Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)

Course Title

Basics of Finite Element Analysis

Lecture – 08

Weighted Integral Statements

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Hello, welcome to basics of finite element analysis, today what we are going to cover is stuff related to weighted integral residues and we will try to establish as to why we have a need for these weighted integral, weighted integral residues or weighted integral errors.

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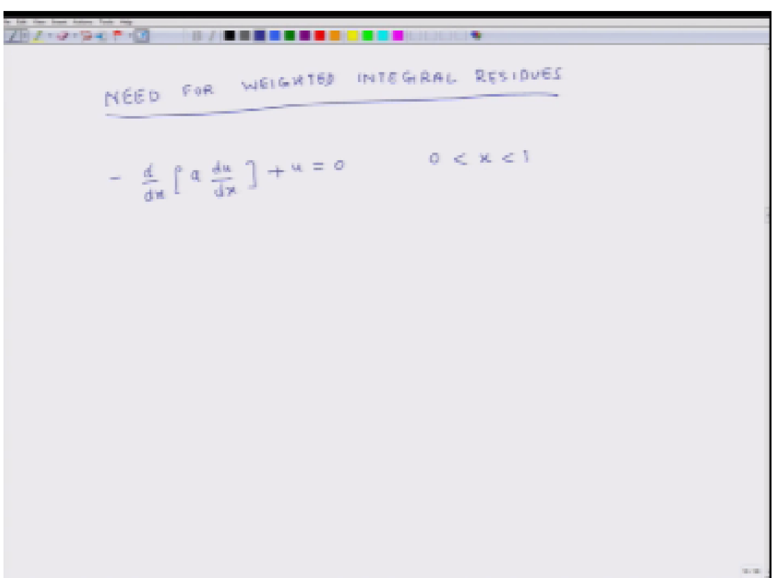

So that is what we are going to talk about need for weighted integral residues, so let us consider this equation -d over dx a it could be a function of x times du over dx + u =0 and this governing equation is valid for this domain, you should note that in the domain I have not included the boundary points, the governing equation is valid only in the domain and domain does not include boundary points and the boundary conditions.

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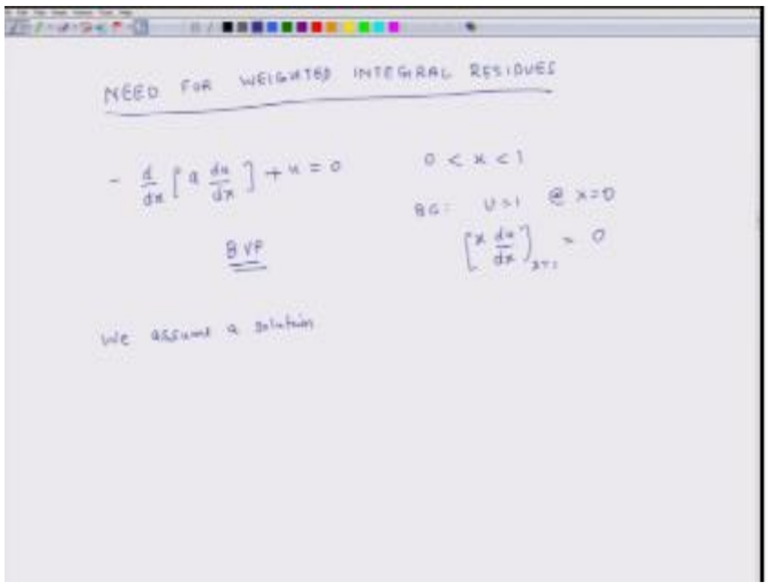

Are u is equal to 1 at x is equal to 1, excuse me 0 and x times du over dx at x is equal to 1 is 0, so this is a boundary value problem because I have specified conditions on both the boundaries okay, in boundary value problems we have to know the conditions on all the boundaries, so let us say that we assume a solution and here we are not going to break it up into small elements we are just considering the whole domain as one single element, so we assume a solution.

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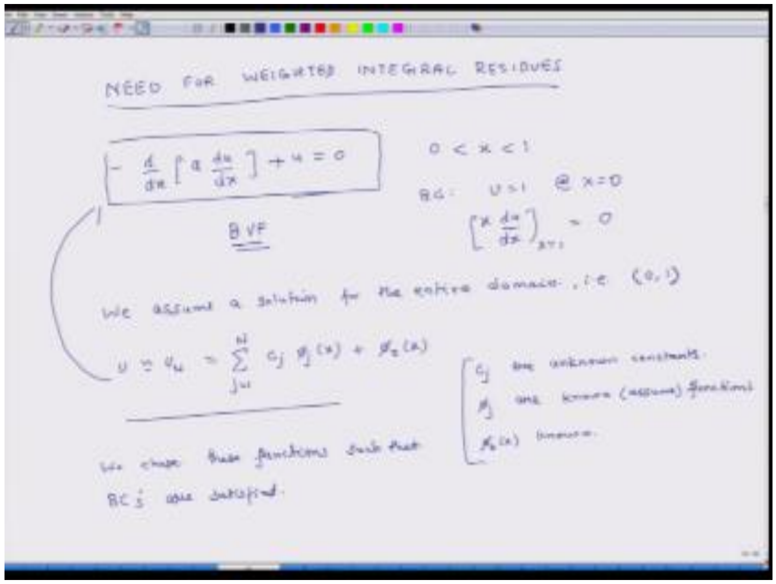

For the entire domain that is 0 to 1, so these curved brackets means that the domain does not include the endpoints okay, if it was a rectangular bracket then it means that the domain includes the endpoints, if it was a rectangular bracket on left and curved bracket on right then it includes 0 but it does not include 1 so this is some convention and we say that u is approximately equal to some u_N some function and what is, it is a function of c_j $\varnothing_j$ x plus some.

Some other function $\varnothing_0$ so I am just assuming it there is no reason that I have to have this form but this is for illustration purpose j is equal to one to n okay, and here c_j are unknown constants okay. $\varnothing_j$ are assumed functions known and actually we assume them so these fees are known and actually assumed functions okay, and same thing of $\varnothing_x$ some known and we choose and when this is happening we choose this functions such that BC's are satisfied.

We choose the functions so that the boundary conditions are satisfied why do we choose it like that, because this governing differential equation it is not valid on boundary conditions right. And we will be anyway plugging this into here to ensure that the governing equation has to be satisfied so that work we are going to do anyway so the, so while we are choosing the function just we have to make sure that.

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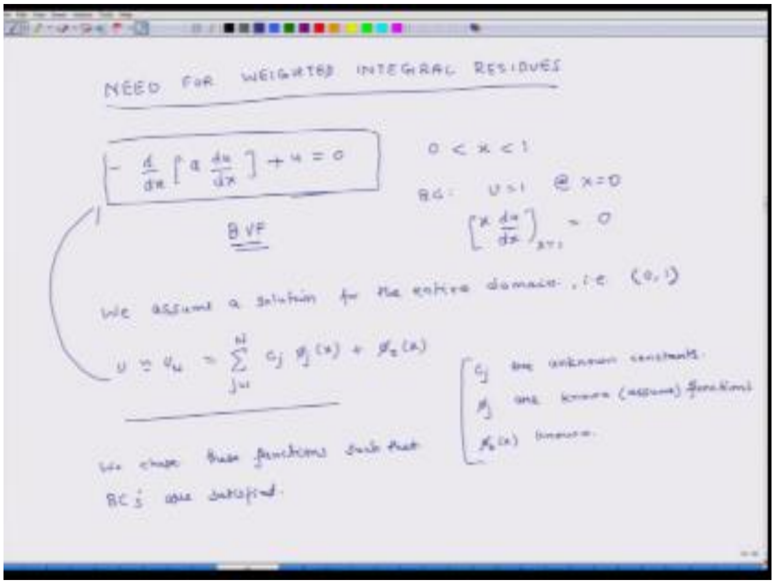

These functions satisfy the boundary conditions because we are going to ensure that that governing equation is going to get satisfied anyway later, so we have to choose it such that boundary conditions are satisfied.

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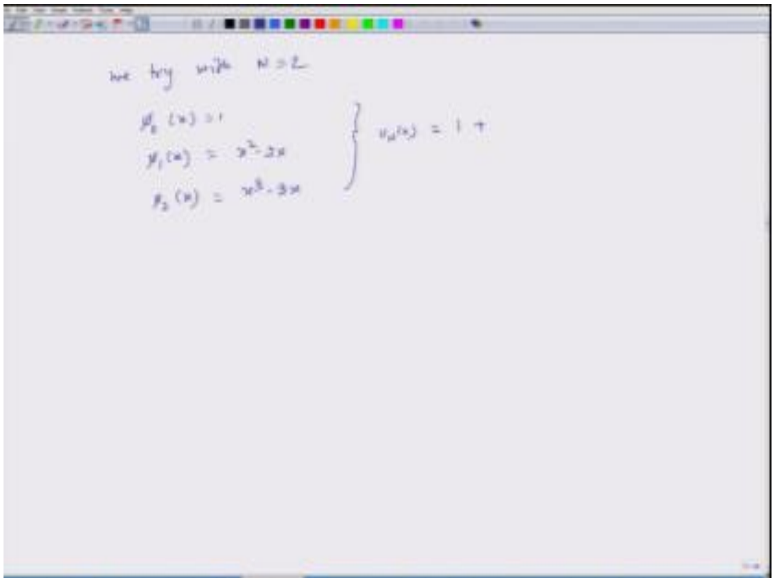

So let us say so that is what we will do, so let us we, we try with N equals 2, so we will choose 2 functions okay, and this is what we choose $Ø_x$ is one which is a constant $Ø_1(x)$ is equal to $x^2 – 2x$ and $Ø_2(x)$ is $x^3 – 3x$ so my u_n x is what 1

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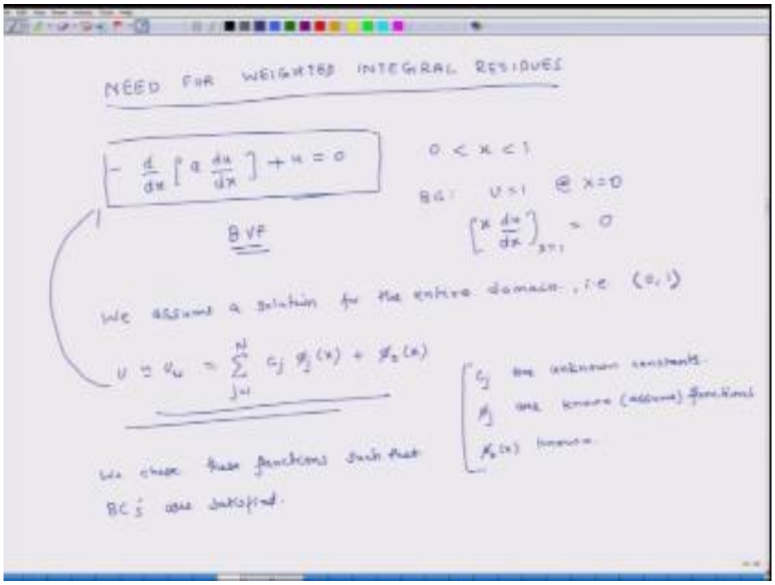

So I will use this equation you know I am going to use this equation.

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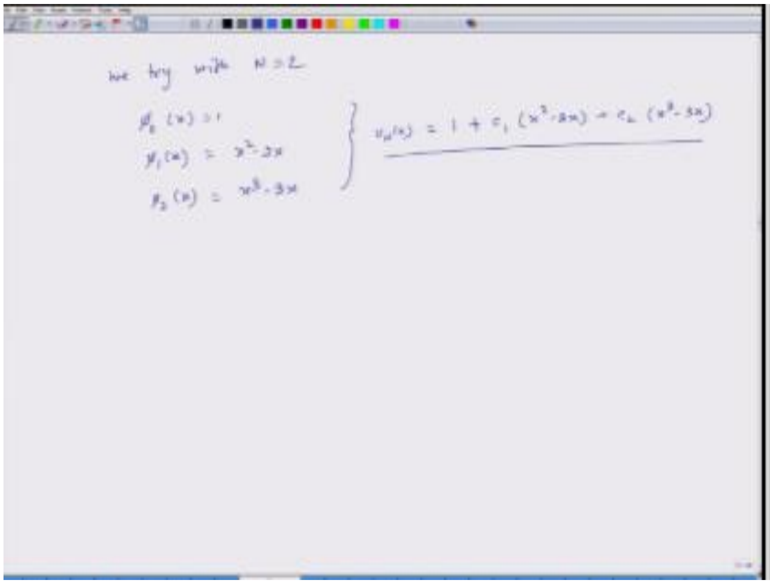

So it is $1 + c_1x^2 – 2x + c_2x^3 – 3x$, so this our equation now we see whether this function meets the boundary conditions or not if we do not, if it does not meet then we do not go further we go and pick up another function .

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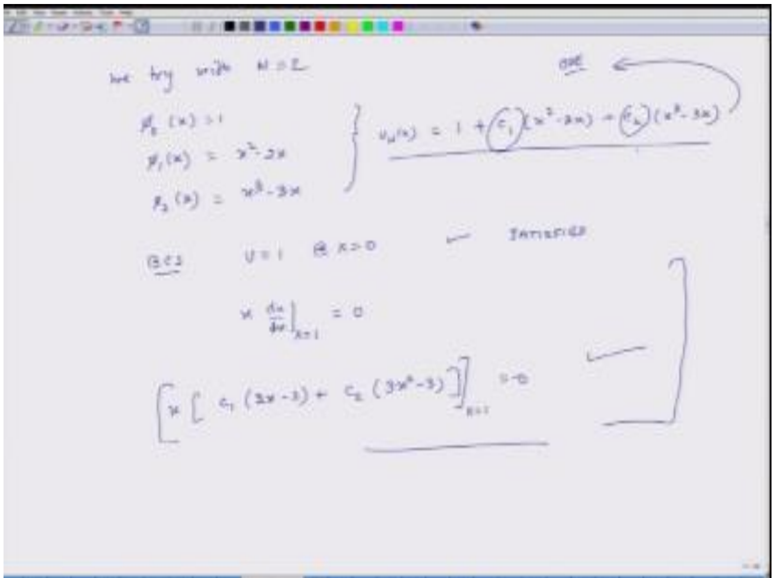

So BCS, first BCS u = 1 at x is equal to 0 so we put x is equal to 0 in this entire function and it is satisfied right. The second boundary condition is that x d_u over d_x at x is equal to 1what was it this is equal to 0 right. So we differentiate this equation and then multiply it by x, so we get x times c_1 2x-2 + $c_2 3x^2$ - 3 and the value of this at x is equal to 1 is when we calculate it comes out to be 0. So both the boundary conditions are satisfied okay.

Now what we do is now we do not know the value of c_1 and c_2 right, we do not know these values so we have to figure out these values we have made sure that the boundary conditions are satisfied so the way we figured out that way we figure out is that we plug this equation back in the ordinary differential equation, and then see so we plug this back in the ordinary differential equation and develop conditions to calculate c_1 and c_2 so that is what we do.

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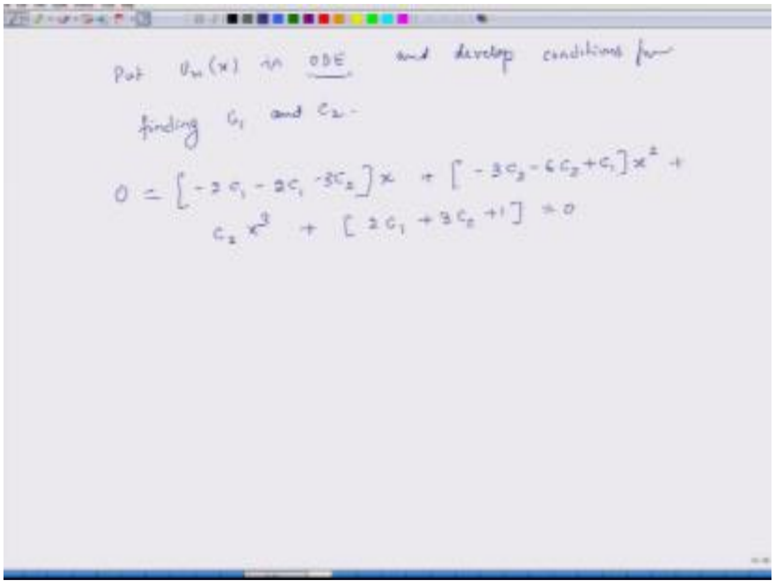

So we put u_N where N is equal to 2 in ODE and develop conditions for finding c_1 and c_2 okay, so I have so what I will do is I will not do the detail math that is fairly straightforward but I will write the result, so what I get is that my final equation is 0 equals - $2c_1$-$2c_1$-$3c_2$ this is the constant term associated with x + [- $3c_2$-$6c_2$ +c_1] this is the term associated with excuse me, x^2 + c_2x^3 so I have 3 terms related to this independent variable x and then a constant term.

$2c_1$ +$3c_2$ +1 is equal to 0 okay, everyone understands how we can get this equation all we are doing is putting this expression.

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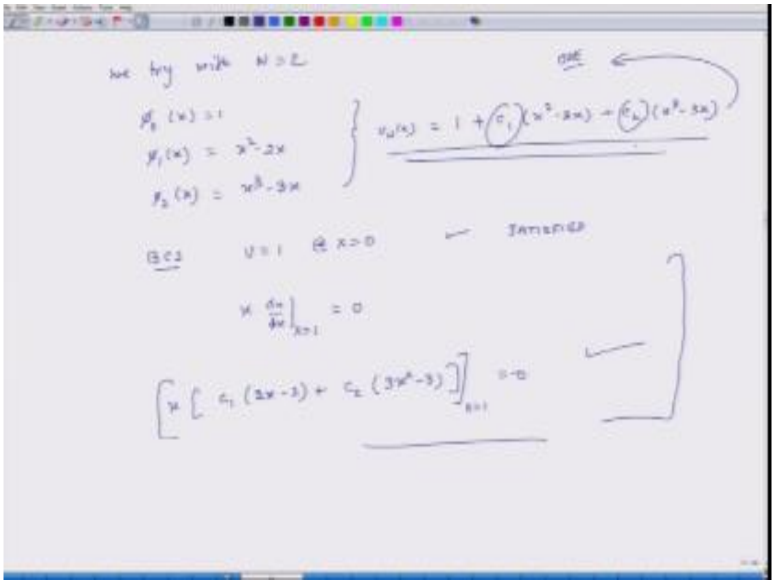

For u_N.

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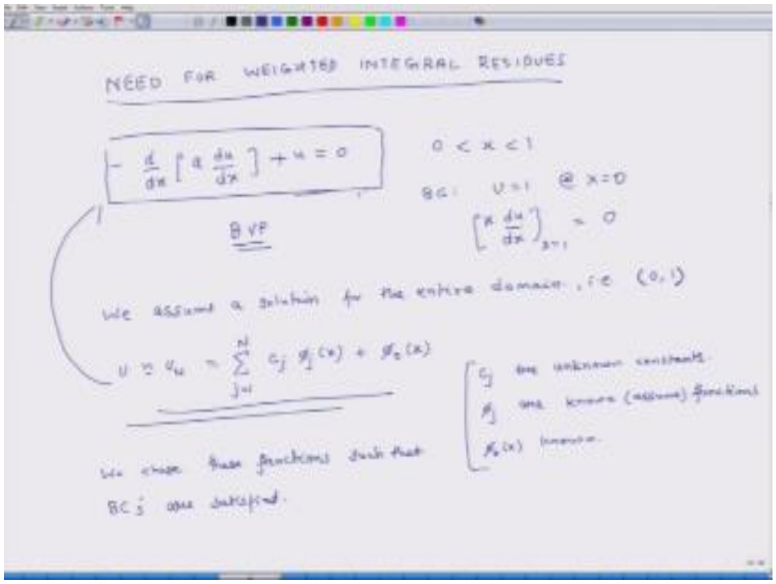

In the partial differential, in the differential equation and when we plug in all that.

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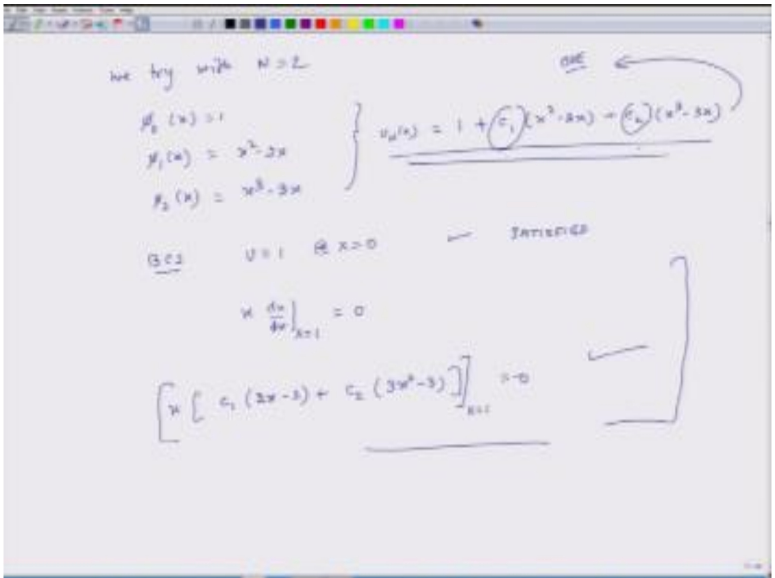

And do the math I get this equation.

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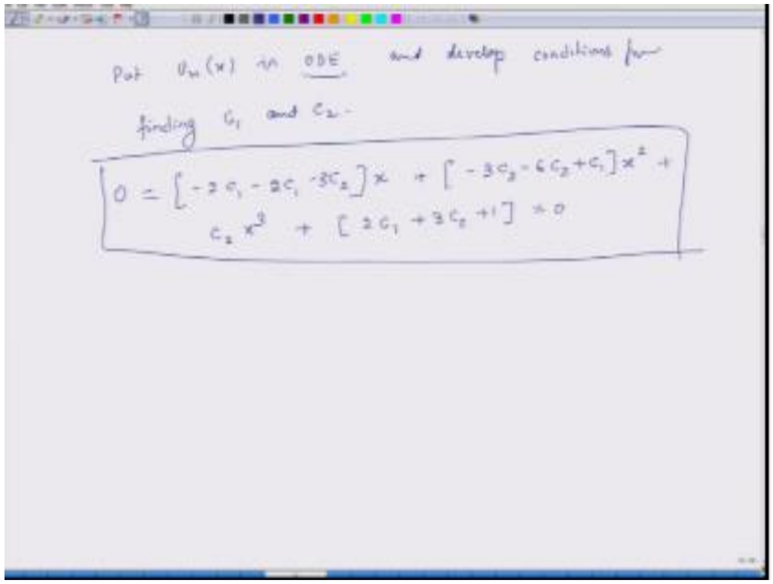

Now this ordinary differential equation is valid for the region 0 to 1, the domain is 0 to 1.

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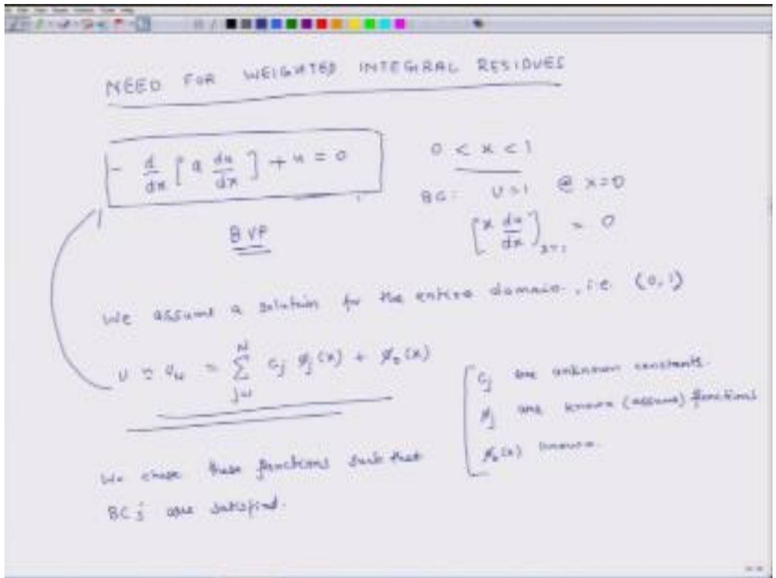

Right, domain is 0 to 1 right.

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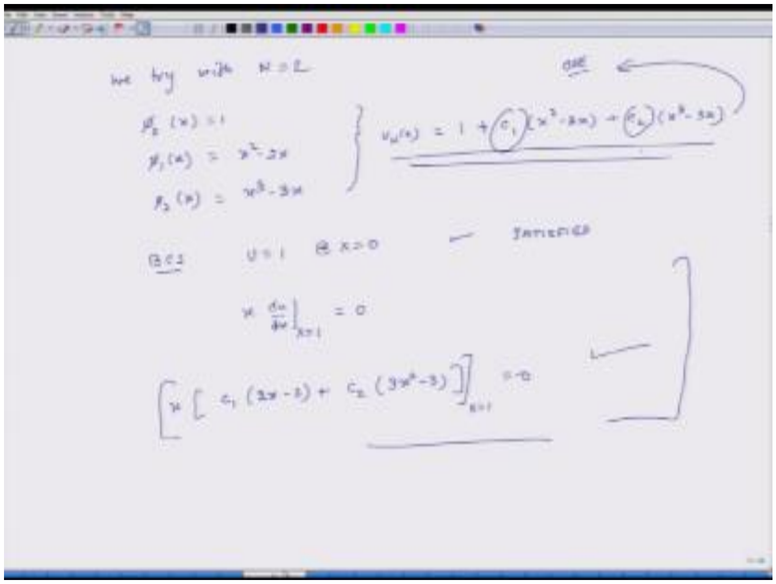

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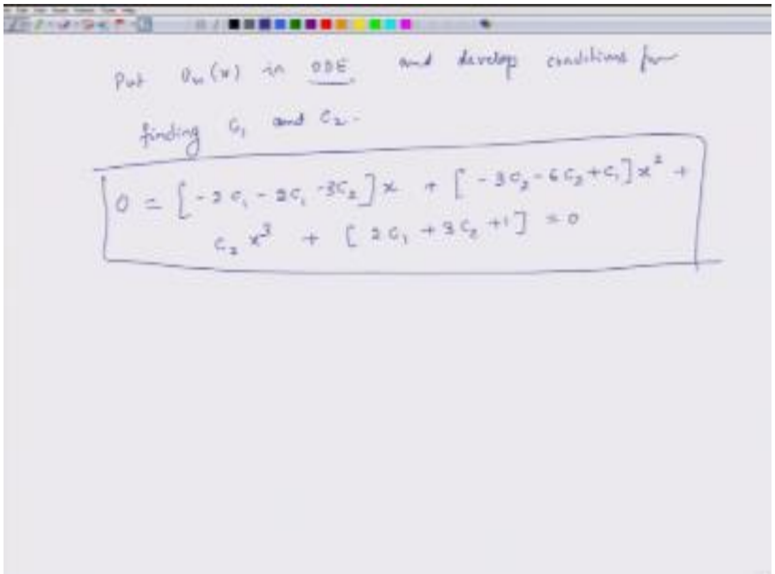

Which means that this ordinary differential equation has to be satisfied for what range of values, for all possible values between 0 and 1 for all possible values of what, for all values of x right, the domain is x is between 0 and 1 so it has to be satisfied for all possible values in the domain that x is between 0 and 1 now that can happen only.

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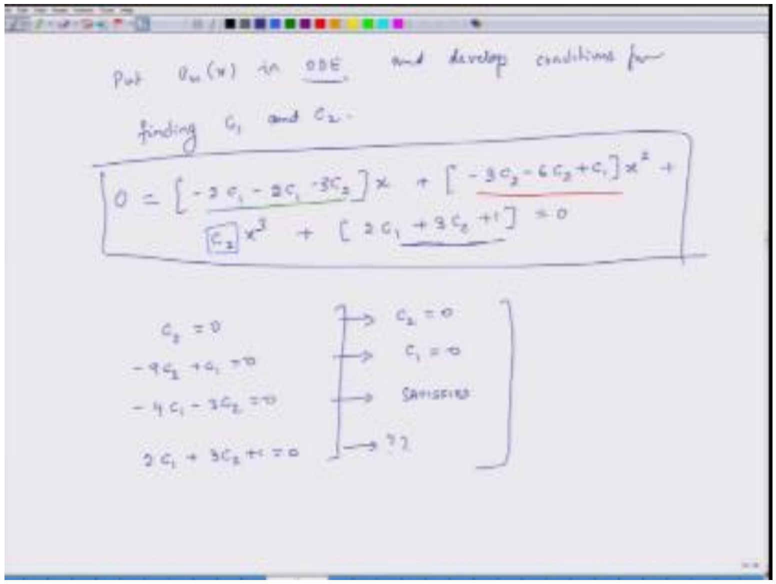

If this term is individually 0, right if this term is individually 0, and if this term is individually 0, and if this term is individually 0, only then it can be true for all possible values of x otherwise it is not possible that is only thing, so from that so now I have what 1, 2, 3, 4, 4 equations how many unknowns are, there are 3 known's okay, we will write down these equations, so c_2 is equal to 0 I get from the cubic term then from the quadratic term I get $-9c_2 + c_1$ is equal to 0 and then from the linear term I get- $4c_1$- $3c_2$ is equal to 0.

And then from the constant term I get $2c_1 + 3c_2 + 1$ equals 0 right, which means this gives me $c_2 = 0$ when I plug in here I get $c_1 = 0$ right, and then this is satisfied but what about this equation, it is satisfied or not, it is not satisfied right, so and so what this shows is so then we have to now go back and maybe figure out some other function because this function does not satisfy the ordinary differential equation, it satisfy the boundary condition but it does not satisfy the ordinary differential equation right. So that is the problem.

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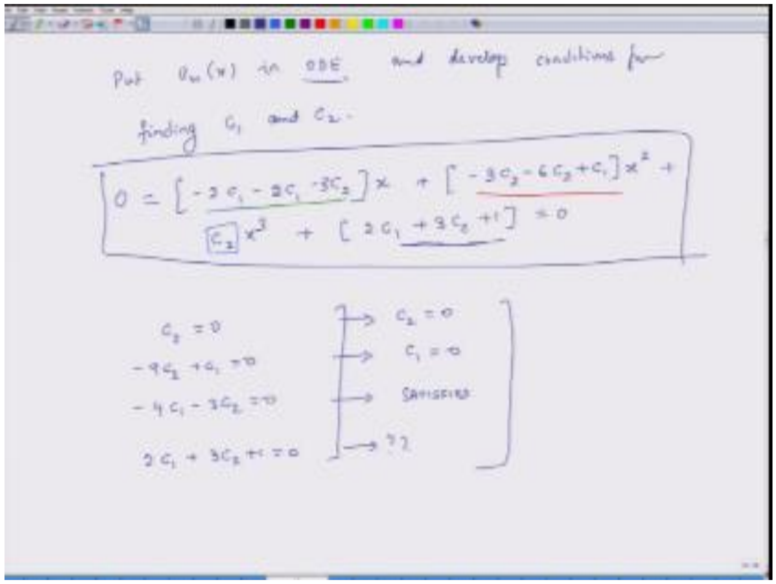

And so this is the problem we run into if we try to solve some of these.

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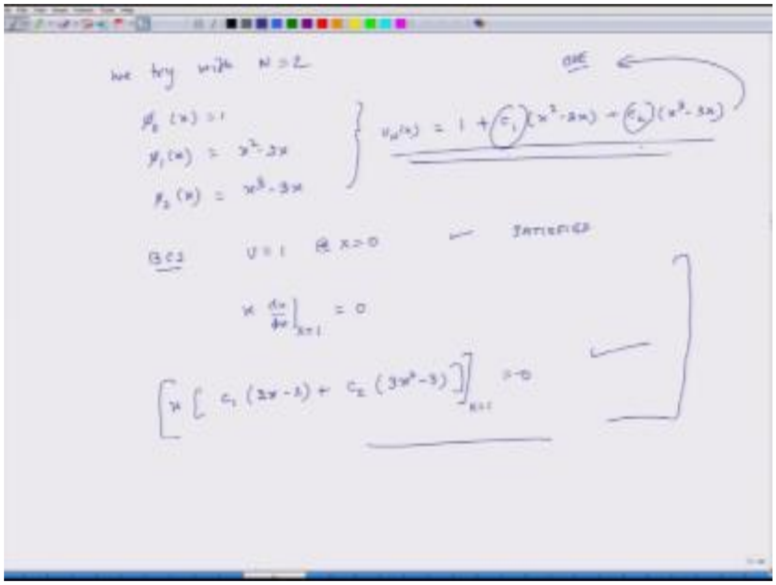

Equations.

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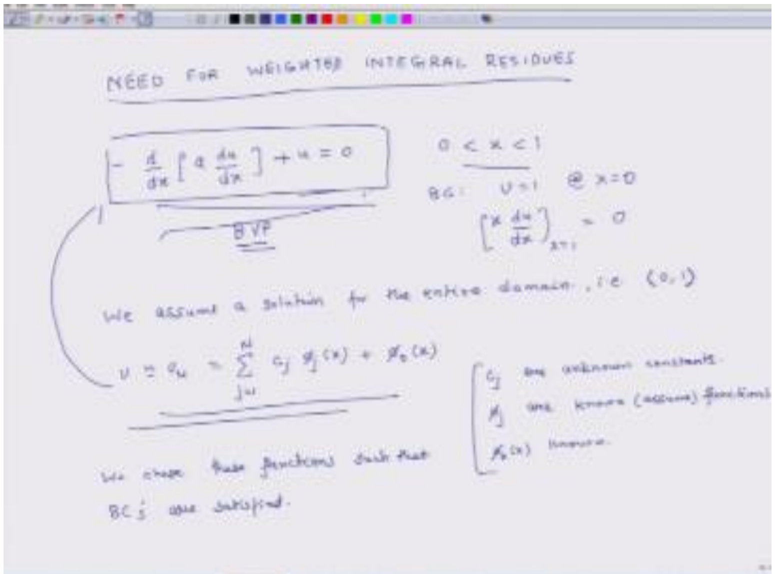

And this is a relatively simple equation, it is a relatively simple equation but we are having a.

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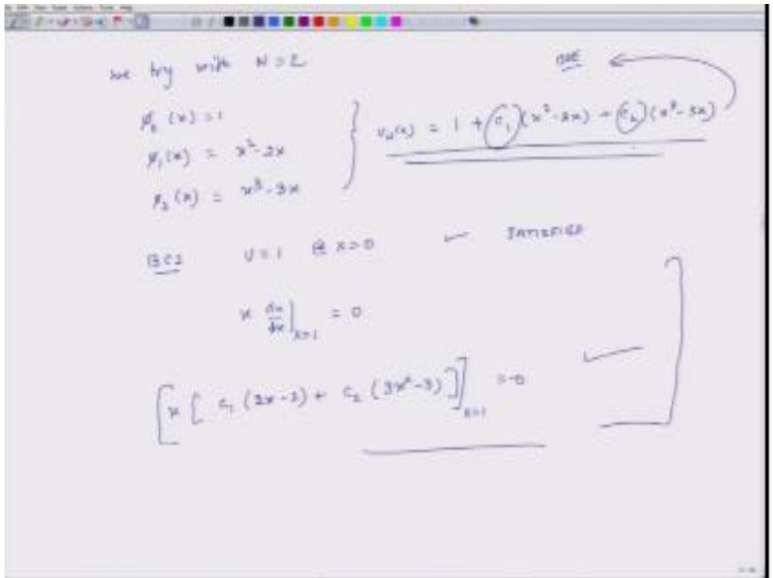

Little hard time solving this equation.

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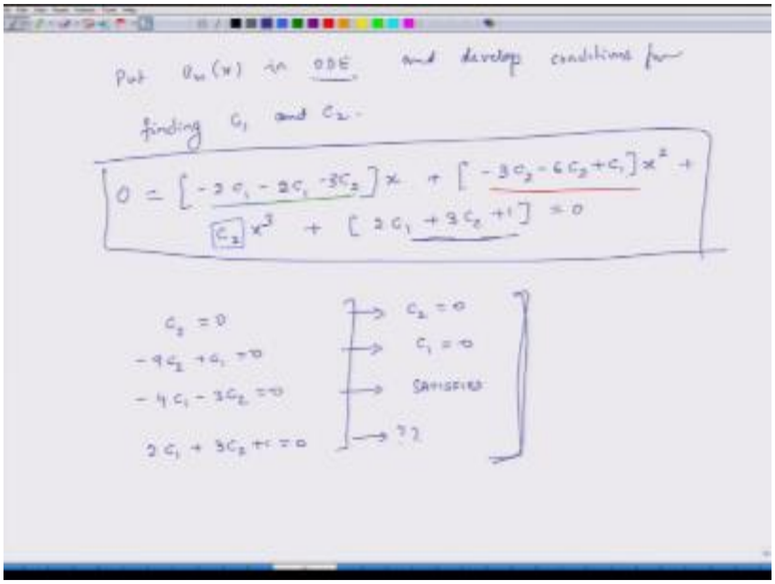

In an exact way okay, so that is the reason.

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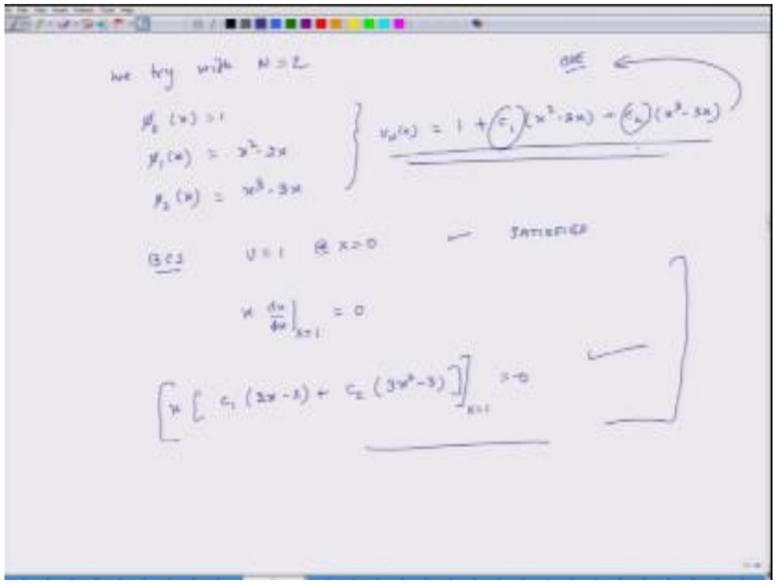

That.

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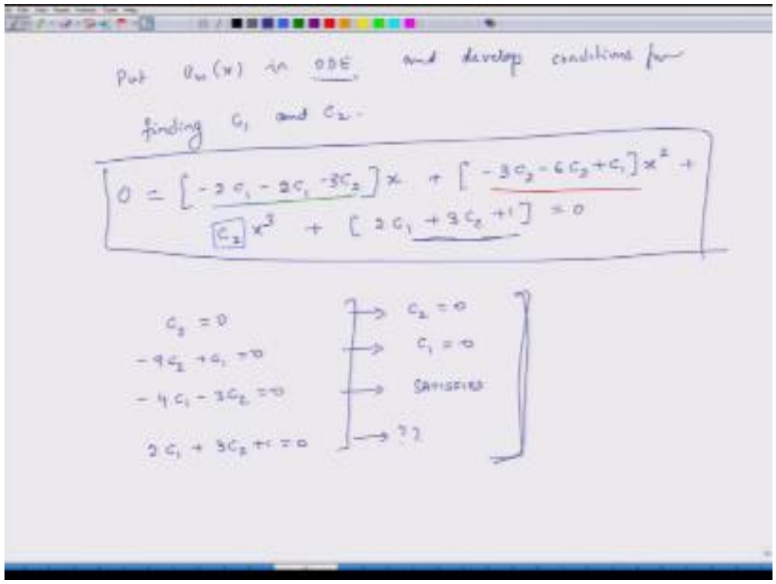

And if these equations these conditions are not satisfied what does that mean that the error in the differential equation will be nonzero for some range of conditions, for some conditions, for some conditions that error is also known as a residue in the remaining part of the course we will use this term residue a lot error is a, because this is what is left it is also known as residue, so that is the thing so because of this problem we say okay.

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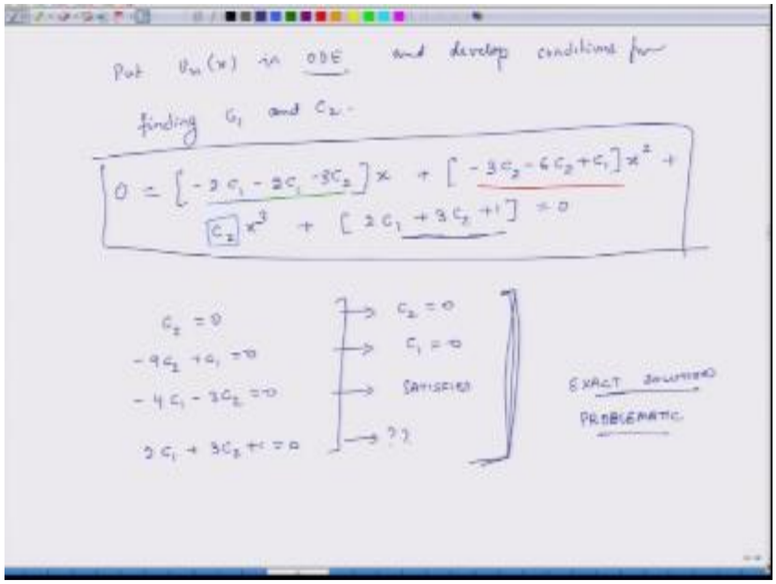

We may have a problem hence getting exact solution is problematic, if these conditions were satisfied then error would have been 0 at all values for all values of x, for all values of x in the range 0 to 1, if they are right now they are not being satisfied which means for all values of x it is not possible for me to get a 0 error, so if.

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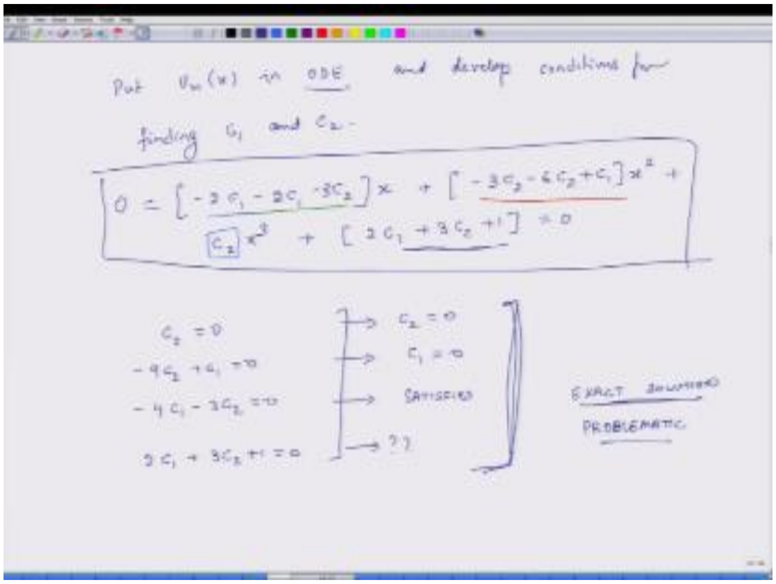

So that is why exact solution is problematic so I say okay I am having problems with having getting an exact solution so I will stop worrying about it I will say that okay instead of having.

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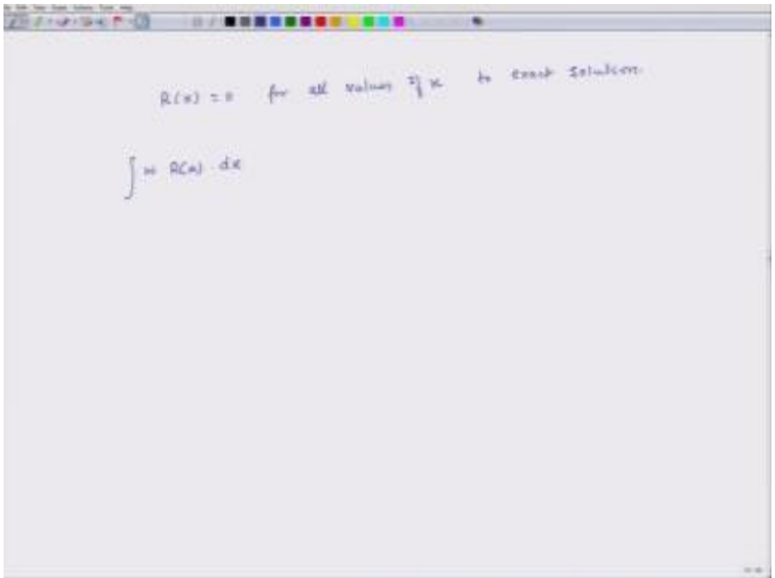

Residue is 0 for all values of x to get exact solution I will say okay, I will not worry about this first statement rather what I will say is I will multiply my residue R(x) and I multiply it by some weight function and we will . I had discussed this earlier in last week also I am again repeating it and then maybe next week we will actually look at the mathematics why do we multiply it by weight function.

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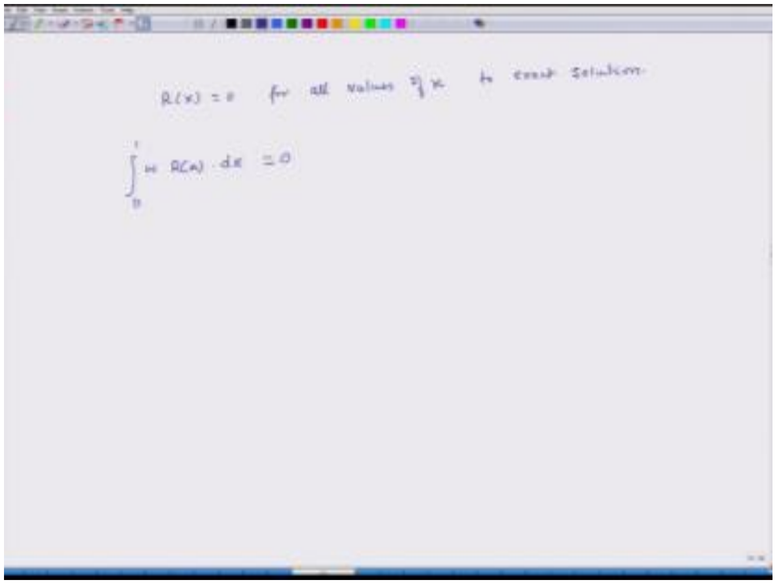

So instead so what I will do is I will integrate this error over the domain which is 0 to 1 and an integral sense not in a point-by-point sense, but in an integral sense the residues residue is 0 in an integral sense and it is not just an integral sense.

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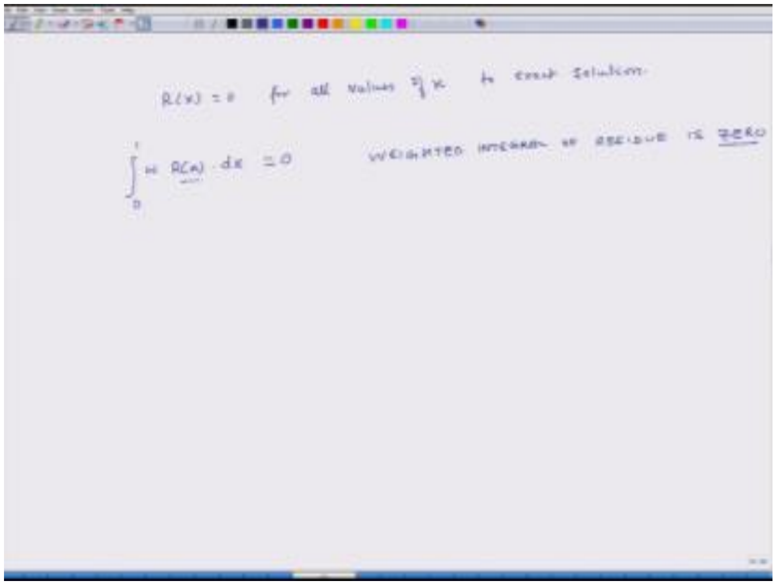

But it is weighted integral of residue is 0, weighted integral of residue is 0, first thing intuitively you can look at that if now I use this approach on an element then this weighted integral of residue on small elements if it is 0 then chances are that it will become slowly very close to accurate because on every small, small element the weighted integral is 0 so my solution will approach to 0 situation and this w.

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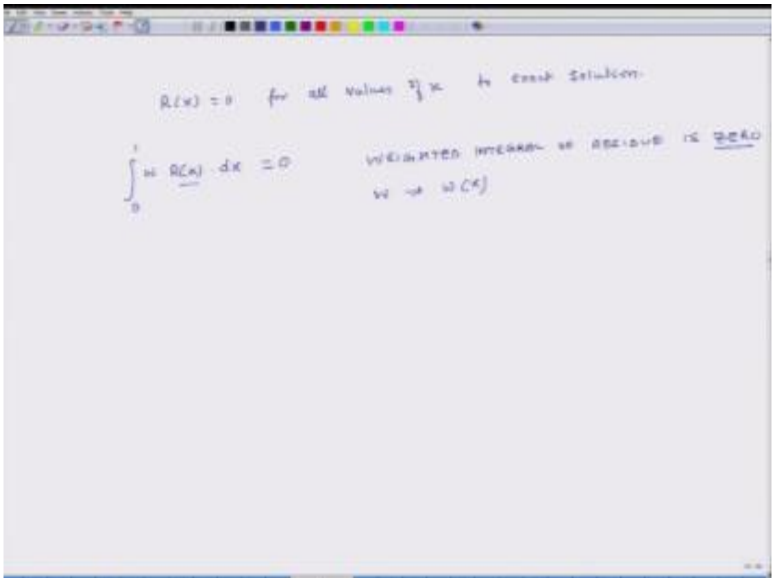

Is actually a function of x, in this case it is a function of independent variables, in two-dimensional it could be a function of x and y, in three dimension it could be a function of x, y and z okay, so this is a function and it is called a weight function.

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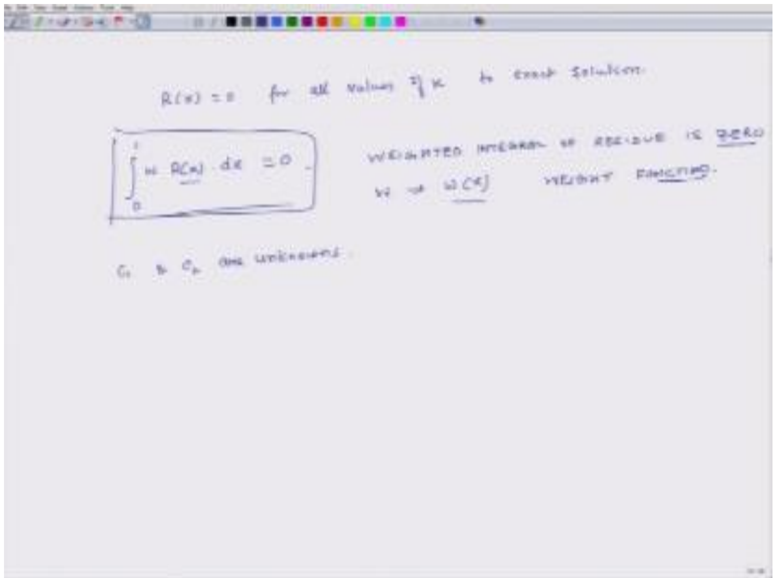

Okay, now we had two conditions two constant in we had two unknowns right c_1 and c_2 are unknowns, c_1 and c_2 are unknowns, so what I do is I get one equation if I multiply this if I use this equation we assume some weight function w_1 I will get one equation from here then I assume another function w_2 and I get another equation from here so in that way I get 2 equations and 2 unknowns, we will do that math very quickly so we assume.

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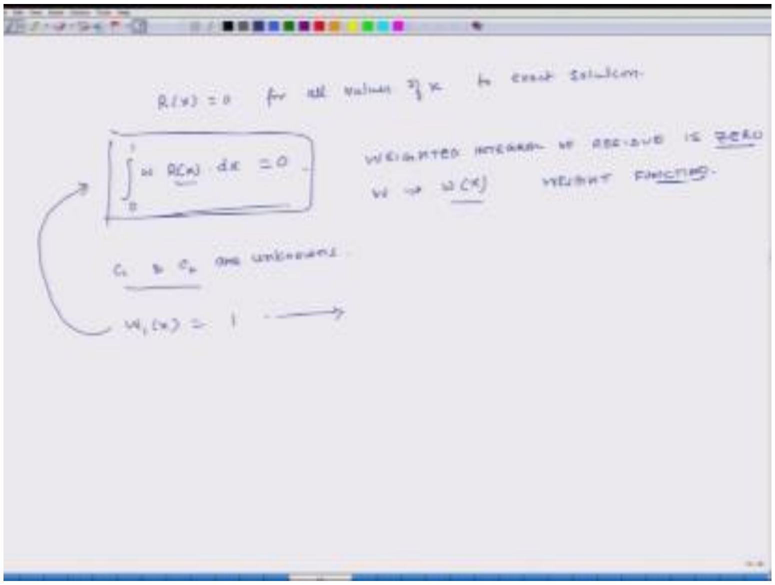

That w_1 (x) is nothing but a constant so I assume it is x and when I do this and I plug it in this equation and how do I calculate residue.

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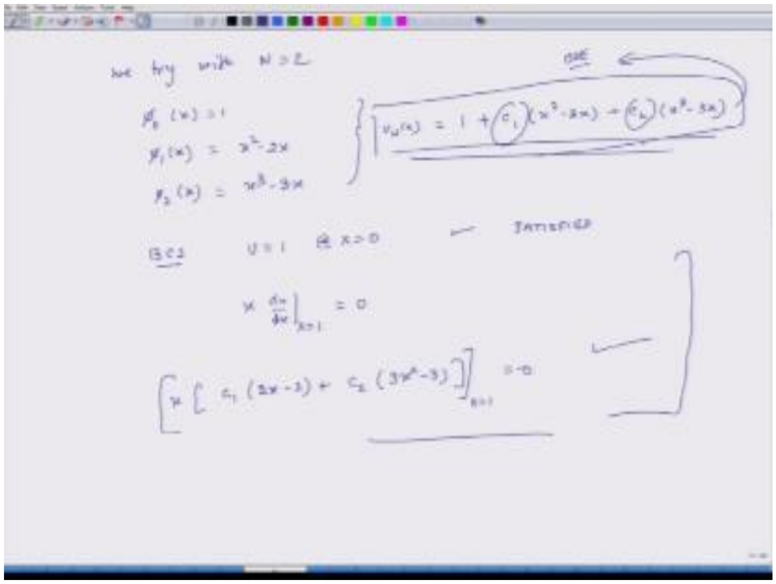

I calculate residue by plugging this equation.

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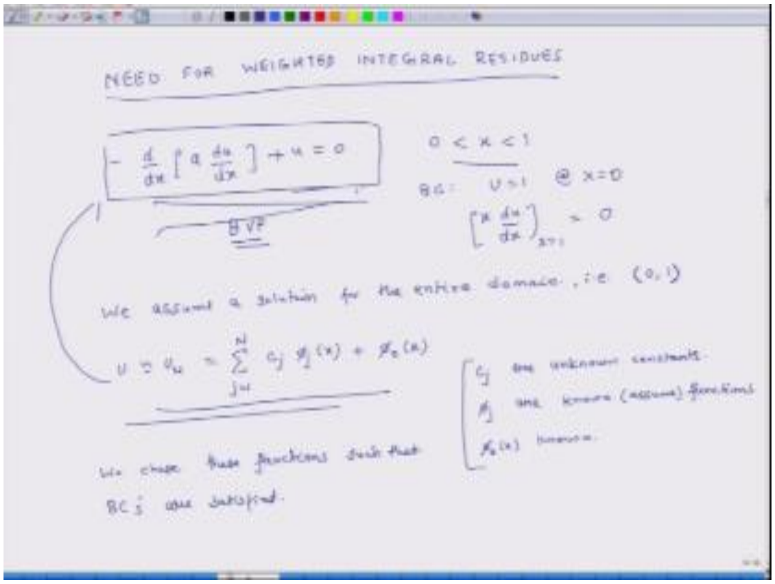

In the differential equation because it is not being exactly satisfied it is there is a residue.

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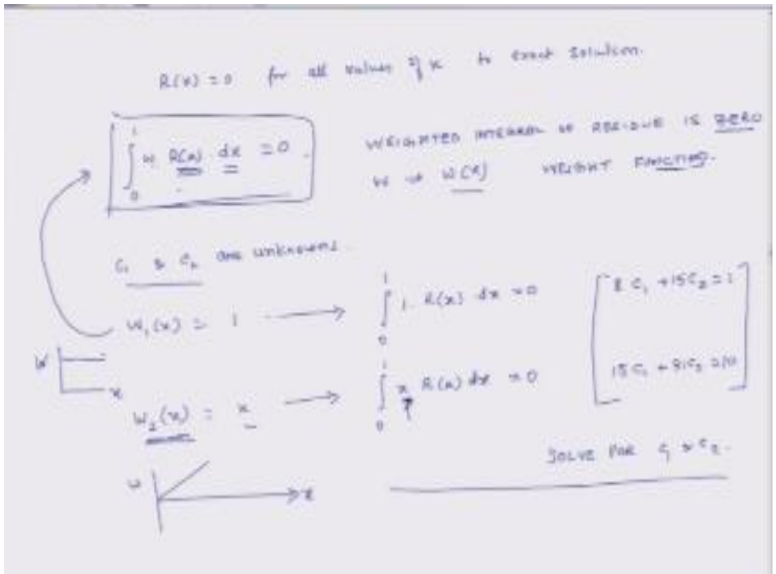

So I put that residue here and I get one equation and so the my first equation is 0 to 1, weight function is 1 times residue of x, d_x is equal to 0 and from this the relation I get is $8c_1$+ $15c_2$ is equal to 1. This is the first equation I get then I get I assume another weight function, now there is a method for picking these weight functions we will discuss that later so I pick up another weight function and here I just say that okay it is a constant x okay.

And from this I get second equation 0 to 1 x times residue of x dx is equal to 0. But here I am saying x the weight function is same as x, this is the assumed form of weight function, okay. Now x is position, x-axis is position. So weight function if I plot in this case, it is like this, okay. In this case if I plot weight function with respect to x it is like this. So from this I get another equation $15C_1$ + $31c_2$ is equal to 10.

Now these are two equations mutually independent, with two variables. So I can get the value of, solve for c_1 and c_2 okay. This method of applying weight functions, multiplying weight functions to the residue and then integrating the product over the domain, see this is my domain dx is the domain, if it is was a 2D surface then my domain would have been dx times dy, right? So this method is known as variational method.

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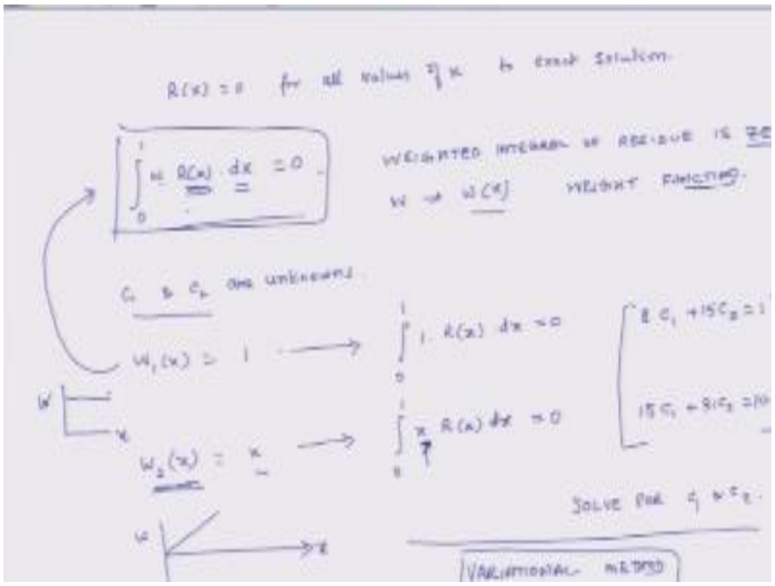

And you will learn about this later a variational approach okay. Now there are different variational approaches and what they differ in is how do you pick the weight functions? The choice of weight functions, different methods, but the overall method is same, that you find the residue multiplied by a weight function, in an integral sense you make the error zero in an integral since not in a point-by-point sense.

And then you apply different weight function so you get different equations if you have n unknowns you multiply by n weight functions, there is a logical way of finding these weight functions. And then you get n unknowns n equations a symbol at least on the n.

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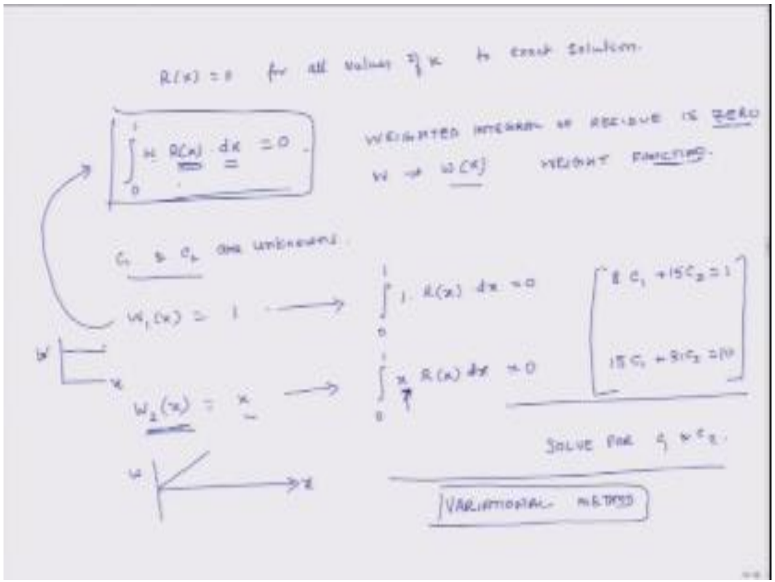

So here we had done the mathematics for the whole domain, in finite element we do it element by element same method, we find residue for each element multiply weight function integrate it, get equations at element level, find all the element equations, assemble them, apply boundary conditions then solve it, that is the thing. So this is called variational method.

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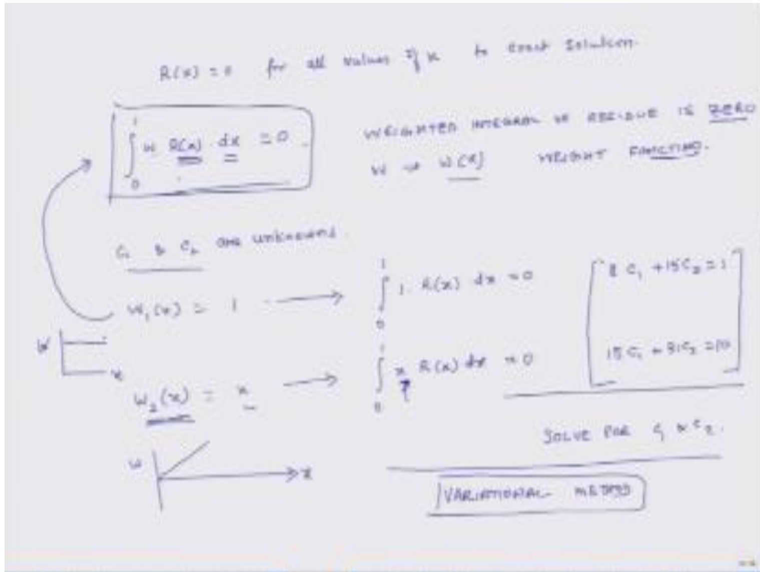

And there are different as I mentioned earlier, different ways to pick up weight functions and that is why there may be several flavors of variational method. So this is brings us to the conclusion of this lecture and we will continue our discussion next week, look forward to seeing you tomorrow, thanks.

<u>Acknowledgement</u>
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