Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)
Course Title
Basics of Finite Element Analysis

Lecture – 47
Temporal approximation for hyperbolic problems

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Hello, welcome to basics of finite element analysis and in today's lecture what we will discuss is another set of equations known as hyperbolic equations and the important thing about these equations is that they involve second order derivatives in time, so in case of parabolic equations which we have discussed in the last several lectures, 2, 3 lectures we figured out how to handle first order derivatives of time and how to integrate equations ordinary differential equations in time of first order using the alpha family of approximations, what we will do today is hyperbolic equations.

And these, as these equations have second order of derivatives in time we will use a different integration method known as a Newmark family of integration to convert these partial differential ordinary differential equations second order ordinary differential equations in time to algebraic equations.

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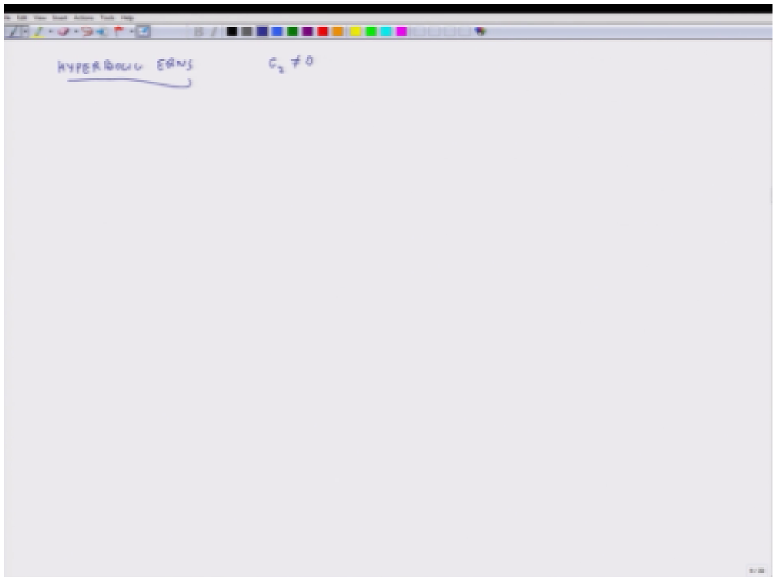

So as I mentioned what we are going to discuss today are hyperbolic equations and the conditions for a hyperbolic equation is that c_2 should not be equal to 0.

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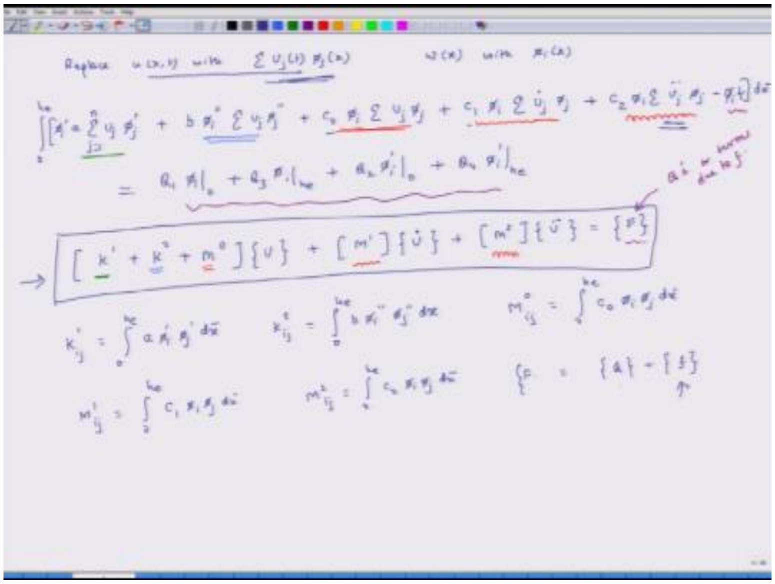

So what is c_2, c_1, c_0, c_1, and c_2 are listed in our original differential equation so this should not be = to 0 because that is what gives us the second order derivative in time.

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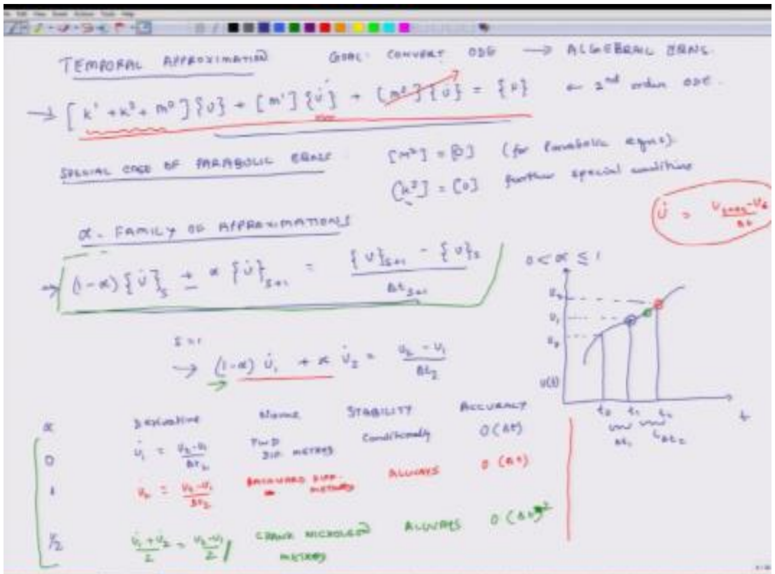

So but.

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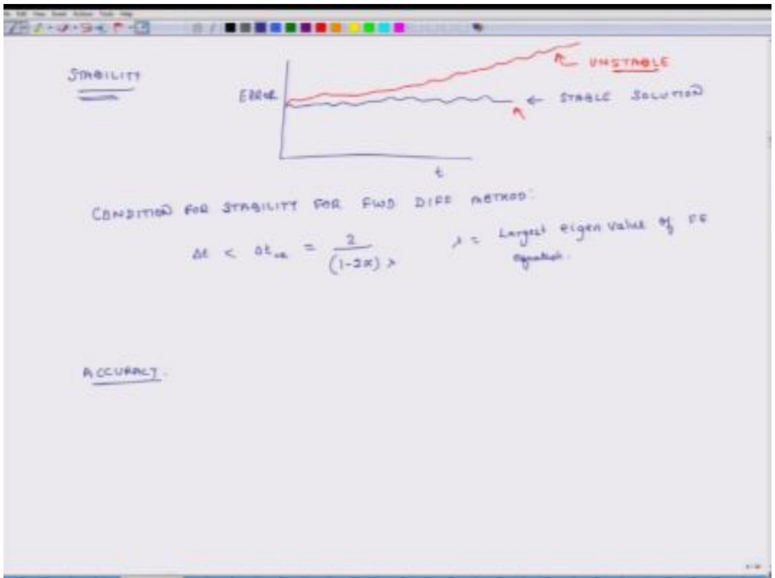

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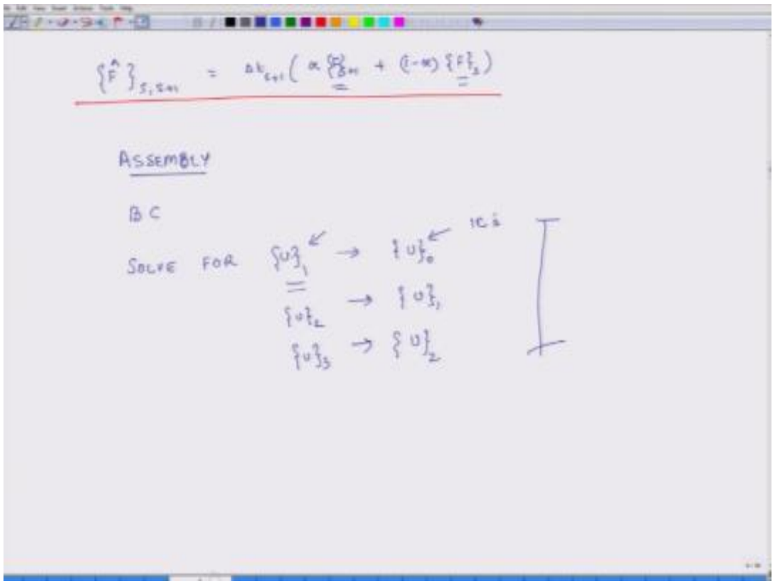

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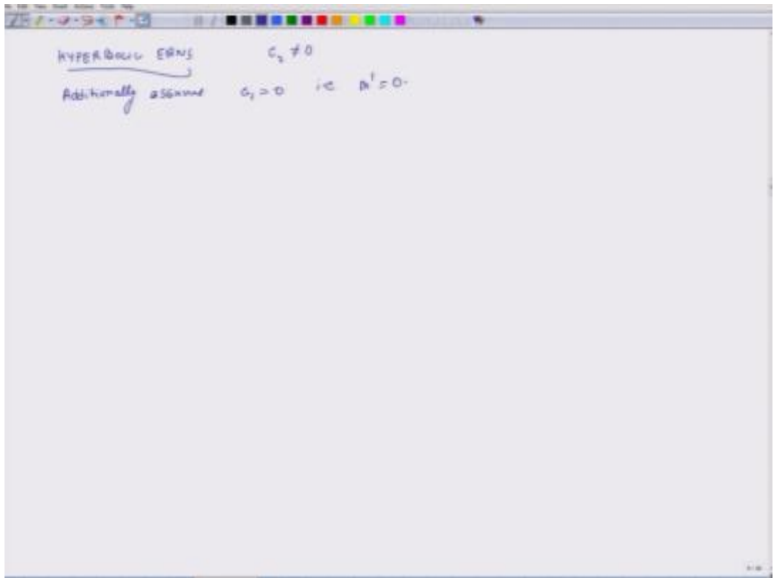

In context of today's class so this is the condition but we will also additionally assume, assume that c1=0 and that implies m` = 0 and not m` m1 m1=0.

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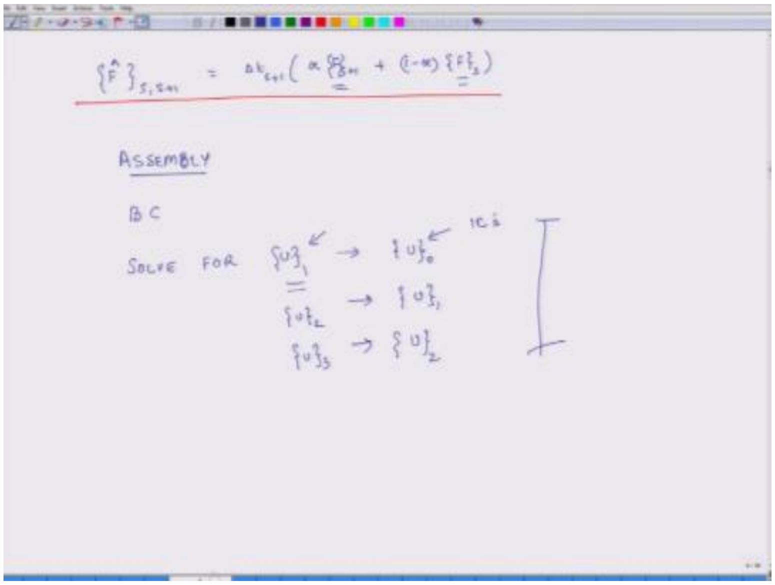

Physically what it means.

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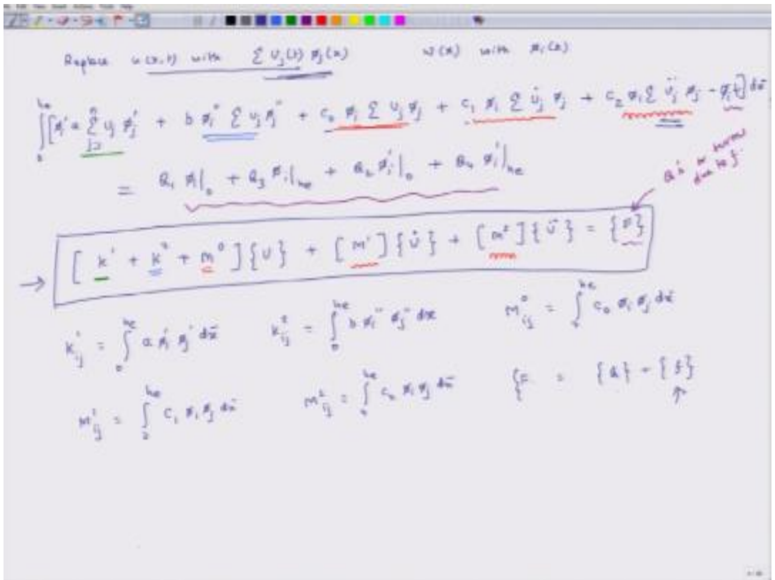

Is that this term.

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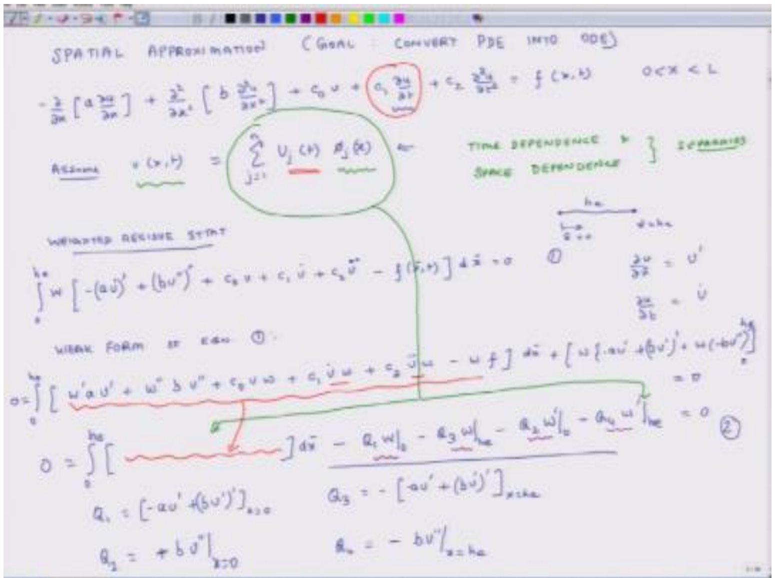

This term is 0 in the special type of hyperbolic equations which we will discuss today and physically what this means is c_1 is some constant times du and dt if u is displacement then du and dt would be velocity and so c_1 times velocity is in several systems representative of viscous forces or damping forces, so in context of the discussion we will have in our class today we will say that.

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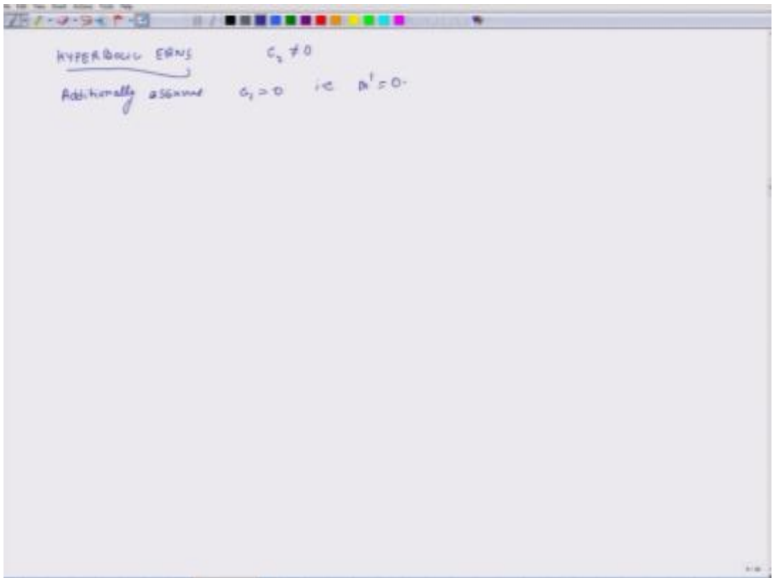

The damping of viscous effects in our system are negligible or they can be ignored, so for such a system.

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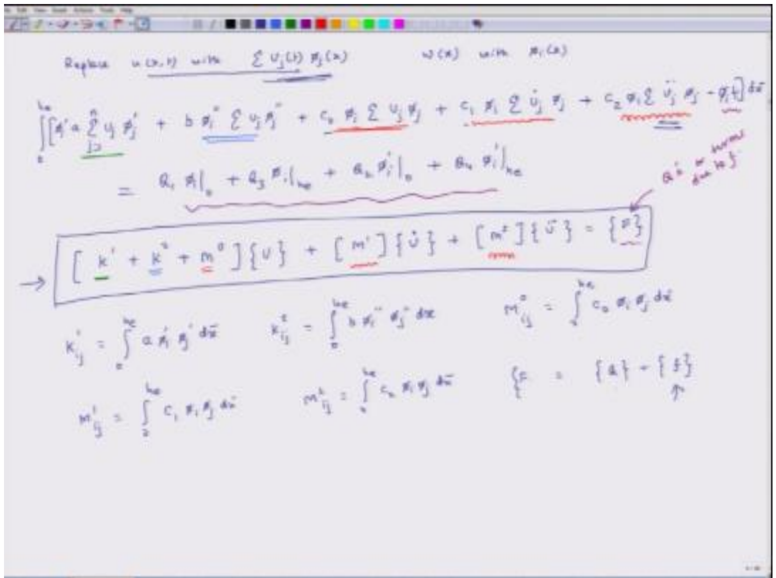

The equation, this entire equation

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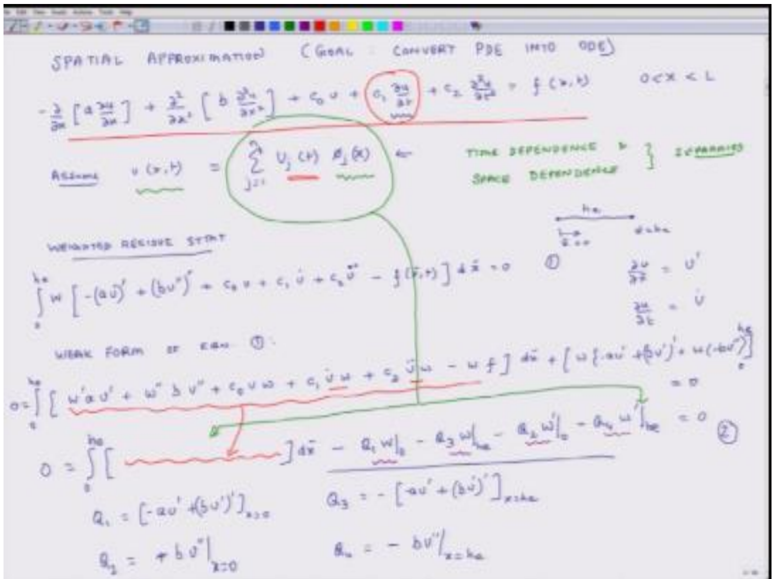

This entire equation it essentially

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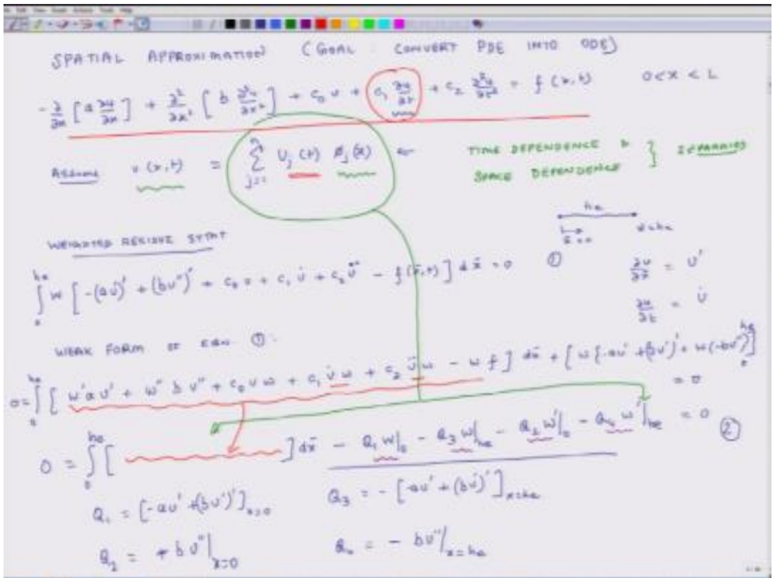

Reduces to

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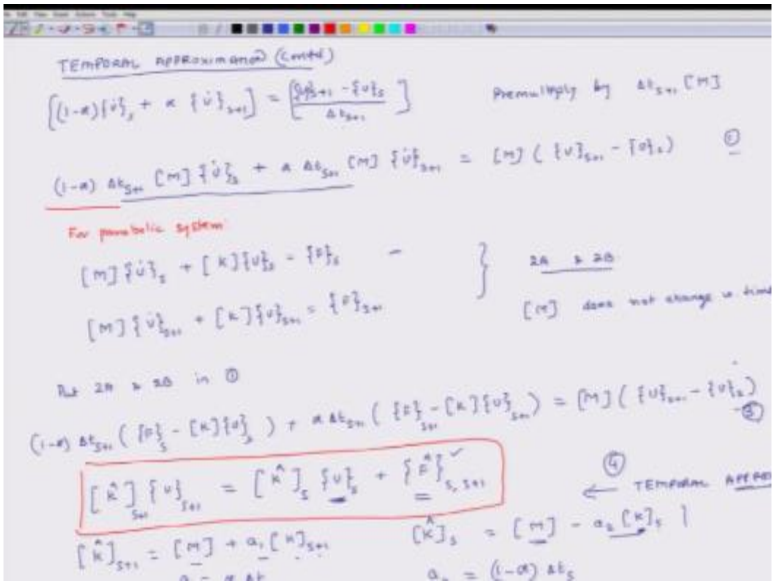

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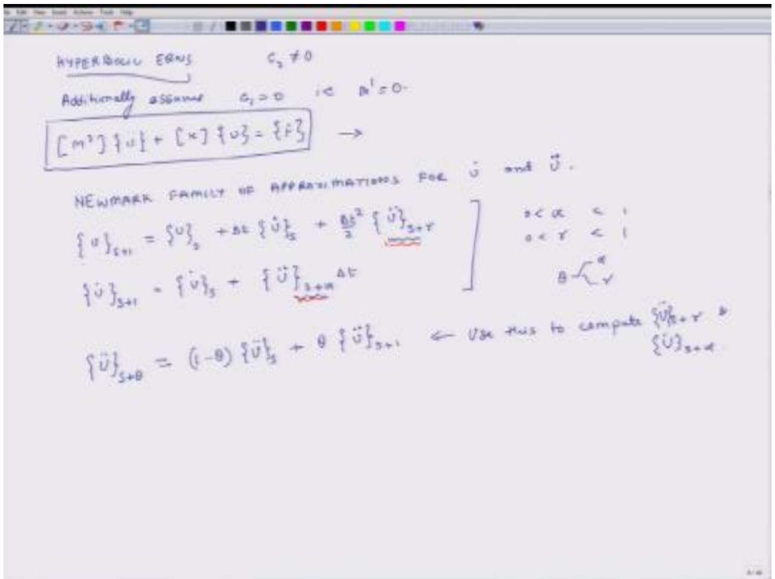

This form, so that is the hyperbolic equation and how we arrive at this equation we have seen earlier through our regular method of the special approximation. so now our aim is that we have to first integrate this, so in this equation we have m2 which is the mass matrix k gave which is the stiffness matrix and F which is the force vector, so all these things are known and we have to now express or find out the value of u at different nodes for all periods for all time instant and our aim will be to get that information by integrating it.

So that we get a set of algebraic equations, so those algebraic equations we will get through an integration scheme known as Newmark scheme, so we get it through Newmark family of approximations for u dot and the second derivative which is u double dot and the relations are U_{S+1} = U at time step S +Δt times U dot at time step S+Δt^2 over 2 times U at time step S+γ so γ is a different term I have introduced and we will explain what that means, and similarly the velocity or the first derivative in time at step number S+1=u at times step S +$\ddot{U}$ at time step S+ α times Δt.

So an α is between 0 and 1 and so is γ, α could be a number between 0 and 1 and so could γ so the question is, so I have, so what this family of approximation helps us do is it helps us calculate u and u dot at time step S+1. On the right side we know u dot and u at time step S, we also know the value of Δt we can choose it, we also know the acceleration said time step S but we do not understand what is how to calculate u at S+ γ and U at S+ α okay so that is, so this is actually the second derivative.

So to calculate S $\ddot{u}$ at S+ γ or S+ α we use another formula described in the same for min the same family of approximations and it says that the second derivative of U at S+ θ now θ could be either α or it could be γ so we will give a relation for this and we can use this relation to calculate $\ddot{U}$ at S+γ and $\ddot{U}$ at S+ α so U at S+ θ=1- θ times $\ddot{U}$ at S +θ times $\ddot{U}$ at S+1, so use this to compute US+γ and US+ α, so if I + θ= α then I get US+ α if I get θ= γ then I get S+γ.

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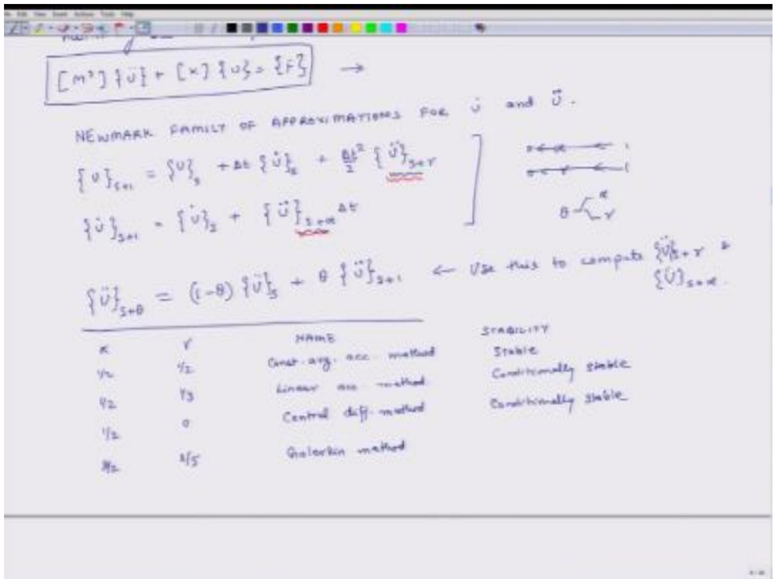

Now the next question is what should be the values of θ, α and γ okay what should be the values of θ and γ so for that we will develop a table, so α and γ name of the method and stability, so in the first case α =1/2 γ =1/2 this method is known as constant average acceleration method, and the stable stability thing is that it is always stable so I do not have to worry about Δt it will always be stable, if I take Δt as very large it may become inaccurate but it will not become unstable so I can use this method.

Or I can also use a α =1/2 γ =1 3^{rd} and this method is known as linear acceleration method, this is conditionally stable. Another set of α and γ could be ½ γ could be 0 this is called central difference method, this is also conditionally stable and their option could be 3 over 2 and 8 over 5 and this is known as Galerkin method. So one small error which I made was that these conditions are not accurate α they can exceed 1 but they have to be more than 0.

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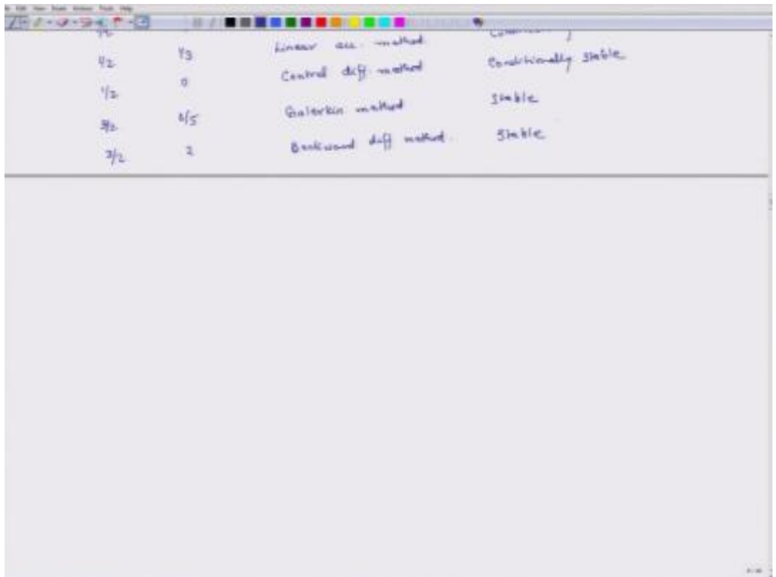

So in this Galerkin method this is all again stable and another value could be 3 over 2 and this is 2 this is known as backward difference method and this is also stable.

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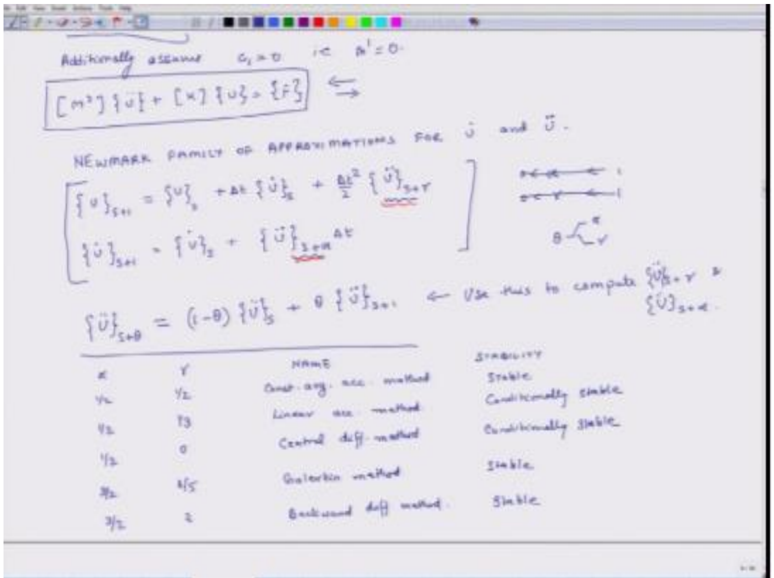

So this is, this gives us the overview and what it says is that if I have to solve this equation m_2 times $\ddot{U}$+k times U=F I can express $\ddot{U}$ which are the second order derivatives in time using these 2 relations okay, I mean I can I have to move this you using these two relations and then using n, and once I do that I will be able to convert this equation into a set of algebraic equations, so and in while using these I have to choose.

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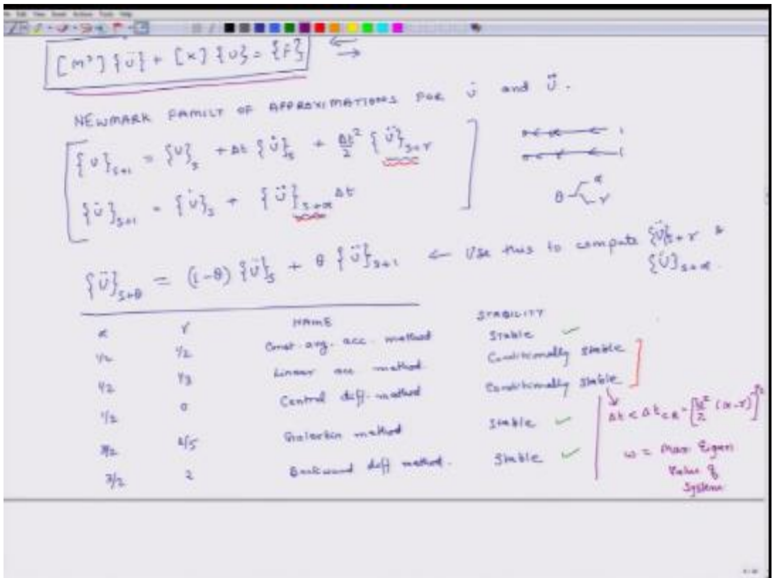

An appropriate value of γ and an appropriate value of α and I have several choices and out of these 5 choices two are conditionally stable and the remaining 3 are always stable. In this case the conditional stability is when, so what is the conditional stability, when Δt is less than Δt critical and this equals $\Omega^2/2$ times $\alpha - \gamma$ the whole thing based to the power of ½ okay, where Ω is maximum Eigen value of system, so once again first if we have to use these 2 methods linear acceleration central difference method.

Which are conditionally in stable we have to ensure that the time increment is less then time critical and that equals square root of Ω^2divided by 2 times $\alpha - \gamma$ and Ω is the maximum Eigen value of the system so I have to calculate the Eigen values by putting F=0 in this equation.

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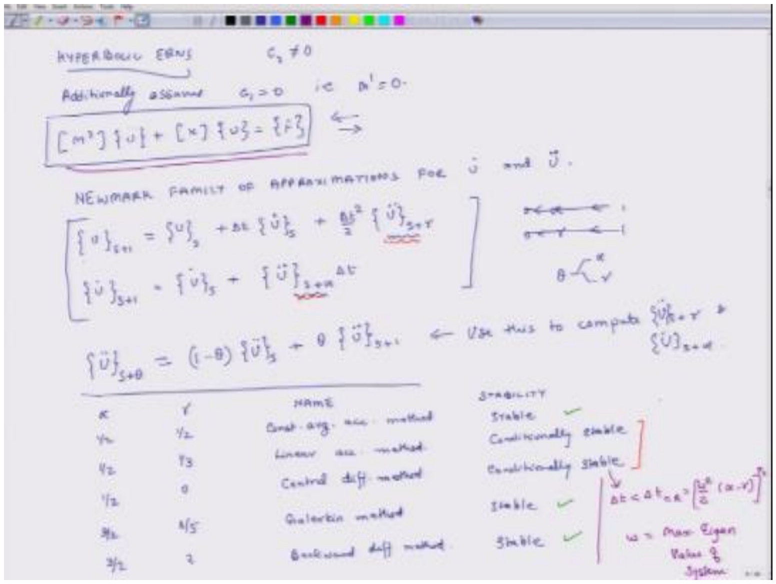

So next what I will do is I will straighten away write the solution of this equation because you know the philosophy and the theory and at this stage for the purpose of the – we will straight away write the solution of this equation.

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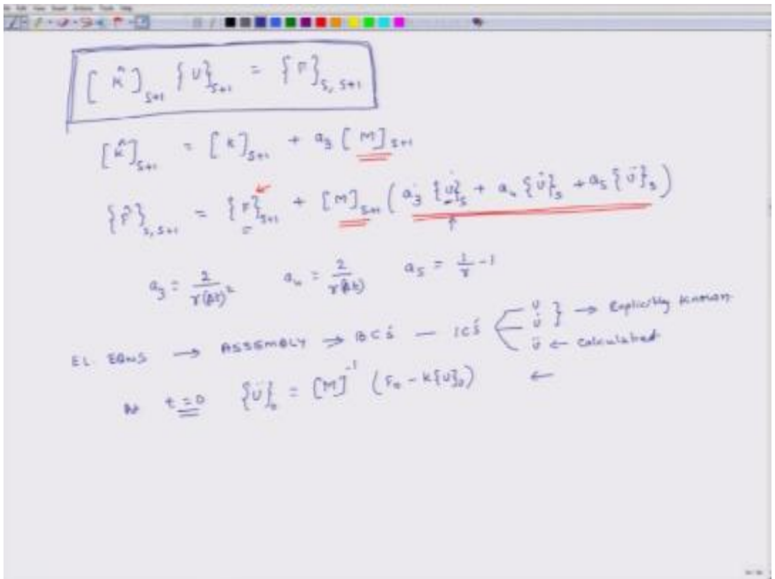

And the solution is K^ so this is for the hyperbolic equation where the damping is also 0 so K^ evaluated at S+1th time step times U evaluated at S+1 time step=F corresponding to S and S +1, and here K^ evaluated, so this is the equation we have to solve, so if I can calculate the left side vector the right side vector F and on the left side if I can calculate k^ it gives me an assembly element level equations I can assemble that do all the boundary conditions essential to this application and then I will be able to figure out U.

At S+1th time step okay, but we have to define what are K^'s and F at S and S+1 so K^=K evaluated at S+1th time step plus a constant a_3 times mass matrix evaluated at S+1 time step and typically this mass matrix does not change with time so we do not have to worry, I mean once we calculate it remains the same, but the force vector it certainly changes with time and F corresponding to S and s+1 is defined as F evaluated at S+1th time step and we know how to calculate this.

+M_{S+1} multiplied by a3, us+a4 $U^{\cdot}$ S+a5 U.S okay, now as I said typically mass does not change with times so this is more less fixed so we do not, once we calculate it remains the same in a lot of situations for all the things. Force we know that this dependents on F and Q's right and if we know the values of Q's from the last time and also we know that the F force vector is known function so I can calculate this function also, this vector F also, but these terms have to be known to calculate F at S, S+1 okay so I have to know the position.

If U is the position if U represents position then I have to know U that is the position at last previous time step I have to know $\ddot{\bar{U}}$ or velocity at previous time step and I have to know acceleration at previous time steps, oh by the way a4 I have to define a3, a3=2 over γtimes Δt^2 a4=2 over γ times Δt and a5=1 over γ- 1 so all these a_3,a_4,a_5 they are known because this γ is known Δt is known everything but I have to know the solution at the previous time step, now when I solve for this equation how do I do that.

I first do element equations, then I do assembly, then I do boundary conditions, then I do boundary conditions, but I have to know to solve for the next time step I have to know the solution at the previous time step, so I have to know so what that tells means that I have to know and then I apply ICS and I have so what are the 3 ICS I need U,U. and U.. okay, now these two they have to be explicitly known, this U.. at the last time step I can calculate so for instance at time so this is actually calculated.

At time t=0, for instance at time t=0 first thing what we will do is we will calculate U and how we will U we will calculate U.. at time step 0 as such that M inverse – F_0 – K through this relation right because our original equation is, what is our original equation.

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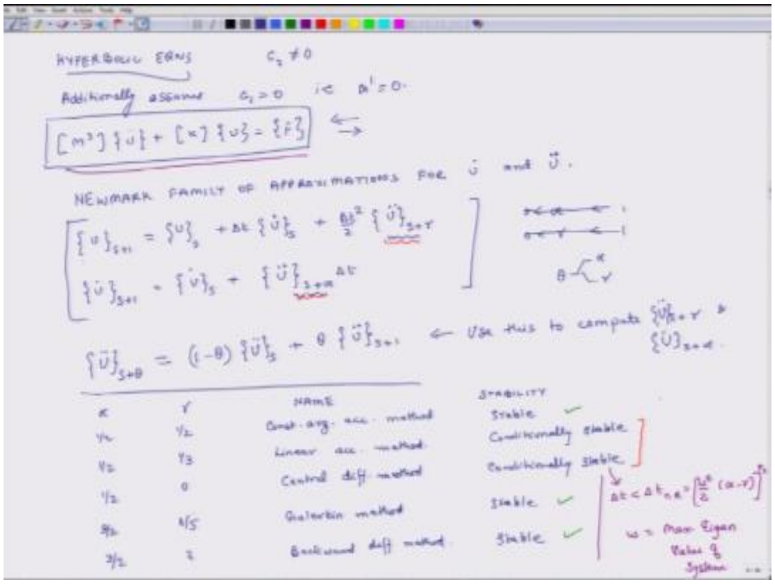

M2 U..+K_u=F so I use the same equation.

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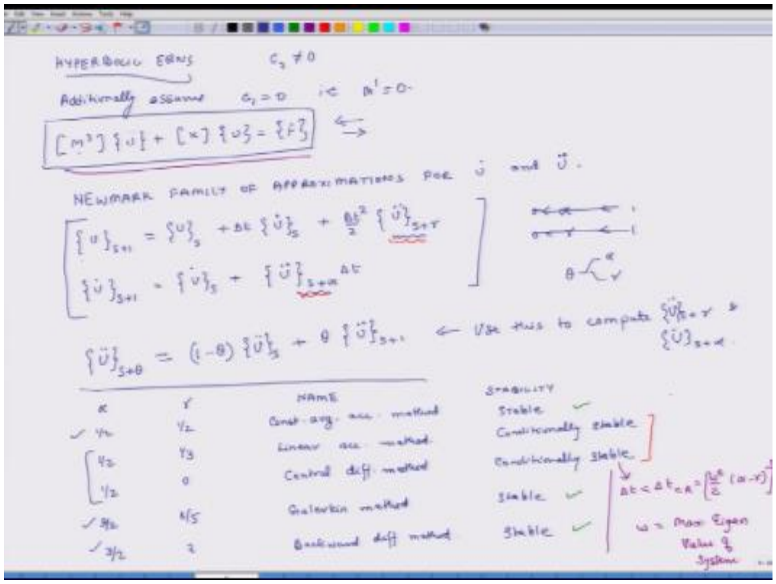

And so this should be, so I use the same equation to calculate acceleration set at time t=0. I plug these, these, these values are so this acceleration, velocity, and U, U, U prime U, U.. and the acceleration into this equation and then using that approach I find U_1 in terms of $U_{0,}$ U0. and U0.. okay, then I do go to the next time steps, I calculate U2 in terms of U_1, U1., U1.. and so on and so forth so this is how I keep on marching forward in time.

And the last thing is that at the end of each time step at the end of each time so suppose I have calculated U_1 then I have to calculate for the next time step I have to calculate velocities and accelerations, so I calculate that and I march forward I may keep on marching forward and our solutions will be stable if we use these 2values of γ and α they may be conditionally stable if I use these 2 one of these 2 sets of α and γ provided my time increment is smaller than the critical time step which as calculated here.

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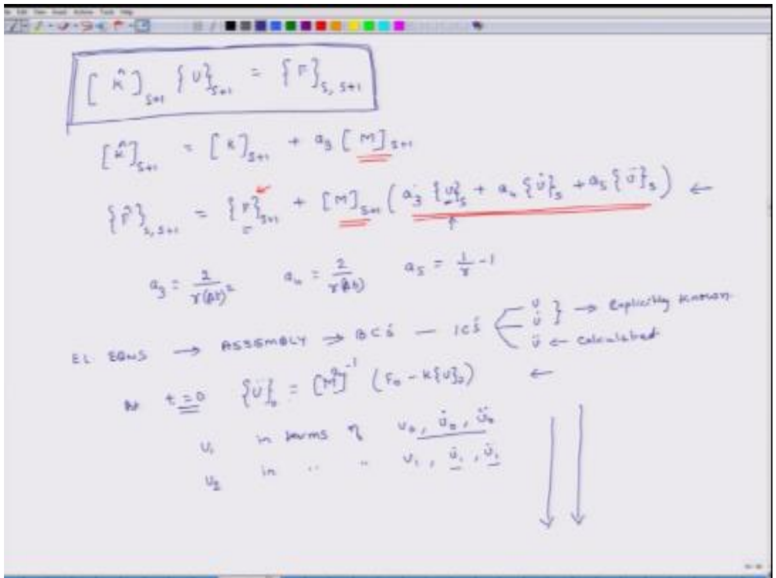

So this is the conclusion of our today's discussion on hyperbolic equations and in tomorrow's lecture we will close the entire discussion on FEA, but before that we will also cover two small topics known as explicit method and implicit method, so with that we conclude for today and we will meet once again tomorrow, so thank you very much, bye.

Acknowledgement

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