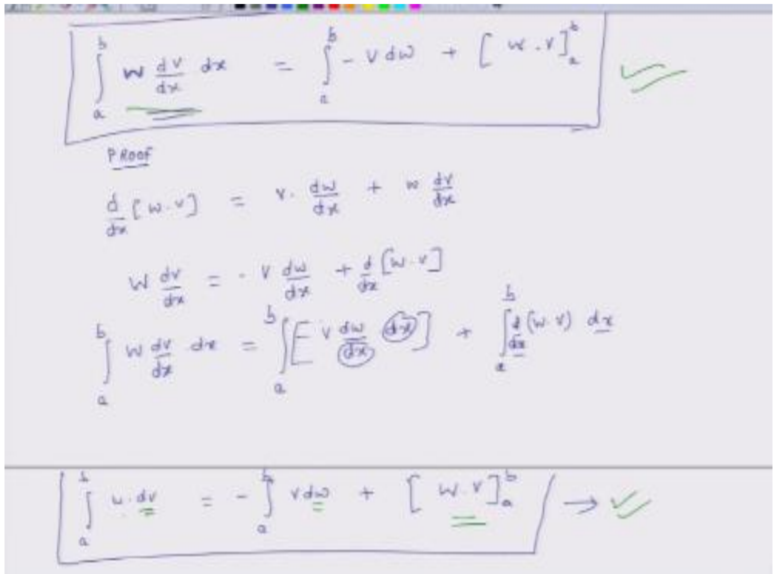

So this is far, this function w times dv/dx.

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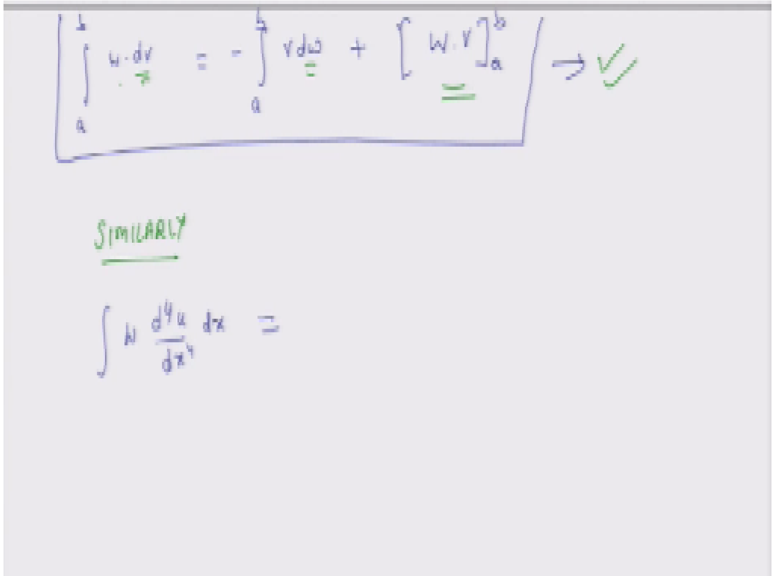

Now we will do this one more time, okay. So similarly I can have another relation suppose I have a fourth order differential w times d^4 u/dx^4, dx suppose I want to integrate this thing. Then how do I do this? Before that actually let us do the second order thing.

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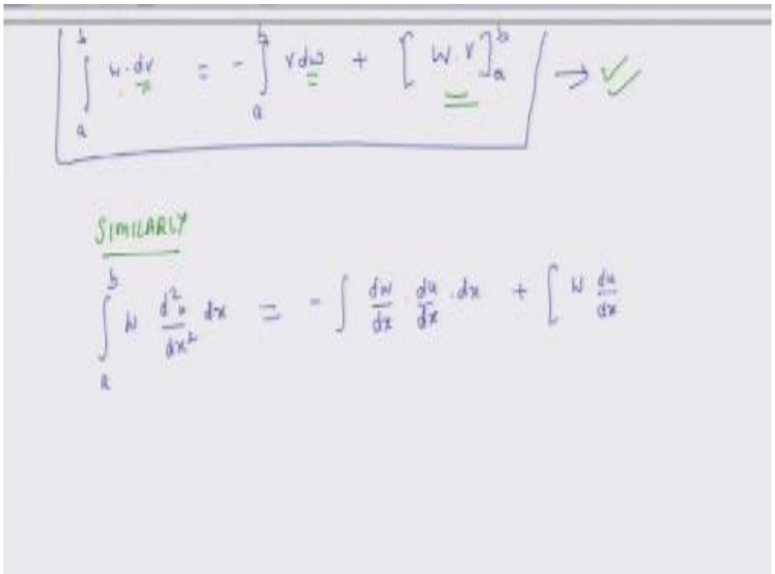

So this is equal to minus this is a to b, Oh I am sorry u. So this is equal to dw/dx times du/dx times dx plus w times du/dx a to b, how did I get this?

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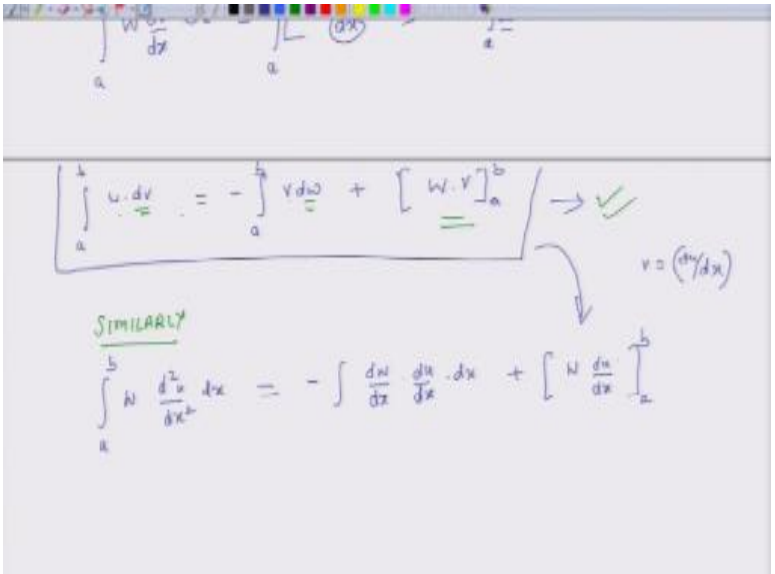

It is basically coming out directly from this thing. Here what we have done is, here this is double derivative of U. So I can assume that v = du/dx, okay. I can assume that v = du/dx, I plug this thing here, so here I get V it gets replaced by du/dx, w is same and instead of v it is du/dx, okay.

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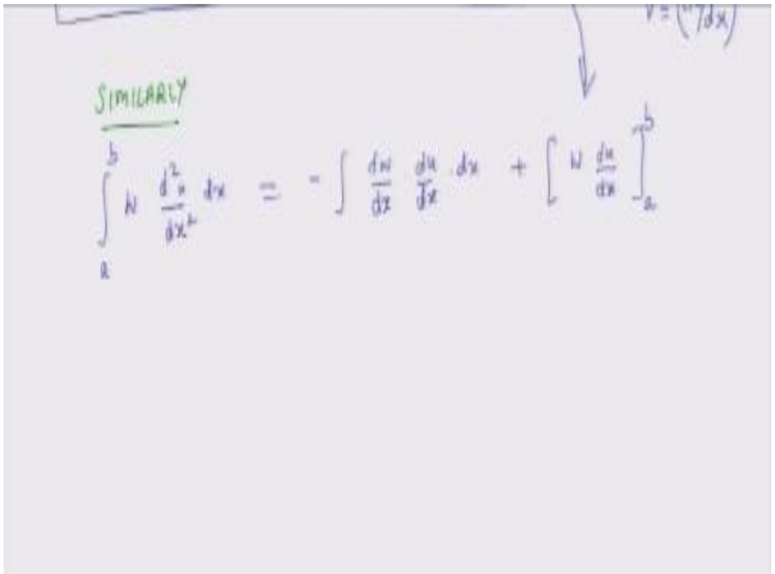

Now think about it, a lot of our equations in one-dimensional for instance if I am trying to pull a bar intention. It is governed by second order differential equation.

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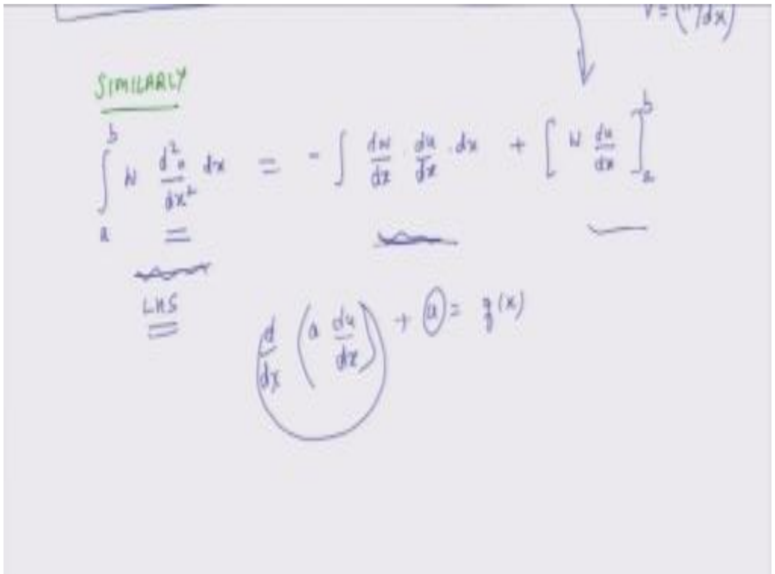

For of this type, (a du/dx) and then d/dx + u is equal to some function q something like this, okay. Similarly the heat conduction equation, in one-dimensional is a second order differential equation, okay. So you will get in several equations second order derivatives, second -- you will get -- you will also get u times w. So when I am trying to compute residue I will get a second order derivative multiplied by w.

And of course in this case u times multiplied by w, when I do this Math, what happens is, that this side is equal to the right side, and here if I am computing the left side, you remember that we assume -- we assume different values of fees, right? For a linear element it could be linear in nature or quadratic or things like that.

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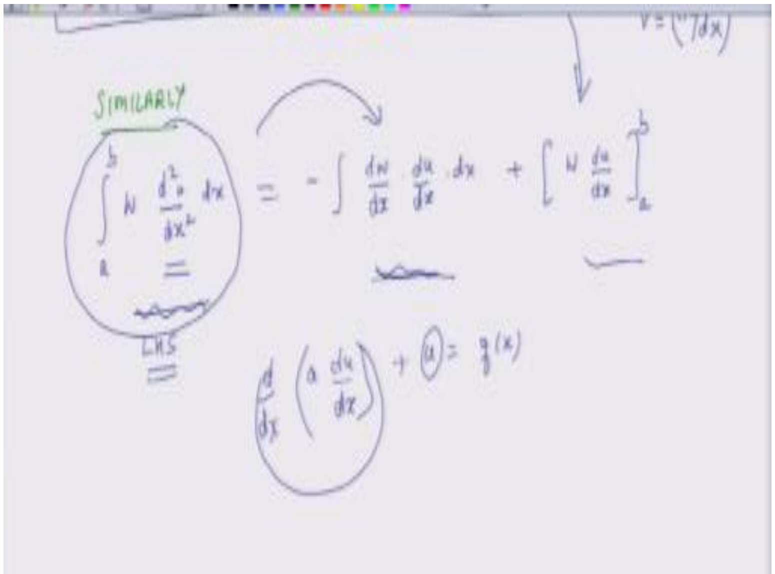

So here the requirement one mathematical requirement you will see directly here, is that u has to be atleast of class two, okay. But if I transform this integral to this type, then u only has to be of class one entire thing, okay. So that is one thing that is one thing that my requirements of differentiability, here for u have gone down by one, this is one thing.

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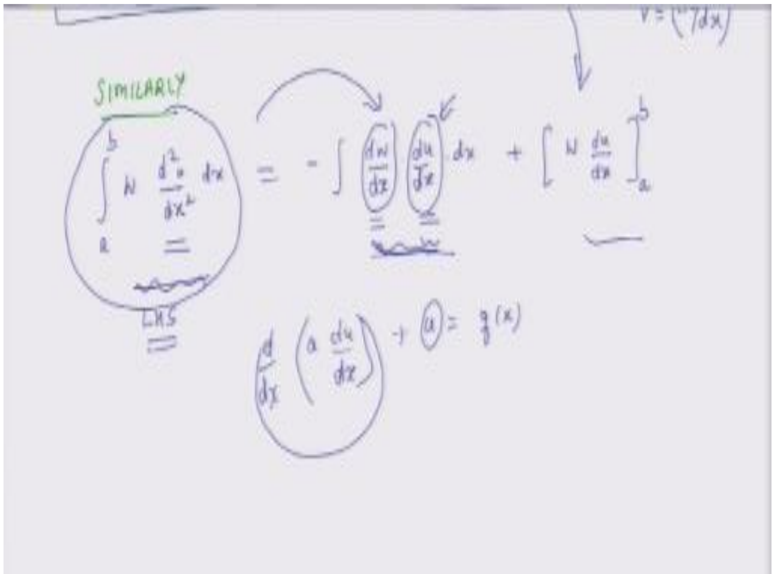

And the second thing is, this is symmetric, okay. So right now I will not call it symmetric, but the requirement on w is one class one requirement on u is also class-one, okay. Now later in the FEA course what you will see is, that this balancing of differentiability on the weight function and, and this other function is related to the error.

So balancing of differentiability on weight function and the unknown variable, if it is balanced then it leads to symmetric matrices, it leads to symmetric matrices. We will see this later, we will actually do the mathematics we will see this later. And when we have symmetric matrices, then solving symmetric matrices is much easier than solving non-symmetric matrices, you know equations which have non symmetric matrices is more complicated.

Matrices are symmetric then it is faster. So that is another thing and then there are some more advantages of this thing.

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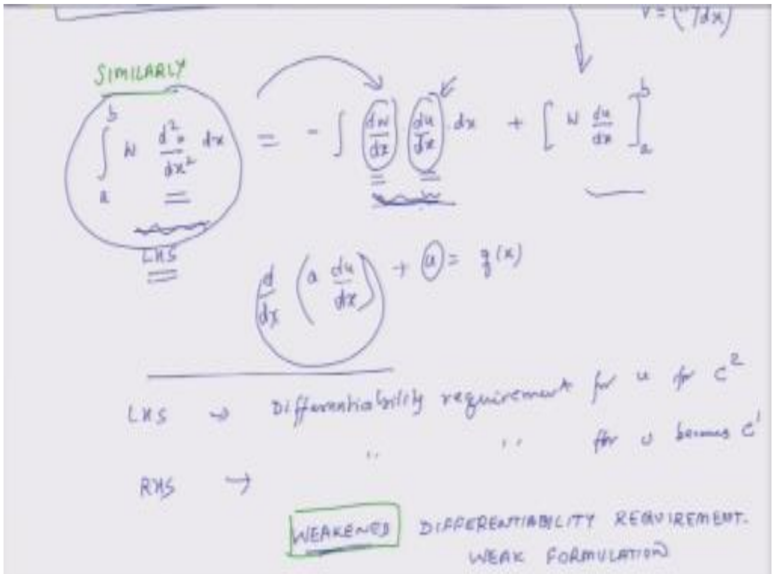

So this is, so what we have done is, in this case in that L.H.S, differentiability requirement, for u was c^2 we needed a c^2 class of a function. If we go -- if we transform this to the right side, differentiability requirement for u becomes c^1, I mean if it is c^2 no problem I mean it will do the job, okay. What we have done is, we have weakened differentiability requirement.

We have weakened this term come -- it comes in a lot of finite element literature we have weakened the requirement of differentiability, okay. And that is why, this method once we weaken it is known as weak formulation. It is not that the answers are not poor or something like that; the differentiability requirement has been weakened. So this is known as a weak formulation.

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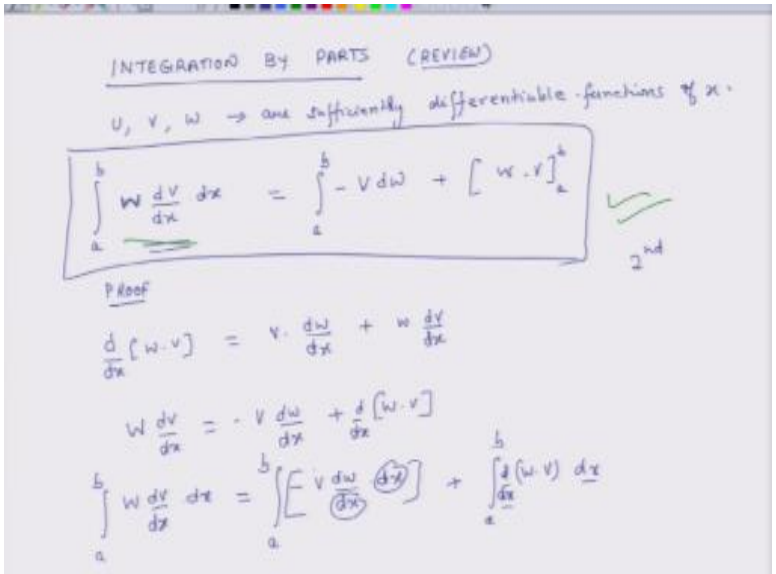

So this is for second order, so you see the benefit of this for second order derivatives, okay. Now we will check one more thing.

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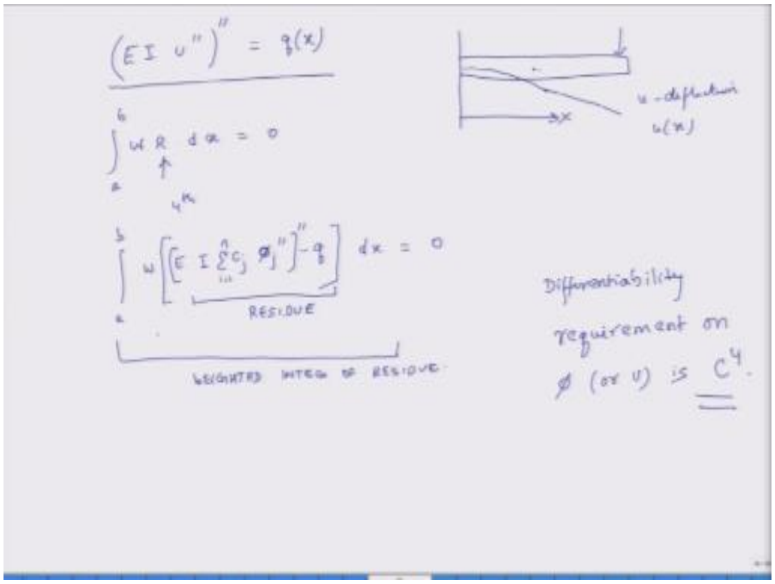

Consider a beam, the equation for beam is what? I will not use w let us say u is the deflection. This is the equation for a beam, right? If u is the deflection of the beam i is the moment of inertia, this moment of inertia can change with respect to the length of the beam, suppose there is a beam. I am putting some force here, right? Because of this, this is my x-axis deflection is happening here.

So that this deflection at some point is u, is deflection and u is a function of x and q is the loading, this loading could be a point load or a distributed load, okay. So this is, so this is what this is the fourth order equation, it is a fourth order equation. So when I construct its residue and I multiplied by some weight function and I integrate it let us say between the length 0 to l which is the domain.

Or if I am doing it for an element then it will be from h1 to h2, right? It does not matter, I mean the theory right now the theory is I am talking about it does not matter. So actually to make it more generic I will say I will integrate it from a to b. What is the differentiability requirement for r, in this case? It will be fourth-order fourth right why? Because when I -- w, what is r? r is E_i C_j $ø_j$ Σ i is equal to 1 to n, you differentiate it two times minus q dx, right? This is my residue that is the residue. So I am basically the relation says integral of w times Ei and then Σ of c_j $ø_j$, double prime, prime.

C_j are constants $ø_j$ I have to differentiate two times and then from the right side I bring in qx. So if this is not exactly zero then there will be a residue so that is why it is called residue, and then differentiate this whole expression twice with and then integrate this thing with respect to x. So this is y and this entire thing is weighted integral of residue, is a weighted integral of residue.

So in this case differentiability requirement, on ø or I can also call u because ø is a u is a function of ø right, is what? Fourth order, right? So this has to be the fourth class, oh yes you are right, yeah, yeah, yeah you are right, that is there, okay. So it is fourth order.

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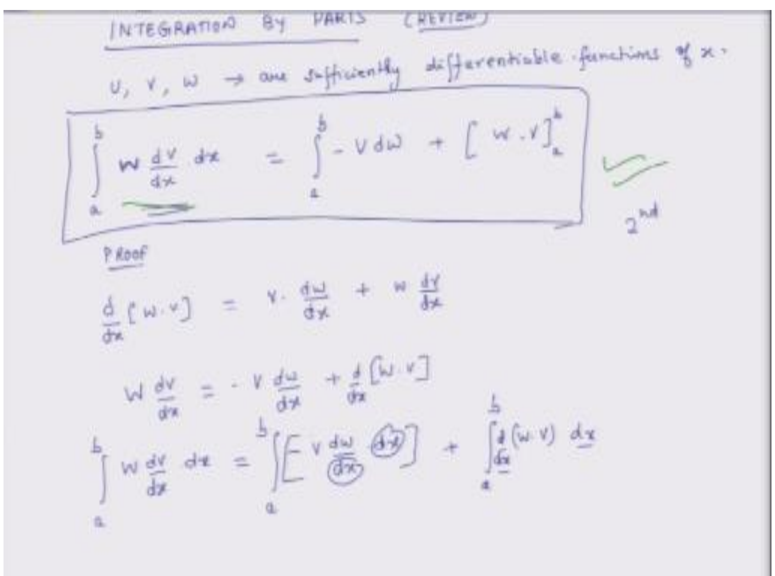

Now using this method this integration by parts if I integrate it once by parts.

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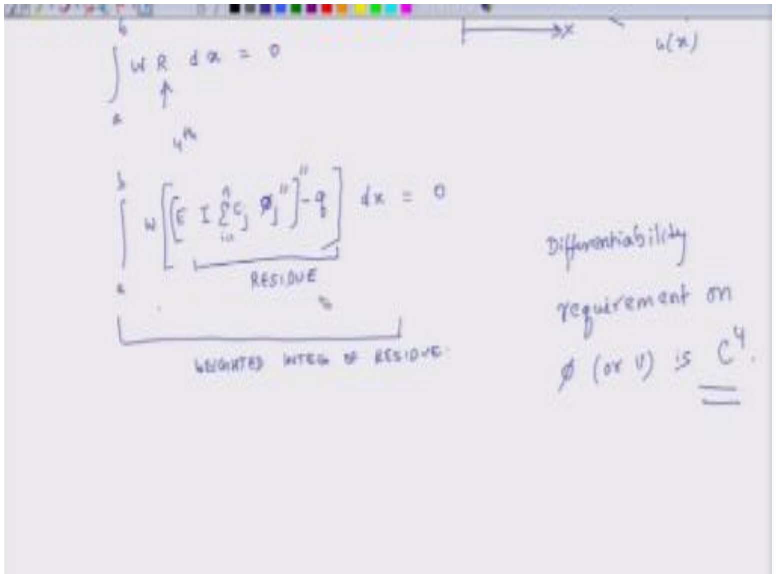

I will reduce the differentiability to three and if I integrate it two times, then I will reduce it to two, right. So that is what I will do and that will conclude this lecture.

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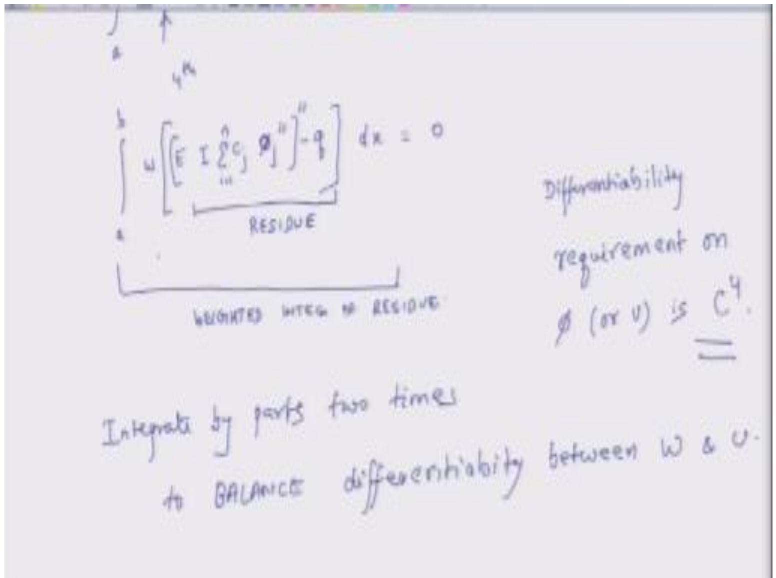

So if we -- so we integrate it by parts, two times to balance differentiability between w and u you.

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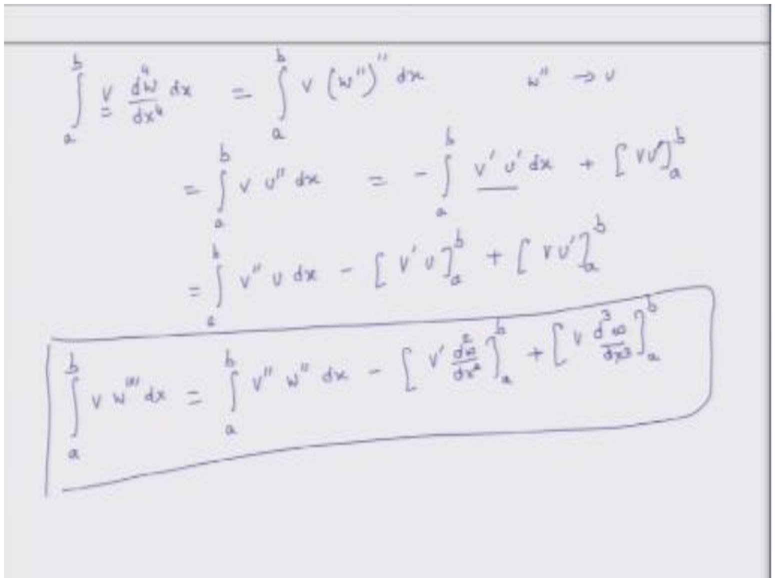

So that is what we will do. So, so suppose there is a fourth order derivative and you will apply this result directly to the beam equation if you want and there is a -- let us say in this case weight function is v, I should have written it as w, okay. And the unknown is w, okay. So this I can express it as a to b. So here weight function is v, w is the unknown function. So v(w")" dx and let us say that w" I call it as u, okay.

So then this becomes a to b, v u" dx and when I integrate it by parts I get minus a to b, v' u' dx plus [v, u] a to b. This is vu', okay. So now it is v' and 'u. So between u and v it is balanced, but u is itself a double derivative so it is, here on u my continuity requirement is three and on v it is one. So now I integrate it one more time. So now what I do is, a to b -- so I integrate it by parts one more time, okay.

So I get minus -- so again this minus again I get another minus that goes away so I get v', v" u dx minus [v' u] a to b plus [v, u'] a to b, okay. And now what I do is, I replace u by w. So I get integral a to b, v" w " dx - [v' dw/dx] a to b plus [v d^2 / dx^2], oh actually this should be cube, a to b. So I will write down the left side also v, okay. So if there is a fourth order equation, then I can -- I have to integrate it two times to shift the differentiability and make the system balanced. So that the differentiability requirement on my weighting functions and the unknown function is same.

So we have seen that how we do it on second order function and how do we going to do on a fourth order differential equation, second-order could be a heat conduction equation, heat transfer, I am not heat conduction -- heat transfer equation fourth order could be like a beam or elasticity prolong pace like that, okay. So this concludes our lecture for today we will continue this discussion in context of 2d systems tomorrow, thanks.

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