Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)
Course Title
Basics of Finite Element Analysis

Lecture – 44
Spatial approximation

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Hello, welcome to basics of finite element analysis, today is the eighth week, second day of this particular course. In the last lecture we had discussed some basics about of time dependent problems and also we had provided an overview as to how these time dependent problems are solved. What we will do today is that we will execute the first step of that overall method and in context of the differential equation which we had explained earlier we will do a special approximation for that particular partial differential equation. And as a consequence of that we would get rid of the partial derivative and change it to, change the PD into an ordinary differential equation.

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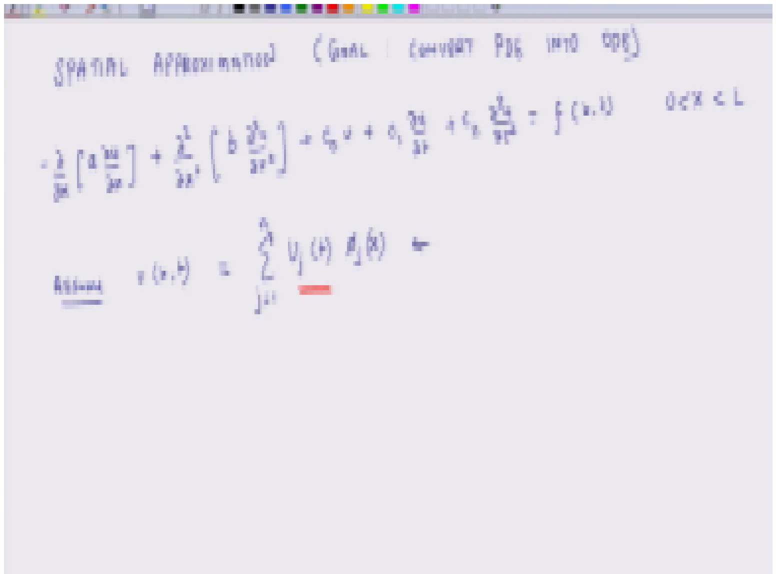

So our team for today is special approximation and our goal is convert PDE into ODE, okay. So we will rewrite that the original differential equation which was ∂/∂x [a ∂u/∂x] plus second derivative with respect to x of b ∂^2 u/ ∂x ^{2} and then I have c0u + c1 ∂u/∂t + c2∂^2 u/ ∂t ^{2} equals a source term which depends on position and time and this equation is valid for values of x which lie between l and 0.

And then of course there are boundary conditions and initial conditions which we had listed in the last lecture for purposes of brevity I am not going to write them here. So for special approximation what we do is that we assume a form of u. So we know that u is dependent on x and time. So u is the unknown, we know a, b, c0, c1, c2, f in this equation, what we do not know is u.

And u is varying with respect to x and time. So we are trying to figure out what this function is. So we assume that u is a function of, an explicit function of time so I will call it Uj and an explicit function of x. And it could be an sum of a lot of these functions. So here j is equal to 1 to n. So this particular assumption is fundamentally different from the assumptions which we made when we did not have time dependent problems, there this term Uj was not changing with respect to time.

Here the term Uj which physically would represent displacements of nodes or the values of primary variables at specific nodes that can fluctuate with time, and that makes sense. So suppose there is a beam which is vibrating like this then the nodal displacements of the beam are not fixed in space, they are functions of time okay.

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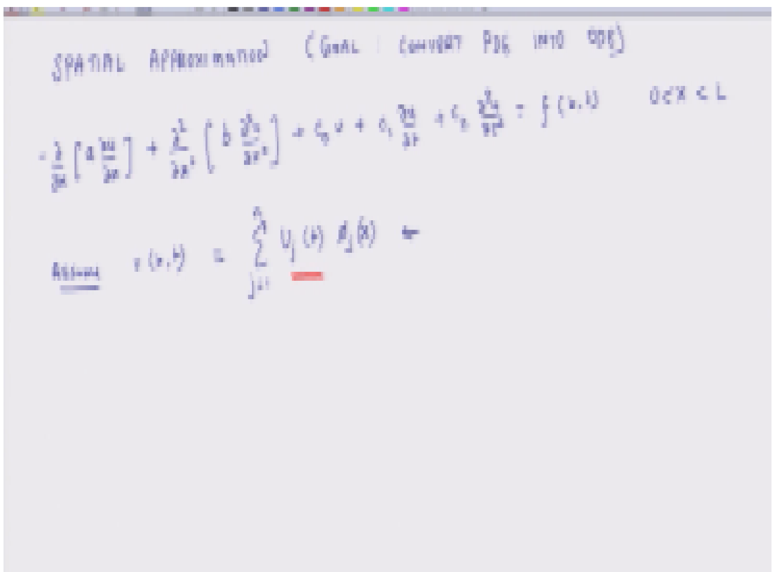

So, so we are assuming that these nodal displacements at least in context of beam or nodal primary variables they are functions of time and the variation from one node to the other note happens in the same way.

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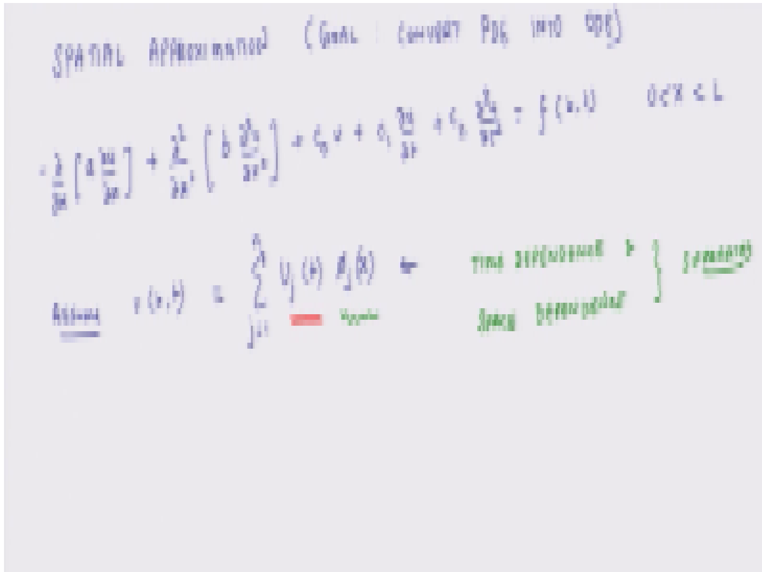

As we assumed in case of static problems where time was not a parameter, so this is an important thing. The second, so this is one thing, the second thing we have assumed here is that the time dependency, time dependency or time dependence and space dependence they are separated, they are separated, what does that mean? What that means is that there is one function which purely which is Øj which purely depends on X and there is another function Uj which purely depends on time.

So they are mutually separated so here we have assumed that variables are separable, this is what we have assumed that variables are separable and that makes our mathematics easier. Now this variable separable approach it makes sense if that reflects reality and in that case things the solutions we get are very accurate, I mean if I keep on increasing my number of nodes things become more and more accurate.

But in certain problems these variables may not necessarily be separable. So in that, those cases the nodal primary variables may not only be dependent on time but they may also depend on the positions especially nonlinear systems the position of node may itself change in our, in a beam when the beam fluctuates up and down by enlarge the position of a particular node let us say this is a beam and this is one particular node which we are interested in, the x-coordinate of this beam when it is when it vibrates by a little bit by little amount.

The x-coordinate does not change much. So its nodal displacement is not a strong function of x, it is primarily a function of time okay. But if violet is pending and if it changes significantly then there could be situations that these nodal primary variables are not only dependent on time but because their coordinates themselves are shifting so in that case they are also dependent on x. So in those kind of complex problems which are significantly nonlinear.

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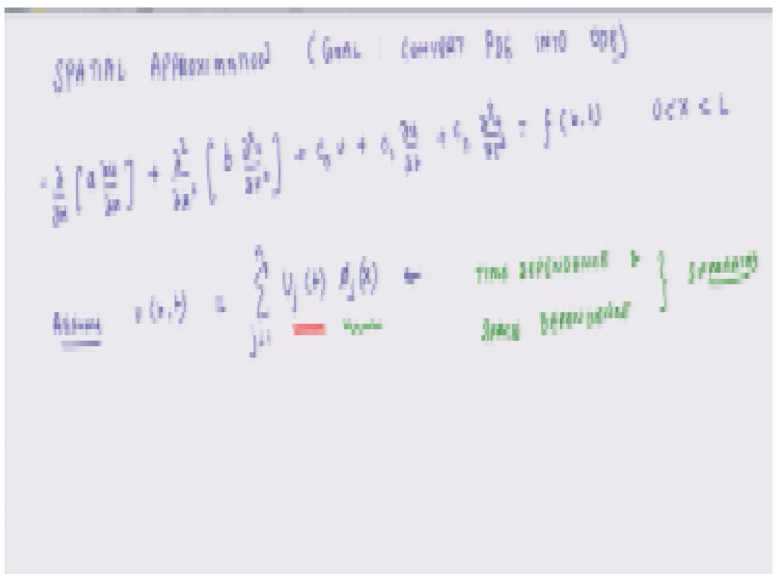

This variable separable approach may not necessarily work and give us very good results but even in those situations it will give us moderately decent results in a large number of circumstances. So in the context of this week of lectures we are looking at those class of problems.

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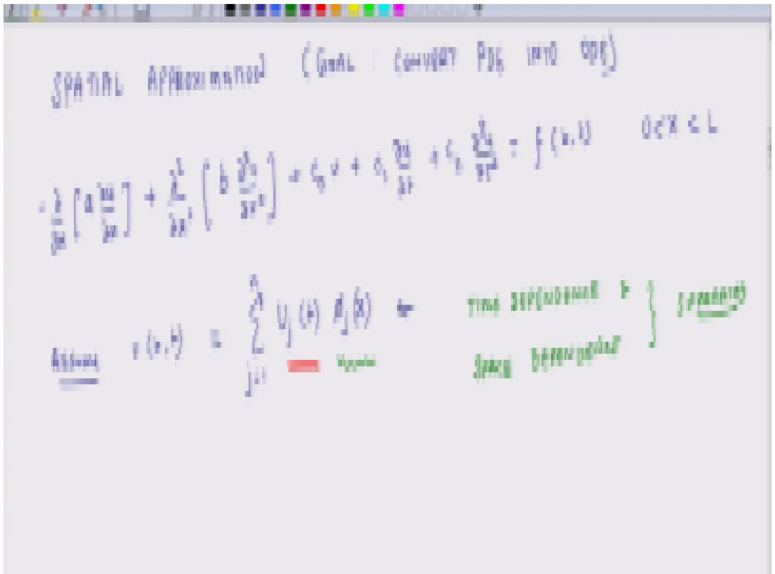

Where the time dependent part and the space dependent part of the solution can be separated out easily because that helps us convert this partial differential equation right away into a, an ordinary differential equation. So this is something extremely important we have to understand and now what we will do is.

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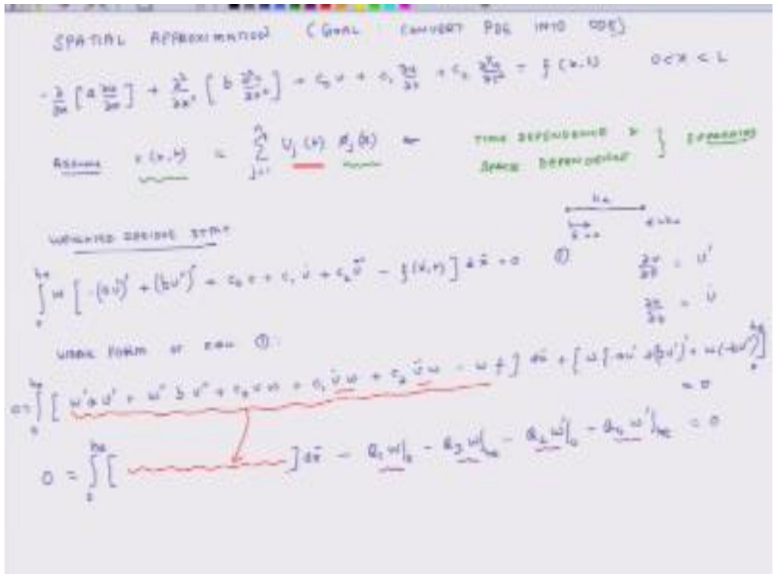

We will plug this form of solution into, into this integral equation and equate it to zero. So what we are going to do is we are going to plug this expression for u into the partial differential equation and calculate the residue, multiply that residue by a weight function, integrate that entire thing over the domain and equate it to zero. So let us say, so let us say I have an element and its size is h_e and I am going to work in local coordinate system.

So here my x- bar is equal to 0 and here x- bar is equal to h_e, okay. Couple of other things, going forward for purposes of brevity I may sometimes use $\partial/\partial x$ maybe written as prime. So if I have $\partial u/\partial x$ then I may express it as u prime. Similarly if I have $\partial u/\partial t$ then I may express it as u-dot, okay. And specifically I am going to express not X but $\partial u/\partial x$ bar because my local coordinate system is in terms of a variable x- bar.

So we are going to equate the weighted residue to 0. So this is my weighted residue statement and the weighted residue statement is zero to h_e integral some weight function W okay. In parenthesis I have my $(-au)'$ and the entire thing is differentiated with respect to x plus $(bu)''$ entire thing differentiated twice with respect to x- bar plus $c_0u + c_1u. + c_2..$ – f(x, t), t x-bar is equal to zero. So this is my weighted residue statement and now what I am going to do is we can the differentiability conditions on the partial derivatives for X, okay.

So we are going to develop a weak formulation for this. So weak formulation, weak for, so let us say this equation 1, soweak form is again integral 0 to h_e, W will also have to be differentiated so [w a u′ + w″ b u″ + c0u w + c1u. w + c2u.. w – w(x)] integrated with respect to x- bar and then when some I will get some boundary terms but before I write those boundary terms I wanted to explain that when we are weakening the differentiability we are weakening it only on the x parameter.

We are not weakening it on the time parameter. So we are not, because my W the weight function we are assuming that purely depends on x, okay. So we do not shift, shift the differentiability operator to W, when in context of time we only shift it in context of position of x. So then in addition to this I have several boundary terms and those terms are W [- au′ + bu′] the entire thing differentiated once plus w[- bu″].

And these boundary terms I am going to evaluate at locations h and h_e and this entire expression is equal to 0, so this entire expression equals zero. So I will rewrite this expression 0 h_e and whatever is here the same thing goes in this bracket so this entire expression comes here so I am not going to repeat it, but I will now more explicitly state the boundary terms.

So these boundary terms are Q1 w evaluated at zero minus Q3 w evaluated at h_e minus Q2 w′ evaluated at zero minus Q4 w″ evaluated h_e, excuse me this should be only the first derivative and this entire thing equals zero. So these are my boundary terms and these when we do the assembly then we, when we assemble all these elements level equations then the value of these boundary terms essentially comes from our boundary conditions, okay.

So when we are, so till so far we had do not see the need the for initial conditions explicitly because all what we have done till so far is we have integrated this partial differential equation in space. So we are getting several boundary terms and these boundary terms comes from, get handled by boundary conditions. When we integrate this equation in time then we will see that the need for initial condition also comes into picture.

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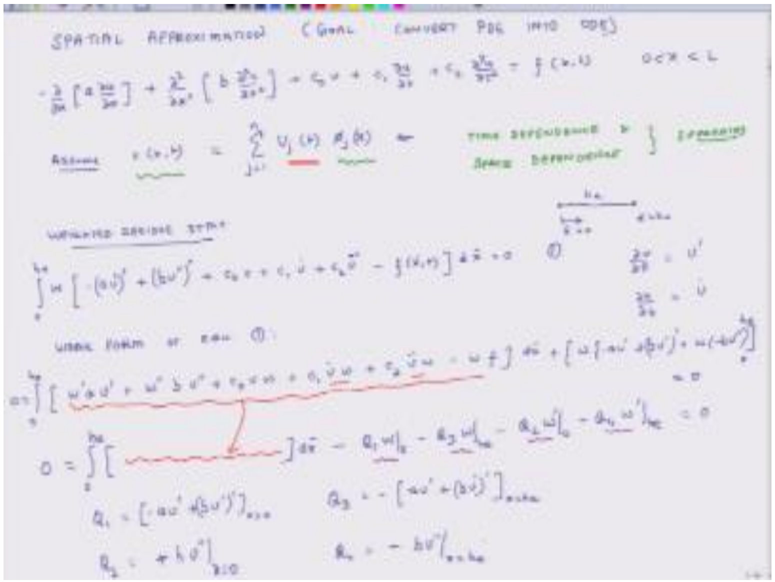

So here Q1 equals $-au' + bu'$ the entire thing differentiated twice again and this is evaluated at x is equal to 0. Similarly Q3 equals $-[a - au' + bu'']$ evaluated $x = h_e$. And Q2 equals $-bu''$ evaluated at $x = 0$ and Q4 equals now there should be a plus sign here and minus vu prime time evaluated $x = h_e$.

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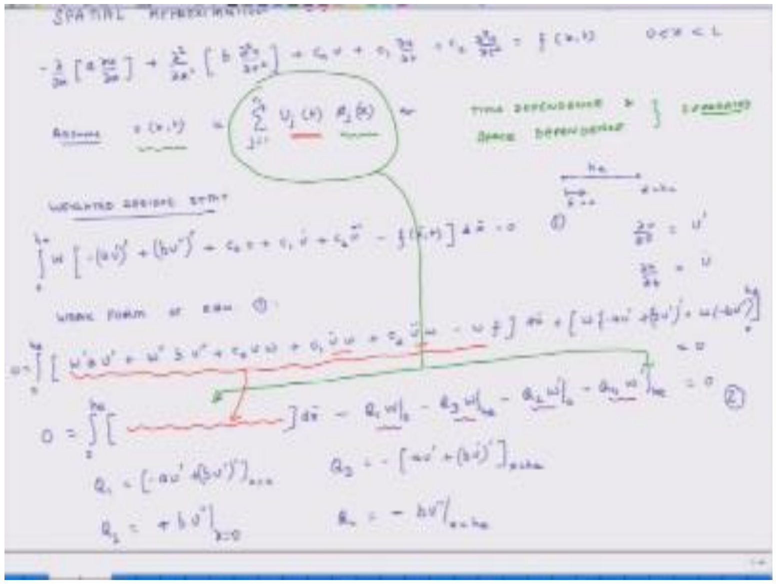

So we have weakened the differentiability operator and now what we will do is, so this is let us say this is equation two and what we are going to do is we are going to use this form of u and we will substitute it in equation 2 in this equation okay.

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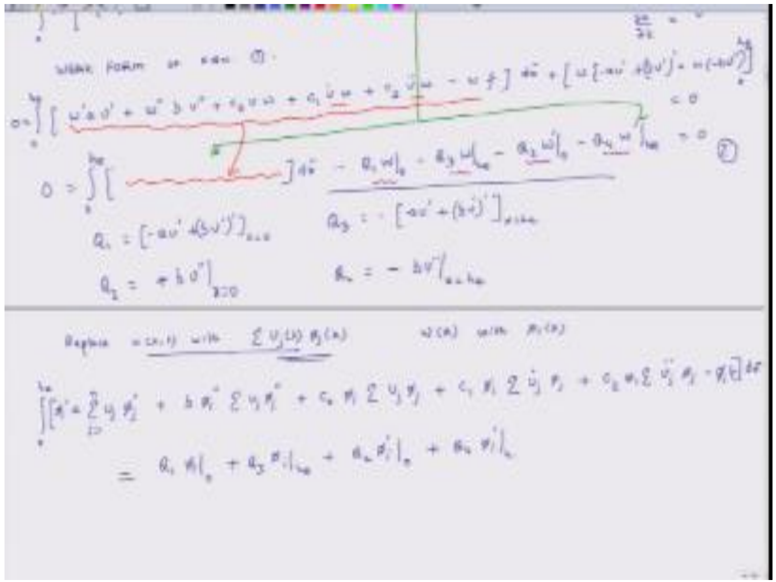

So let us do that, so what we are doing is we are going to replace u(x, t) with ∑(u, j) which is a function of time, times Øj which is a function of x, okay. So this is one substitution we will do and the other thing we will do is we will replace w which depends only on x with Øi which is a function of x. So I am going to rewrite my equations again. So for purposes of reference we are looking at this equation which has lot of terms underlined in red.

So w prime is nothing but Øi prime times a, times u ′, so u′ is ∑(uj) and for purposes of brevity I will not explicitly write uj as uj as a function of time and then Øj and j equals 1 to n and and this is differentiated in next plus bØi second derivative times ujØj second derivative plus c0Øi ujØj plus c1Øi ∑ of, and here u is differentiated in time. So when I differentiate this expression in time I get derivative of uj.

So I get uj dot and then I get Øj + c2Øi summation of and once again the c2 term has second derivative in of u in time. So I get uj differentiated twice in time, times Øj – wf dx, the whole thing is integrated over not dx but dx bar. And then of course I get and this w is actually Øi. So it is Øi times f. And this and then so this entire thing equals. So I am now shifting all these boundary terms to the right side of equation. So I get Q1 Ø1 evaluated 0 plus Q3 Øi sorry Øi evaluated at h_e plus Q2 Øi′ evaluated at zero plus Q4 Øi′ evaluated at h_e.

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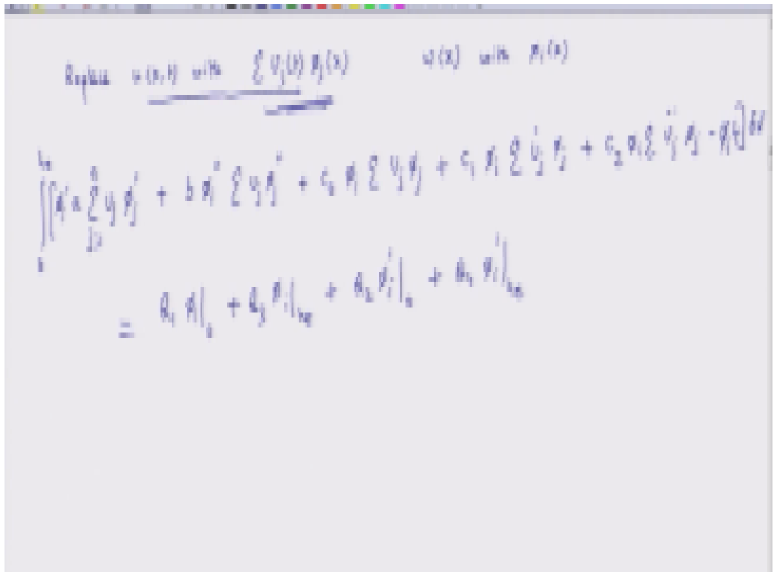

So this is my weak form and I have in this weak form I have also included the special approximation by and also in such a way that variables related to time and x are separated, okay.

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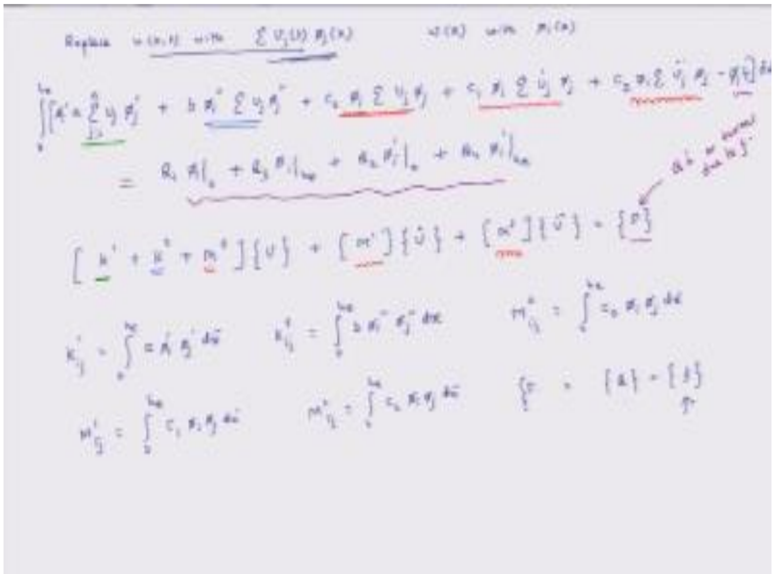

So the overall displacement function or u function primary variable function is our explicitly time component and explicitly space component and they are not intermingled. So this equation I can express it in this form k1 + k2. So basically these are three matrices plus m0 times u plus m1 times u. plus m2 times u.. is equal to a force function. So what is m0? So if I integrate these terms I get a matrix.

And from there I get m0. If I integrate these terms which are associated with velocity variable uj. then I get this m1 matrix. And if I integrate terms related to the second derivative of u then I get m2. And then finally I have two stiffness matrices, the first stiffness matrix comes from these terms. So that is K 1 and the second stiffness matrix comes from the terms underlined in light blue so it is here.

And the force term is this entire thing plus the contribution of Qi times f, okay. So this term has Q's and terms u2f. So this is there, so explicitly we will write down the relations quickly kij1 equals integral of aØiØj dx 0 to h_e kij2 is equal to 0 to h_e b times Øi ″ Øjdx" then m0 ij equals integral 0 to h_e, c0 ØiØj dx bar m1 ij equals 0 to h_e, c1 ØiØj dx bar and m2 ij equals 0 to h_e, c2 ØiØj dx bar, okay.

And important thing to notice okay, so then fi is equal to a q matrix plus an f matrix. This f matrix is time dependent and also x dependent and because the solutions of the system change with time so q can also change with time. So q can also vary with x and the time, this is important to note. Finally in this entire system we have assumed that a, b, c0, c1, c2, they can change with x.

So they can also they are also not necessarily position independent and some of the stiffness terms, some of the stiffness terms can not only depend on x because these constants but they can also vary with time, these terms a, b, can change with time. So k1 and k2 can change with time so this is important to note. So this completes are spatial approximation method and what we will do tomorrow is so what we have done is that.

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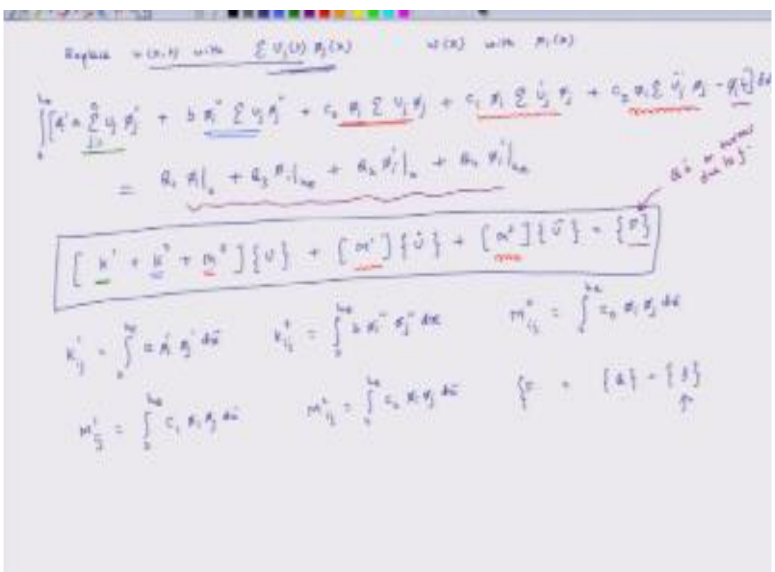

We have converted the entire partial differential equation into an ordinary differential equation because now we have nodal displacements in vector u, nodal velocities which are derivatives in time and nodal accelerations which are secondary derivative system time. So now the differentiability with respect to x is gone what we are left with now is only differentiability in time. So now we have reduced that PD second order partial into an ordinary differential equation of second order. So we will continue this discussion tomorrow and look forward to seeing you tomorrow, thanks.

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