

Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)
Course Title
Basics of Finite Element Analysis

Lecture – 48
Explicit and implicit method, diagonalization of mass matrix, closer

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Hello, welcome to basics of FEA, today is the last day of this course and in today's lecture we will cover two themes, explicit schemes and implicit schemes and then after discussing these two particular topics we will close our discussion for this course.

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EXPLICIT & IMPLICIT SCHEMES

$$[M] \{ \ddot{u} \} = [K] \{ u \} + \{ F \}$$
$$[K] \{ u \}_{n+1} = [K] \{ u \}_n + A [K] \{ u \}_n \quad A = \frac{(I - \alpha) K}{M}$$
$$[K] \{ u \}_{n+1} = [K] \{ u \}_n + A [K] \{ u \}_{n+1} \quad A = \frac{(I - \alpha) K}{M}$$

So what we are going to discuss is, and implicit schemes. And we will explain it by an example so that things will become clearer. So we will consider the example of the parabolic system of equations. So our parabolic equation we had defined as it was expressed like this. So this equals

okay. So this is the equation we had developed about three, four lectures back in time and here we had explained that $K^{\wedge}$ evaluated as $(s+1)$ time step equals $m1 + a1 [k]$ at $s+1$.

And $k^{\wedge}$ at s th time step is equal to $m1-a2 [k]$ at s and similarly we had, so this is our basic differential equation not differential the algebraic set of equations F was also defined earlier. Now $A1$ was defined as α times $\Delta t(s+1)$ and $a2$ was defined as $1-\alpha$ times Δt time at $s+1$. Now this α relates to the alpha family of approximations which we had discussed earlier.

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Handwritten notes on the slide:

HYPERBOLIC EQNS $C_1 \neq 0$
 Actually assume $C_2 = 0$ i.e. $\rho' = 0$.

$[M^*] \{\ddot{u}\} + [K^*] \{u\} = \{F\}$

NEWMARKA FAMILY OF APPROXIMATIONS FOR $\ddot{u}$ and $\dot{u}$.

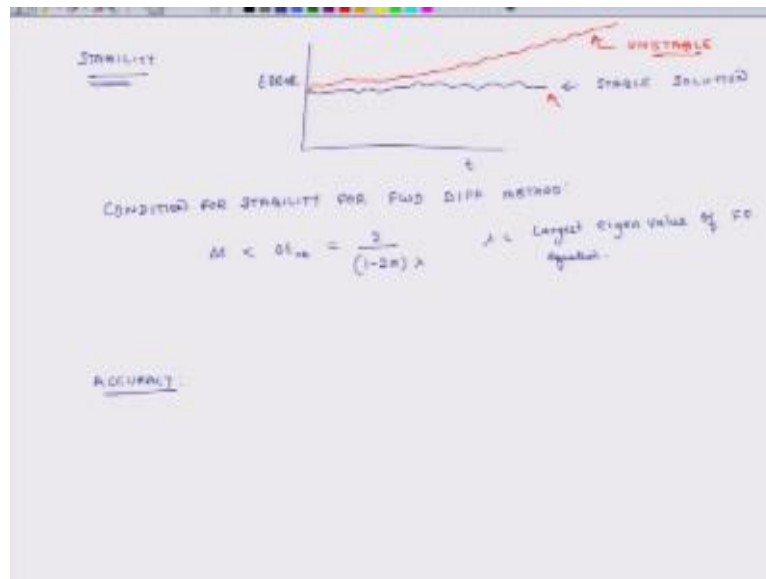
$\begin{bmatrix} \{u\}_{s+1} = \{u\}_s + \Delta t \{\dot{u}\}_s + \frac{\Delta t^2}{2} \{\ddot{u}\}_{s+\alpha} \\ \{\dot{u}\}_{s+1} = \{\dot{u}\}_s + \Delta t \{\ddot{u}\}_{s+\alpha} \end{bmatrix}$

$\{\ddot{u}\}_{s+\alpha} = (1-\theta) \{\ddot{u}\}_s + \theta \{\ddot{u}\}_{s+1}$ Use this to compute $\{\ddot{u}\}_{s+\alpha}$

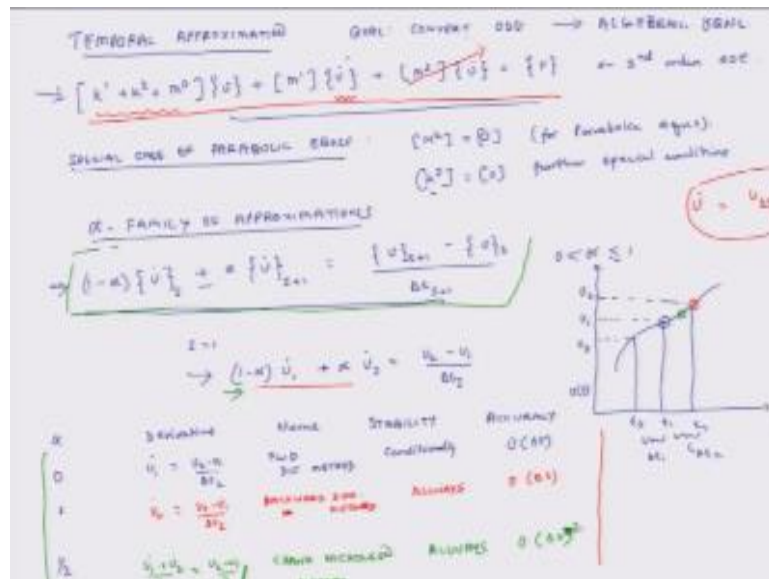
θ	γ	NAME	STABILITY
✓ $1/4$	$1/2$	Newmark-beta method	Stable ✓
$\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$	$1/3$	Linear accel. method	Conditionally stable ✓
	0	Central diff. method	Conditionally stable ✓
✓ $1/6$	$1/5$	Galupkin method	Stable ✓
✓ $2/3$	2	Backward diff. method	Stable ✓

$\Delta t < \Delta t_{crit} = \frac{2}{\omega_n}$ i.e. ω_n = Max Eigen Value of System

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And there were several choices of alpha's we could make which you see here. So alpha could be zero for forward difference method it could be one for backward difference method, it could be half for Crank Nicolson method, we had also said that it could be two thirds for Galerkin method. So we could choose one of these values of alpha and solve the equation which we have discussed earlier.

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$$\boxed{[K]_{s+1} \{U\}_{s+1} = \{F\}_{s+1}}$$

$$[K]_{s+1} = [K]_{s+1} + a_3 [M]_{s+1}$$

$$\{F\}_{s+1} = \{F\}_{s+1} + [M]_{s+1} (a_3 \{U\}_s + a_4 \{\dot{U}\}_s + a_5 \{\ddot{U}\}_s) \leftarrow$$

$$a_3 = \frac{\Delta t}{r(h)} \quad a_4 = \frac{\Delta t^2}{r(h)} \quad a_5 = \frac{\Delta t^3}{r} - 1$$

EL EOMS $\rightarrow$ ASSEMBLY $\rightarrow$ B.C. $\rightarrow$ IC $\rightarrow$ $\begin{cases} \dot{U} \\ U \end{cases} \rightarrow$ Explicitly known
 $\dot{U} \leftarrow$ Calculated

$$\{U\}_s = [M]^{-1} (F_0 - K\{U\}_0) \leftarrow$$

in terms of $U_1, \dot{U}_1, \ddot{U}_1$
in ... " $U_2, \dot{U}_2, \ddot{U}_2$

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EXPLICIT & IMPLICIT SCHEMES

$$\left[\hat{K} \right]_s \{u\}_s = \left[\hat{K} \right]_s \{u\}_s + \{f\}_{s+1} \quad - \text{Eq. 1}$$

$$\left[\hat{K} \right]_{s+1} = \left[m' \right] + a_1 \left[\hat{K} \right]_{s+1} \quad \left[\hat{K} \right]_s = \left[m' \right] - a_2 \left[\hat{K} \right]_s$$

$$a_1 = \frac{h}{\Delta t} \Delta t_{s+1} \quad a_2 = (-h) \Delta t_{s+1}$$

if $h=0 \quad a_1=0$

$$\left[m' \right] \{u\}_{s+1} = \left[\hat{K} \right]_s \{u\}_s + \{f\}_{s+1}$$

$$m' = \begin{bmatrix} \text{Symmetric} \end{bmatrix}$$

Now consider what happens if $\alpha = 0$. So if $\alpha = 0$ then $a_1 = 0$ which means that in the $k^\wedge$ this term goes away because a_1 is zero. So then this equation, so this is equation one it can be written as m_1 times $u(s+1)$ equals $k^\wedge$, $k^\wedge$ times u at s time step plus $f^\wedge$ s plus 1th time step.

That here the left side of the equation the $k^\wedge$ matrix becomes essentially m_1 matrix and it does not change with time, it does not change with time, okay. Now in general, in general my mass matrix not in general my mass matrix is always a symmetric. This mass matrix, why is it symmetric? Because when I see its definition of m_1 .

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Replace $u(x)$ with $\sum U_j(x) \phi_j(x)$ $u(x)$ with $\phi_j(x)$

$$\int_a^b \left[\frac{1}{2} \left(\sum_{j=1}^n U_j \phi_j' \right)^2 + b \sum_{j=1}^n U_j \phi_j'' + c_2 \sum_{j=1}^n U_j \phi_j + c_1 \sum_{j=1}^n U_j \phi_j + c_2 \sum_{j=1}^n U_j \phi_j' \right] dx - \int_a^b f(x) \sum_{j=1}^n U_j \phi_j dx$$

$$= \frac{1}{2} U^T K U + U^T F$$

$$\rightarrow \left[\frac{1}{2} K^1 + \frac{1}{2} K^2 + M^2 \right] \{U\} + [M^1] \{\dot{U}\} + [M^2] \{\ddot{U}\} = \{F\}$$

$$K_{ij}^1 = \int_a^b \phi_i \phi_j' dx \quad K_{ij}^2 = \int_a^b \phi_i \phi_j'' dx \quad M_{ij}^1 = \int_a^b \phi_i \phi_j dx$$

$$M_{ij}^2 = \int_a^b \phi_i \phi_j' dx \quad \{F\} = \{a\} + \{b\}$$

Then $m1$ corresponds to this function. So it is $\phi_1 \phi_i$ times ϕ_j . So $m1$ is always symmetric matrix.

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EXPLICIT & IMPLICIT SCHEMES

$$\boxed{[K] \{u\}_{n+1} = [K] \{u\}_n + \{F\}_{n+1}} \quad \text{--- EQ. 1}$$

$$[K]_{n+1} = [M'] + a_1 [K]_{n+1} \quad (K)_2 = [M'] - a_2 [K]_2$$

$$a_1 = \frac{\Delta t}{\tau} K_{n+1} \quad a_2 = \left(\frac{-\Delta t}{\tau} \right) K_{n+1}$$

If $\alpha = 0 \quad a_1 = 0$

$$[M'] \{u\}_{n+1} = [K] \{u\}_n + \{F\}_{n+1}$$

$$M' = \begin{bmatrix} \text{Symm} \\ \text{diag} \end{bmatrix} \quad \text{But it is NOT diagonal matrix.}$$

But it is not so this M' is but it is not, it is not diagonal matrix. It is symmetric but not diagonal, if we can somehow make this matrix diagonal, then what will happen? What we will be able to do is, that we will have non zero terms here, suppose the matrix is 3/3 or 4/4 suppose it is four-by-four then what will happen? So I am not saying the M' is diagonal, suppose we are somehow able to modify M' into a diagonal mass matrix.

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EXPLICIT & IMPLICIT SCHEMES

$$\boxed{\begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix}_{s+1} = \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix}_s + \begin{bmatrix} \hat{F} \\ \hat{G} \end{bmatrix}_{s+1}} \quad \text{--- Eqs. 1}$$

$$\begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix}_{s+1} = \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix}_s + \frac{\Delta t}{2} \begin{bmatrix} \hat{F} \\ \hat{G} \end{bmatrix}_{s+1} \quad \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix}_s = \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix}_{s+1} - \frac{\Delta t}{2} \begin{bmatrix} \hat{F} \\ \hat{G} \end{bmatrix}_{s+1}$$

$$\frac{\Delta t}{2} \hat{M} = 0 \quad \hat{a}_1 = 0$$

$$\begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix}_{s+1} = \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix}_s + \frac{\Delta t}{2} \begin{bmatrix} \hat{F} \\ \hat{G} \end{bmatrix}_{s+1}$$

$$\hat{M}' = \begin{bmatrix} \text{Sym} \\ \text{ctrl} \end{bmatrix} \quad \text{But it is NOT diagonal matrix.}$$

$$\begin{bmatrix} \hat{u}_1 & 0 & 0 & 0 \\ 0 & \hat{u}_2 & 0 & 0 \\ 0 & 0 & \hat{u}_3 & 0 \\ 0 & 0 & 0 & \hat{u}_4 \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \\ \hat{u}_4 \end{bmatrix}_{s+1} = \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \\ \hat{a}_4 \end{bmatrix}_{s, s+1}$$

Element level

And let us say the diagonal mass matrix is a four-by-four term. So this term is not zero so all these are zeroes. But the diagonal term, all the terms in this are zero except for the diagonal terms. So the diagonal term is a non-zero number, this is a non-zero number, this is a non zero number, this is a non-zero number, okay. So basically I am rewriting equation one with a diagonal mass matrix.

Which is not same as m_1 and I have not explained how I am going to make it diagonal but somehow by magic I am able to make it diagonal then this is multiplied by u_1, u_2, u_3, u_4 , let us say these are four nodes of an element, these are evaluated at time step $(s+1)$ and this equals this $k \wedge \text{times } U \text{ times } F$, so essentially this will be some vector, right? Which is a combination of s and $s+1$ time step, right? Associated with s and $s+1$ time step.

So this is some number let us call it a_1, a_2, a_3, a_4 and these numbers are n_1 non zero numbers. So n_1, n_2, n_3, n_4 , okay, so this is at an element level. Now when I assemble all these element level matrices I get the assembly level then I do boundary condition I apply initial condition do all that good stuff but essentially what I will be left with is.

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EXPLICIT & IMPLICIT SCHEMES

$$\boxed{[K] \{u\}_{n+1} = [K] \{u\}_n + \{F\}_{n+1}} \quad \text{--- Eq. 1}$$

$$\{u\}_{n+1} = [M]^{-1} [K] \{u\}_n + [M]^{-1} \{F\}_{n+1}$$

$$a_1 = \frac{1}{n} \Delta t^2 \quad a_2 = (-\frac{1}{n}) \Delta t^2$$

If $\Delta t = 0 \quad a_1 = 0$

$$[M] \{u\}_{n+1} = [K] \{u\}_n + \{F\}_{n+1}$$

$M = \begin{bmatrix} \text{Sym} \\ \text{diag} \end{bmatrix}$ But it is NOT diagonal matrix.

$$\begin{bmatrix} N_1 & 0 & 0 & 0 \\ 0 & N_2 & 0 & 0 \\ 0 & 0 & N_3 & 0 \\ 0 & 0 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix}$$

$i, i+1$ Element level

A set of equations of n where this matrix, the mass matrix on the left side will be diagonal. So I do not have to invert this assembled mass matrix to solve for u_1, u_2, u_3, u_4 because what is the first equation? N_1 times u_1 equals A_1 . So I can directly calculate u and s , A_1/N_1 understood? So if so what does that mean? That if I choose α to be zero and what that means is that my K matrix becomes same as M .

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EXPLICIT > IMPLICIT SCHEMES

$$\left[\begin{matrix} M \\ 0 \end{matrix} \right] \begin{Bmatrix} u \\ \dot{u} \end{Bmatrix}_{t_{n+1}} = \left[\begin{matrix} M \\ 0 \end{matrix} \right] \begin{Bmatrix} u \\ \dot{u} \end{Bmatrix}_t + \begin{Bmatrix} F \\ 0 \end{Bmatrix}_{t_{n+1}} \quad \text{--- EQ. 1}$$

$$\begin{Bmatrix} \dot{u} \end{Bmatrix}_{t_{n+1}} = \begin{Bmatrix} M^{-1} \end{Bmatrix} + \alpha_1 \begin{Bmatrix} \dot{u} \end{Bmatrix}_{t_{n+1}} \quad \begin{Bmatrix} \dot{u} \end{Bmatrix}_t = \begin{Bmatrix} M^{-1} \end{Bmatrix} - \alpha_2 \begin{Bmatrix} \dot{u} \end{Bmatrix}_t$$

$$\alpha_1 = \frac{M}{M} \Delta t_{n+1} \quad \alpha_2 = (1-\alpha) \Delta t_{n+1}$$

If $M = 0 \quad \alpha_1 = 0$

$$\begin{Bmatrix} M^{-1} \end{Bmatrix} \begin{Bmatrix} u \end{Bmatrix}_{t_{n+1}} = \begin{Bmatrix} M^{-1} \end{Bmatrix} \begin{Bmatrix} u \end{Bmatrix}_t + \begin{Bmatrix} F \end{Bmatrix}_{t_{n+1}}$$

$M^{-1} = \begin{bmatrix} \text{Symm} \\ \text{diag.} \end{bmatrix}$ But it is NOT DIAGONAL MATRIX.

$$\begin{bmatrix} M_1 & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 \\ 0 & 0 & M_3 & 0 \\ 0 & 0 & 0 & M_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}_{t_{n+1}} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}_{t_{n+1}}$$

Element level
EXPLICITLY SOLVE FOR u

And then if I do some special trick to make my mass matrix as diagonal then I can explicitly solve for u 's, I can explicitly solve for u and that is a much faster way because to invert a matrix is much more numerically computationally intensive but I can explicitly solve for u and that is a very fast method of solving the problem, okay. So this method where we choose α to be zero and somehow diagonalise the matrix is known as explicit method of solving the problem, a time dependent problem so this description is valid for time-dependent problems.

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EXPLICIT & IMPLICIT SCHEMES

$$\left[\begin{matrix} \hat{u} \\ \hat{v} \end{matrix} \right]_{n+1} = \left[\begin{matrix} A \\ B \end{matrix} \right] \left[\begin{matrix} u \\ v \end{matrix} \right]_n + \left[\begin{matrix} F \\ G \end{matrix} \right]_{n+1} \quad - \text{FEQ. 1}$$

$$\left[\begin{matrix} \hat{u} \\ \hat{v} \end{matrix} \right]_{n+1} = \left[M' \right] + \cancel{A_1} \left[\begin{matrix} u \\ v \end{matrix} \right]_{n+1} \quad \left[\hat{u} \right]_n = \left[M' \right] - A_2 \left[u \right]_n$$

$$A_1 = \frac{\Delta t}{\Delta x} A \Delta y_{n+1} \quad A_2 = \frac{(\Delta t)^2}{\Delta x} A \Delta y_{n-1}$$

If $\Delta t \rightarrow 0 \quad A_1 \rightarrow 0$

$$\left[\underline{M'} \right] \left[\begin{matrix} \hat{u} \\ \hat{v} \end{matrix} \right]_{n+1} = \left[\begin{matrix} \hat{u} \\ \hat{v} \end{matrix} \right]_n + \left[\hat{F} \right]_{n+1}$$

$M' = \begin{bmatrix} \text{Sym} \\ \text{anti} \end{bmatrix}$ But it is NOT diagonal matrix.

$$\begin{bmatrix} H_1 & 0 & 0 & 0 \\ 0 & H_2 & 0 & 0 \\ 0 & 0 & H_3 & 0 \\ 0 & 0 & 0 & H_4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}$$

Element level
Explicitly

By default we do not do that, so that default method is known as the implicit method it is an implicit scheme for solving the problem, okay. But in this case this is an explicit method where I am not inverting the mass matrix or the stiffness matrix and I directly solve explicitly solve for the unknowns which I use. So this is known as explicit method. So that is what I mean by explicit scheme and implicit scheme.

In implicit scheme we have to invert we have to invert because the matrix on the left side is not diagonal, in explicit schemes we do not need to do that and we are able to get to that situation by making two things happen, first we choose α to be zero and second one is we diagonalize the mass matrix.

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EXPLICIT & IMPLICIT SCHEMES

$$\boxed{[\hat{u}]_2^{n+1} = [\hat{u}]_2^n + \{\hat{r}\}_{2, n+1}} \quad \text{--- EQ. 1}$$

$$[\hat{u}]_{2, n+1} = [m'] + a_1 \cancel{[\hat{u}]_{2, n+1}} \quad [\hat{u}]_2 = [m'] - a_2 [\hat{u}]_2$$

$$a_1 = \frac{\Delta t}{\Delta x} A_{2, n+1} \quad a_2 = \left(\frac{-\Delta t}{\Delta x} \right) A_{2, n+1}$$

$\Delta t \neq 0 \quad a_1 \neq 0$

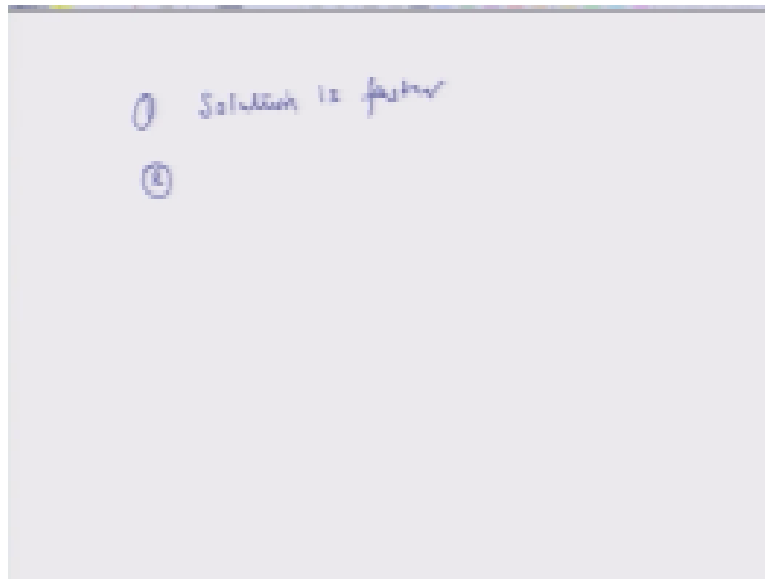
$$[m'] \{\hat{u}\}_{2, n+1} = [\hat{u}]_2^n \{\hat{u}\}_2 + \{\hat{r}\}_{2, n+1}$$

$m' = \begin{bmatrix} \text{Symmetric} \end{bmatrix}$ But it is NOT DIAGONAL MATRIX.

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}_{n+1} = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}_{1, 2, n+1}$$

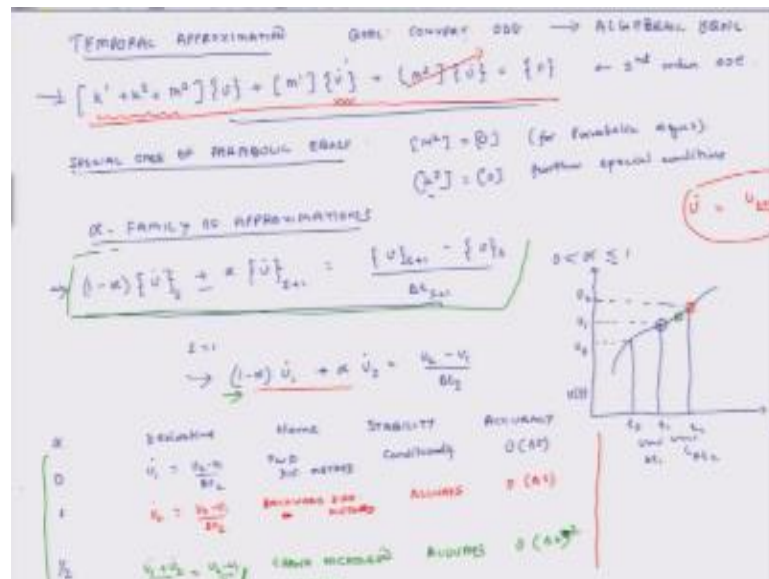
Element level
EXPLICITLY SOLVE FOR u

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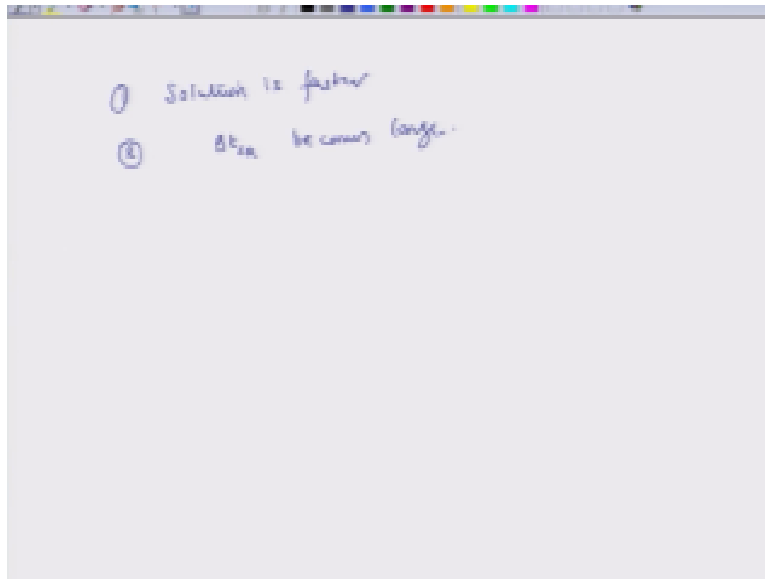
Now the advantage of explicit scheme is one is that solution is faster, okay. Solution is faster that is one advantage, the second thing is that when you diagonalise the mass matrix it just happens that the value of t critical.

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Now if you remember the forward scheme is conditionally stable, okay. Forward scheme is conditionally stable, now that conditional stability means that my time increment has to be smaller than the critical value.

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It just happens that when you diagonalise the mass matrix the value of t critical becomes large. It becomes large. So I can jump from 1 time to second time to third time in larger steps. So I get two benefits one is solution is faster because my solution process is explicit.

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EXPLICIT & IMPLICIT SCHEMES

$$\boxed{[M] \frac{du}{dt} \bigg|_{s+1} = [M] \frac{du}{dt} \bigg|_s + \{F\}_{s+1}} \quad - \text{Eq. 1}$$

$$[M] \frac{du}{dt} \bigg|_{s+1} = [M'] + a_1 \frac{du}{dt} \bigg|_{s+1} \quad [M]_s = [M'] + a_2 [M]_s$$

$$a_1 = \Delta t \Delta t_{s+1} \quad a_2 = (-\Delta t) \Delta t_{s+1}$$

If $\Delta t = 0 \quad a_1 = 0$

$$[M'] \frac{du}{dt} \bigg|_{s+1} = [M] \frac{du}{dt} \bigg|_s + \{F\}_{s+1}$$

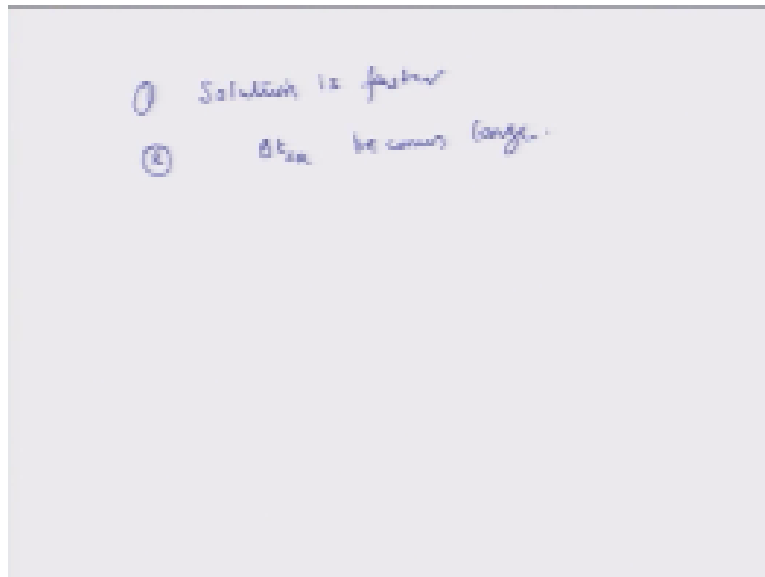
$M' = \begin{bmatrix} \text{Symmetric} \end{bmatrix}$ But it is NOT DIAGONAL MATRIX.

$$\begin{bmatrix} M_1 & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 \\ 0 & 0 & M_3 & 0 \\ 0 & 0 & 0 & M_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}_{s+1} = \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix}_{s, s+1}$$

Element level
EXPLICITLY SOLVE FOR u

I do not have to invert the matrix.

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The second one is t_{critical} becomes large so I can take larger time increments and that also gives me another advantage. So because of these two factors explicit schemes are in general faster ways of solving the problem but they may not be applicable in all sorts of situations in special situations we can use the advantage of this thing, that is what I wanted to talk about explicit and implicit schemes so the next thing in this context is we will learn how to diagonalize, we will learn how to diagonalize the mass matrix okay.

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$$[M]_C = \frac{\rho h_e}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \rightarrow \text{consistent element mass matrix}$$

Way 1 For each row, add up element values and put the sum in diagonal element. Other diagonal elements - put to zero.

$$[M]_D = \frac{\rho h_e}{30} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \frac{\rho h_e}{6} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Row-Sum Lumping

We will so suppose my original mass metrics at let us ay original mass metrics is a 3/3 matrix suppose it is a 3 noded element so let us say and the values are row h_e over 30 and the internal values are 4 2- 1 2 16 2 -1 2 4 okay. Now this is the original mass matrix which I have calculated from the relations which we discussed earlier using the weak formulation. So this is known as consistent mass matrix, actually consistent element metrics because the way we calculated these numbers is consistent with the original theory which we have developed.

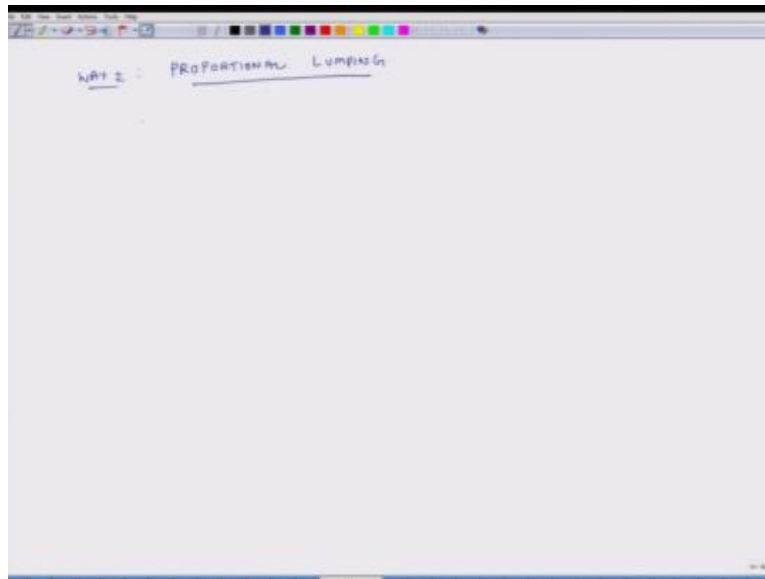
But now we will do some approximations and we will try to diagonalize it, there are two ways to diagonalize way 1 okay, what you do is you add up so for each row add up element values and put the sum in diagonal element, other non diagonal elements, values are put to 0 so for non diagonal elements you put the value to 0 so what do we do, so this is consistent so I am putting this subscript C so, so what we do is for each row we add up the element values and we put the sum of those values in the diagonal element and all other non diagonal elements are set to 0.

So this is my consistent mass matrix M_C and my diagonal matrix and I will call it M_D and some people call it M_L as because we are lumping all the elements in one position so M_L or M_D and this is equal to row h_e over 30 and what is the sum of all the elements in the first row it is 4+2 + -

1 so it is 5, so that is the diagonal element value and other 2 values are 0, the second row if I add up all the elements it is $2 + 16 + 2$ so that comes to 20 so that goes in the second diagonal element and other values are 0.

In the third row the sum is $-1+2+4$ so it is 5 and the non diagonal elements are 0 okay, so this I can simplify it as $\text{row } h^e \text{ over } 6$ 1000, 4100 excuse me this should be 0 1 this way of diagonalizing the mass matrix is known as row sum lumping, is known as row sum lumping because we are summing up all the row elements and are lumping it into one particular position of the diagonal, one diagonal element okay so we can lump it in this way.

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Now the other way to do it is, so second way and this is known as proportional lumping okay so in proportional lumping

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Handwritten notes on a digital whiteboard showing the assembly of a consistent element mass matrix $[M]_G$.

The first equation shows the assembly of the consistent element mass matrix:

$$[M]_G = \frac{\rho h}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \rightarrow \text{consistent element mass matrix}$$

Below this, the text "WAY 1" is written. To its right, a note explains the process: "For each row, add up element values and put the sum in diagonal element. Non diagonal elements: put to zero."

The second equation shows the assembly of the global mass matrix $[M]_D$ using the row-sum lumping method:

$$[M]_D = \frac{\rho h}{30} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \frac{\rho h}{6} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Below the second equation, the text "ROW-SUM LUMPING" is written and underlined.

Our original mass matrix.

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WAY 2: PROPORTIONAL LUMPING

$$[M]_C = \frac{\rho h_e}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

Add all elements of matrix:

$$4 + 2 + (-1) + 2 + 16 + 2 + (-1) + 2 + 4 = 30$$

$$4:16:4$$

$$1:4:1$$

$$[M]_D = \frac{\rho h_e}{30} \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{bmatrix}$$

$$D_1 = \frac{30 \times 1}{1+4+1} = 5$$

$$D_2 = \frac{30 \times 4}{1+4+1} = 20$$

$$D_3 = 5$$

The non diagonalized version which is M_C was row h^e over 30, 4 2 -1 2 16 2 -1 2 and 4 okay so what we see is we look at the diagonal elements and we see that they are in the ratio is 4:16 is to 4 or you can call it 1:4:1. So what I do is first add all elements of the matrix, so what do I get 4+2+ - 1 +2+16+2+- 1+2+4 so that equals 30, this equals 30 and now I split it amongst the diagonal elements in this proportion so let us say this I call this so M diagonal is equal to row h^e over 30 and I have to figure out what are the values of D_1 , D_2 and D_3 .

So $D_1 = 30 \times 1/1+4+1$ $D_3 = 30 \times 4/1+4+1$ so this equals what is it 5 and this equals 20 and $D_3 = 5$, so this is known as proportional lumping approach it just turns out that.

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$[M]_e = \frac{\rho h^2}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \rightarrow \text{consistent element mass matrix}$

WAY 1 For each row, add up element values and put the sum in diagonal element. Other diagonal elements put as zero.

$$[M]_D = \frac{\rho h^2}{30} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \frac{\rho h^2}{6} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Row-Sum Lumping

In this case the proportional lumping and the row sum lumping

(Refer Slide Time: 20:06)

Handwritten mathematical derivation for proportional lumping of a matrix. The title is "WAT 2: PROPORTIONAL LUMPING".

The matrix $[M]_c$ is given as:

$$[M]_c = \frac{\rho h_e}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

The diagonal elements of the matrix are identified as:

4 : 16 : 4
1 : 4 : 1

The sum of all elements of the matrix is calculated:

$$4 + 2 + (-1) + 2 + 16 + 2 + (-1) + 2 + 4 = 30$$

The diagonal matrix $[M]_D$ is then defined as:

$$[M]_D = \frac{\rho h_e}{30} \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{bmatrix}$$

The diagonal elements D_1 , D_2 , and D_3 are calculated as follows:

$$D_1 = \frac{30 \times 1}{1 + 4 + 1} = 5$$
$$D_2 = \frac{30 \times 4}{1 + 4 + 1} = 20$$
$$D_3 = 5$$

Gives us the same values but this may not necessarily be always true, it may not be always true but in these two met ways we can do the diagonalization, it is important to note that when we do proportional lumping.

(Refer Slide Time: 20:27)

The image shows handwritten notes on a whiteboard. At the top, a matrix $[M]_G$ is defined as $\frac{P_{1p}}{30}$ multiplied by a 3x3 matrix with elements $\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$. An arrow points to the text "consistent element mass matrix". Below this, "WAY 1" is written. A note says "For each row, add up element values and put the sum in diagonal element. Other diagonal elements put to zero." Below this, the matrix $[M]_D$ is shown as $\frac{P_{1p}}{30}$ multiplied by a 3x3 matrix $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. This is then simplified to $\frac{P_{1p}}{6}$ multiplied by a 3x3 matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The final result is underlined and labeled "Row-Sum Lumping Gr".

$$[M]_G = \frac{P_{1p}}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \rightarrow \text{consistent element mass matrix}$$

WAY 1

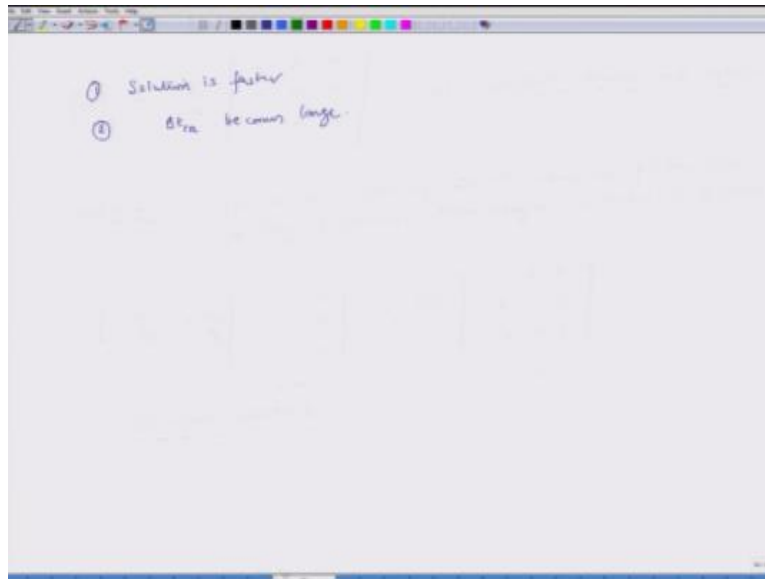
For each row, add up element values and put the sum in diagonal element. Other diagonal elements put to zero.

$$[M]_D = \frac{P_{1p}}{30} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \frac{P_{1p}}{6} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Row-Sum Lumping Gr

Or row sum lumping we have to make sure that.

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(Refer Slide Time: 20:37)

Handwritten notes on a whiteboard showing the assembly of a consistent element mass matrix $[M]_G$.

The initial matrix is given as:

$$[M]_G = \frac{\rho h^3}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \rightarrow \text{Consistent element mass matrix}$$

The elements 4, 2, and -1 in the first row are circled in green.

WAY 1

For each row, add up element values and put the sum in diagonal element. Other diagonal elements put as zero.

The resulting matrix is shown as:

$$[M]_G = \frac{\rho h^3}{30} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \frac{\rho h^3}{6} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The first matrix is underlined in green, and the second matrix is underlined in green.

Row-Sum Lumping Gp

See this mass matrix is multiplied by a vector of unknowns now it just happens that if we are solving a 1D problem and I have only one single primary variable let us say U then does not matter, but in beams there are two primary variables W and W prime okay then we cannot add these terms in a straightforward way because the first term will correspond to the displacement, the second term will respond to the slope, third term, so then we have to when especially.

(Refer Slide Time: 21:19)

The image shows a handwritten derivation for proportional lumping of a mass matrix. At the top, it is titled "WRT 2: PROPORTIONAL Lumping". The first equation is $[M]_c = \frac{\rho h^2}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$, with the diagonal elements 4, 16, and 4 circled in red. To the right, the ratios $4:16:4$ and $1:4:1$ are written in red. Below this, the text "Add all elements of matrix:" is written. The second equation is $[M]_b = \frac{\rho h^2}{30} \begin{bmatrix} 21 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. To the right of this, the calculation for the diagonal elements is shown: $4+2+(-1) + 2+16+2 + (-1)+2+4 = 30$. Then, the diagonal elements are calculated: $D_1 = \frac{30 \times 1}{1+4+1} = 5$, $D_2 = \frac{30 \times 4}{1+4+1} = 20$, and $D_3 = 5$.

When we are doing proportional lumping we have to separately find out the mass and the moment of inertia terms separately and then do proportional lumping accordingly understood, the row is also we have to make sure of that thing. So I think this concludes our discussion for this course, this has been an eight week long course and we started this course by making introductory remarks, we explained the philosophy of finite element method, we explained what are nodes, shape functions, our approximation functions, elements, we figured out how to calculate these approximation functions for different situations.

We also learned stuff about residual approaches, variational methods, weighted residual methods and then how from weighted residual method we come down to weak formulation strategies to solve differential equations and, and then we started applying some of these methods in context of beams, bars, heat conduction problems, and once we were became comfortable with static problems then we started working on Eigen value problems and time-dependent problems, so this concludes our discussion of this course.

And the course comes to an end. I am sure that all of you who have been patiently listening to these eight week long lectures you have benefited from this course and so your exams will be

due on 20th and 27th of March so please prepare for them, all these exams both of these exams, so you will be expected to take only one exam but there will be two dates on which this exam is going to happen, so these exams will be multiple choice questions but to solve these multiple choice questions you will be expected to do detailed calculations, so it will not be just so you have to review your FEA course in detail, practice it, and I am sure that you will do fine and if you have any questions please send us emails or communicate to us and we will be happy to address any of your questions. Once again thank you very much patiently being a part of this course and have a great day and best wishes for future, thank you.

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